J. Korean Math. Soc. **53** (2016), No. 1, pp. 27–43
http://dx.doi.org/10.4134/JKMS.2016.53.1.027

GLOBAL EXISTENCE AND BLOW-UP FOR A DEGENERATE REACTION-DIFFUSION SYSTEM WITH NONLINEAR LOCALIZED SOURCES AND NONLOCAL BOUNDARY CONDITIONS

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Abstract. This paper deals with a degenerate parabolic system with coupled nonlinear localized sources subject to weighted nonlocal Dirichlet boundary conditions. We obtain the conditions for global and blow-up solutions. It is interesting to observe that the weight functions for the nonlocal Dirichlet boundary conditions play substantial roles in determining not only whether the solutions are global or blow-up, but also whether the blowing up occurs for any positive initial data or just for large ones. Moreover, we establish the precise blow-up rate.

1. Introduction

In this paper we study the following degenerate parabolic system with coupled nonlinear localized sources subject to weighted nonlocal Dirichlet boundary conditions:

(1.1)
$$\begin{cases} u_t = \Delta u^m + au^{p_1}v^{q_1}(x_0,t),\ v_t = \Delta v^n + bv^{p_2}u^{q_2}(x_0,t), & (x,t)\in\Omega\times(0,T),\\ u = \int_\Omega f(x,y)u(y,t)dy,\ v = \int_\Omega g(x,y)v(y,t)dy, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $x_0 \in \Omega$ is a fixed point. $m, n > 1$, $a, b, q_1, q_2 > 0$, $p_1, p_2 \geq 0$ which ensure that equations in (1.1) are completely coupled with nonlinear localized reaction terms, while

Received February 26, 2014; Revised September 1, 2015.
2010 *Mathematics Subject Classification.* 34B10, 35K57, 35K65.
Key words and phrases. degenerate reaction-diffusion system, nonlocal boundary conditions, blow-up, blow-up rate.
Supported in part by China NSF Grant No. 11501442, the China Postdoctoral Science Foundation Grant No. 2013M540767, the Shanxi Provincial Postdoctoral Science Foundation, the scientific research program funded by Shanxi Provincial education department No. 14JK1474, and the doctor scientific research start fund project of Xi An University of Science and Technology Grant No. 2014QDJ042.

the weight functions $f(x,y)$, $g(x,y)$ in the boundary conditions are continuous, nonnegative on $\partial\Omega \times \Omega$, and $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy > 0$ on $\partial\Omega$. The initial values $u_0(x), v_0(x) \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ with $0 < \alpha < 1$ are nontrivial nonnegative and satisfy the compatibility conditions.

In the past several decades, there have been many articles deal with properties of solutions to porous medium equations or degenerate parabolic system with a localized source subject to homogeneous Dirichlet boundary condition and to a system of heat equations with nonlinear boundary condition (see [5, 9, 11, 13, 21, 25, 26] and references therein). However, there are some important phenomena formulated as parabolic equations which are coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [4, 6, 7]). In this case, the solution describes entropy per volume of the material. The problem of nonlocal boundary conditions for linear parabolic equations of the form

$$(1.2) \qquad \begin{cases} u_t - Au = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega \varphi(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with uniformly elliptic operator

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

and $c(x)) \leq 0$ was studied by Friedman [15]. It was proved that the unique solution of (1.2) tends to 0 monotonically and exponentially as $t \to \infty$ provided

$$\int_\Omega \mid \varphi(x,y) \mid dy \leq \rho < 1, \quad x \in \partial\Omega.$$

As for more general discussions on the dynamics of parabolic problem with nonlocal boundary conditions, one can see, e.g. [22] by Pao, where the following problem:

$$(1.3) \qquad \begin{cases} u_t - Lu = 0, & (x,t) \in \Omega \times (0,T), \\ Bu(x,t) = \int_\Omega K(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

was considered with

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}, \ Bu = \alpha_0\frac{\partial u}{\partial \nu} + u$$

and recently Pao [23] gave the numerical solutions for diffusion equations with nonlocal boundary conditions.

In [13], Du Lili studied the following degenerate reaction-diffusion system with coupled nonlinear localized sources subject to null Dirichlet boundary conditions:

(1.4)

$$\begin{cases} u_t = \Delta u^m + u^{p_1} v^{q_1}(x_0, t), \ v_t = \Delta v^n + v^{p_2} u^{q_2}(x_0, t), & (x,t) \in \Omega \times (0, T), \\ u = 0, \ v = 0, & (x,t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega. \end{cases}$$

They investigate the influence of localized sources and local terms on global existence and blow-up for this system. Moreover, they establish the precise blow-up rate estimates. In [27], Zheng et al. established global existence and blow-up conditions for solutions to the following semilinear parabolic system with weighted nonlocal Dirichlet boundary conditions:

(1.5)

$$\begin{cases} u_t = \Delta u + u^m \int_\Omega v^n(y,t)dy, \ v_t = \Delta v + v^p \int_\Omega u^q(y,t)dy, & (x,t) \in \Omega \times (0, T), \\ u = \int_\Omega \varphi(x,y)u(y,t)dy, \ v = \int_\Omega \psi(x,y)v(y,t)dy, & (x,t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega. \end{cases}$$

The global solutions and blow-up problems for the degenerate parabolic system with local nonlinearities, localized nonlinearities and nonlinear boundary conditions had also been studied extensively, see [1, 2, 3, 8, 10, 12, 14, 17, 18, 19, 20] and the references therein.

The present work is partially motivated by the above works, especially [13, 27]. We will get blow-up criteria for (1.1) with nonlocal Dirichlet boundary conditions, quite different from situations with the null Dirichlet boundary conditions [13]. We will show that the weigh functions $f(x, y)$ and $g(x, y)$ in the nonlocal boundary conditions of (1.1) play substantial roles in determining not only whether the solutions are global or blow-up, but also whether the blowing up occurs for any positive initial data or just for large ones. Moreover, we establish the precise blow-up rate estimates for all the blowup solutions. Our main results read as follows.

Theorem 1.1. *If $m > p_1$, $n > p_2$ and $q_1 q_2 < (m - p_1)(n - p_2)$, then the nonnegative solution of* (1.1) *is global.*

Theorem 1.2. *Assume $\int_\Omega f(x, y)dy < 1$ and $\int_\Omega g(x, t)dy < 1$ for all $x \in \partial\Omega$. If $m < p_1$ or $n < p_2$ or $q_1 q_2 > (m - p_1)(n - p_2)$, then the nonnegative solution of* (1.1) *is global for small initial data.*

Theorem 1.3. *Assume $q_1 q_2 = (m - p_1)(n - p_2)$,*

$$\int_\Omega f(x, y)dy < 1 \text{ and } \int_\Omega g(x, t)dy < 1$$

for all $x \in \partial\Omega$. If $m - p_1 = q_1$ and $n - p_2 = q_2$, then the nonnegative solution of (1.1) *exists globally provided that a and b are small.*

To describe blow-up conditions for solutions and to estimate the blow-up rate of the blow-up solution, we need the following assumptions on the initial data $u_0(x)$ and $v_0(x)$:

(H1) $\Delta u_0^m(x)+au_0^{p_1}(x)v^{q_1}(x_0)\geq 0$, $\Delta v_0^n(x)+bv_0^{p_2}(x)u^{q_2}(x_0)\geq 0$ for $x\in\Omega$;

(H2 there exists a constant $\delta\geq\delta_0>0$ such that

$$\Delta u_0^m(x)+au_0^{p_1}(x)v^{q_1}(x_0)-\delta u_0^{mk_1+1}(x)\geq 0,$$

$$\Delta v_0^n(x)+bv_0^{p_2}(x)u^{q_2}(x_0)-\delta v_0^{nk_2+1}(x)\geq 0,$$

where δ_0, k_1, k_2 will be given in Section 4.

Theorem 1.4. *If $m<p_1$ or $n<p_2$ or $q_1q_2>(m-p_1)(n-p_2)$, then the solution of* (1.1) *blows up in finite time for large initial data.*

Theorem 1.5. *Assume $p_1>1$ (or $p_2>1$) and the condition* (H1) *holds. If $\int_\Omega f(x,y)dy\geq 1$ (or $\int_\Omega g(x,t)dy\geq 1$) for all $x\in\partial\Omega$, then the solution of* (1.1) *blows up in finite time for any positive initial data.*

Theorem 1.6. *Assume $q_1q_2>(1-p_1)(1-p_2)$ and the condition* (H1) *holds. If $\int_\Omega f(x,y)dy\geq 1$ and $\int_\Omega g(x,t)dy\geq 1$ for all $x\in\partial\Omega$, then the solution of* (1.1) *blows up in finite time for any positive initial data.*

Theorem 1.7. *Assume that $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,t)dy\leq 1$ for all $x\in\partial\Omega$, $q_2+1-p_1, q_1+1-p_2>0$ and assumptions* (H1)–(H2) *hold. If the solution $(u(x,t),v(x,t))$ of* (1.1) *blows up in finite time T', then there exist positive constants C_i $(i=1,2,3,4)$ such that*

$$C_1(T'-t)^{-\frac{q_1-p_2+1}{q_1q_2-(1-p_1)(1-p_2)}}\leq\max_{x\in\overline{\Omega}}u(x,t)\leq C_2(T'-t)^{-\frac{q_1-p_2+1}{q_1q_2-(1-p_1)(1-p_2)}},$$

$$C_3(T'-t)^{-\frac{q_2-p_1+1}{q_1q_2-(1-p_1)(1-p_2)}}\leq\max_{x\in\overline{\Omega}}v(x,t)\leq C_2(T'-t)^{-\frac{q_2-p_1+1}{q_1q_2-(1-p_1)(1-p_2)}}.$$

This paper is organized as follows. In Section 2 deals with the maximum principle and comparison principle used for the model. In Section 3, we consider the global existence and nonexistence of solution of problem (1.1). Section 4 is devoted to the estimate of the blow-up rate.

2. Comparison principle and local existence

In this section, we give the comparison principle to the problem. Let $Q_T=\Omega\times(0,T)$, $S_T=\partial\Omega\times(0,T)$, $\overline{Q}_T=\overline{\Omega}\times[0,T)$.

Definition 2.1. A pair of functions $\underline{u},\underline{v}\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ is called a sub-solution of (1.1) if

$$\begin{cases}\underline{u}_t\leq\Delta\underline{u}^m+a\underline{u}^{p_1}\underline{v}^{q_1}(x_0,t),\ \underline{v}_t\leq\Delta\underline{v}^n+b\underline{v}^{p_2}\underline{u}^{q_2}(x_0,t), & (x,t)\in Q_T,\\ \underline{u}(x,t)\leq\int_\Omega f(x,y)\underline{u}(y,t)dy,\ \underline{v}(x,t)\leq\int_\Omega g(x,y)\underline{v}(y,t)dy, & (x,t)\in S_T,\\ \underline{u}(x,0)\leq u_0(x),\ \underline{v}(x,0)\leq v_0(x), & x\in\Omega.\end{cases}$$

Similarly, a super-solution of (1.1) is defined by the opposite inequalities.

Lemma 2.1. *Suppose that* $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *satisfy*

$$\begin{cases} u_t - d_1(x,t)\Delta u \geq c_1(x,t)u + c_2(x,t)v(x_0,t), & (x,t) \in Q_T, \\ v_t - d_2(x,t)\Delta v \geq c_3(x,t)v + c_4(x,t)u(x_0,t), & (x,t) \in Q_T, \\ u(x,t) \geq \left(\int_\Omega \psi_1(x,y)u^{\frac{1}{m}}(y,t)dy\right)^m, & (x,t) \in S_T, \\ v(x,t) \geq \left(\int_\Omega \psi_2(x,y)v^{\frac{1}{n}}(y,t)dy\right)^n, & (x,t) \in S_T, \\ u(x,0) \geq u_0(x) > 0, \ v(x,0) \geq v_0(x) > 0, & x \in \Omega, \end{cases}$$

where $m, n \geq 1$, $d_i(x,t) > 0$ *in* Q_T, $c_j \in C(Q_T)$ *and* $c_2(x,t), c_4(x,t) \geq 0$ *for* $(x,t) \in Q_T$, $\psi_i(x,y) \geq 0$ *on* $\partial\Omega \times \overline{\Omega}$, $\int_\Omega \psi_i(x,y)dy > 0$ *on* $\partial\Omega$, $i = 1,2$, $j = 1,2,3,4$. *Then* $u, v > 0$ *on* $\overline{Q}_T$.

Proof. Let $M_1 = \sup_{\overline{Q}_T} | c_1(x,t) |$ and $M_2 = \sup_{\overline{Q}_T} | c_3(x,t) |$. Set $w = e^{-\gamma t}u$, $z = e^{-\gamma t}v$ with $\gamma > \max\{M_1, M_2\}$. Then

(2.1)

$$\begin{cases} w_t - d_1(x,t)\Delta w + (\gamma - c_1(x,t))w \geq c_2(x,t)z(x_0,t), & (x,t) \in Q_T, \\ z_t - d_2(x,t)\Delta z + (\gamma - c_3(x,t))z \geq c_4(x,t)w(x_0,t), & (x,t) \in Q_T, \\ w \geq \left(\int_\Omega \psi_1(x,y)w^{\frac{1}{m}}(y,t)dy\right)^m, \ z \geq \left(\int_\Omega \psi_2(x,y)z^{\frac{1}{n}}(y,t)dy\right)^n, & (x,t) \in S_T, \\ u(x,0) \geq u_0(x) > 0, \ v(x,0) \geq v_0(x) > 0, & x \in \Omega. \end{cases}$$

It suffices to show that $w, z > 0$ on $\overline{Q}_T$. Since $u_0, v_0 > 0$, there exists $\delta > 0$ such that $w, z > 0$ for $(x,t) \in \overline{\Omega} \times (0, \delta)$. Suppose for a contradiction that $\overline{t} = \sup\{t \in (0,T) : w, z > 0 \text{ on } \overline{\Omega} \times [0,t)\} < T$. Then $w, z \geq 0$ on $\overline{Q}_{\overline{t}}$, and at least one of w, z vanishes at $(\overline{x}, \overline{t})$ for some $\overline{x} \in \overline{\Omega}$. Without loss of generality, suppose $w(\overline{x}, \overline{t}) = 0 = \inf_{\overline{Q}_{\overline{t}}} w$. If $(\overline{x}, \overline{t}) \in Q_{\overline{t}}$, by virtue of the first inequality of (2.1), we find that

$$w_t - d_1(x,t)\Delta w \geq (c_1(x,t) - \gamma)w, \quad (x,t) \in Q_{\overline{t}}.$$

This leads us to conclude that $w \equiv 0$ in $Q_{\overline{t}}$ by the strong maximum principle, a contradiction. If $(\overline{x}, \overline{t}) \in S_{\overline{t}}$, this results in a contradiction also, that

$$0 = w(\overline{x}, \overline{t}) = e^{-\gamma \overline{t}}u(\overline{x}, \overline{t}) = \int_\Omega \psi_1(\overline{x}, y)w(y, \overline{t})dy > 0$$

due to $\int_\Omega \psi_1(x,y)dy > 0$ on $\partial\Omega$. This proves $w, z > 0$, and in turn $u, v > 0$ on $\overline{Q}_T$. □

Lemma 2.2. *Suppose that* $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *satisfy*

$$\begin{cases} u_t - d_1(x,t)\Delta u \geq c_1(x,t)u + c_2(x,t)v(x_0,t), & (x,t) \in Q_T, \\ v_t - d_2(x,t)\Delta v \geq c_3(x,t)v + c_4(x,t)u(x_0,t), & (x,t) \in Q_T, \\ u(x,t) \geq \left(\int_\Omega \psi_1(x,y)u^{\frac{1}{m}}(y,t)dy\right)^m, & (x,t) \in S_T, \\ v(x,t) \geq \left(\int_\Omega \psi_2(x,y)v^{\frac{1}{n}}(y,t)dy\right)^n, & (x,t) \in S_T, \\ u(x,0) \geq u_0(x) \geq 0, \ v(x,0) \geq v_0(x) \geq 0, & x \in \Omega, \end{cases}$$

where $m,n\geq 1$, $d_i(x,t)>0$ *in* Q_T, $c_j\in C(Q_T)$ *and* $c_2(x,t),c_4(x,t)\geq 0$ *for* $(x,t)\in Q_T$, $\psi_i(x,y)\geq 0$ *on* $\partial\Omega\times\overline{\Omega}$, $\int_\Omega \psi_i(x,y)dy>0$ *on* $\partial\Omega$, $i=1,2$, $j=1,2,3,4$. *Then* $u,v\geq 0$ *on* $\overline{Q}_T$.

Proof. Let

$$u(x,t)=\alpha(x)w(x,t),\quad v(x,t)=\beta(x)z(x,t),$$

where $\alpha(x),\beta(x)\in C^2(\overline{\Omega})$ satisfy

$$\alpha(x)>0 \text{ on } \overline{\Omega};\quad \alpha(x)=2^{1-m},\ \int_\Omega \psi_1(x,y)\alpha^{\frac{1}{m}}(y)dy\leq\frac{1}{2} \text{ on } \partial\Omega, \tag{2.2}$$

and

$$\beta(x)>0 \text{ on } \overline{\Omega};\quad \beta(x)=2^{1-n},\ \int_\Omega \psi_2(x,y)\beta^{\frac{1}{n}}(y)dy\leq\frac{1}{2} \text{ on } \partial\Omega. \tag{2.3}$$

A routine computation shows

(2.4)

$$\begin{cases} w_t-d_1(x,t)\Delta w\geq\big(\dfrac{d_1(x,t)\Delta\alpha}{\alpha(x)}+c_1\big)w+\dfrac{c_2\beta(x_0)}{\alpha(x)}z(x_0,t), & (x,t)\in Q_T,\\ z_t-d_2(x,t)\Delta z\geq\big(\dfrac{d_2(x,t)\Delta\beta}{\beta(x)}+c_3\big)z+\dfrac{c_4\alpha(x_0)}{\beta(x)}w(x_0,t) & (x,t)\in Q_T,\\ w\geq 2^{m-1}\big(\int_\Omega\psi_1(x,y)\alpha^{\frac{1}{m}}(y)w^{\frac{1}{m}}(y,t)dy\big)^m, & (x,t)\in S_T,\\ z\geq 2^{n-1}\big(\int_\Omega\psi_2(x,y)\beta^{\frac{1}{n}}(y)z^{\frac{1}{n}}(y,t)dy\big)^n, & (x,t)\in S_T,\\ w(x,0)\geq u_0(x)/\alpha(x)\geq 0,\ z(x,0)\geq v_0(x)/\beta(x)\geq 0, & x\in\Omega. \end{cases}$$

Define

$$M_1=\sup_{Q_T}\Big|\,\frac{d_1(x,t)\Delta\alpha}{\alpha(x)}+c_1\,\Big|,\ M_2=\sup_{Q_T}\Big|\,\frac{d_2(x,t)\Delta\beta}{\beta(x)}+c_3\,\Big|,$$

$$N_1=\sup_{Q_T}\Big|\,\frac{c_2\beta(x_0)}{\alpha(x)}\,\Big|,\quad N_2=\sup_{Q_T}\Big|\,\frac{c_4\alpha(x_0)}{\beta(x)}\,\Big|\,.$$

Let

$$\widetilde{w}=w+\varepsilon e^{\gamma t},\quad \widetilde{z}=z+\varepsilon e^{\gamma t}$$

with

$$\gamma=\max\{M_1+N_1,M_2+N_2\},\quad \varepsilon>0.$$

Using the inequality

$$(k_1+k_2)^m\leq C(m)(k_1^m+k_2^m),\ k_1,k_2\geq 0,$$
$$0<m<1, C(m)=1;\ m>1,\ C(m)=2^{m-1},$$

and (2.2), for $(x,t)\in S_T$ we have

(2.5)

$$\begin{aligned} &\widetilde{w}(x,t)\\ &\geq 2^{m-1}\big(\int_\Omega\psi_1(x,y)\alpha^{\frac{1}{m}}(y)w^{\frac{1}{m}}(y,t)dy\big)^m+\varepsilon e^{\gamma t} \end{aligned}$$

$$\geq 2^{m-1}\Big[\big(\int_\Omega \psi_1(x,y)\alpha^{\frac{1}{m}}(y)w^{\frac{1}{m}}(y,t)dy\big)^m + \varepsilon e^{\gamma t}\big(\int_\Omega \psi_1(x,y)\alpha^{\frac{1}{m}}(y)dy\big)^m\Big]$$
$$\geq \big(\int_\Omega \psi_1(x,y)\alpha^{\frac{1}{m}}(y)\Big[w^{\frac{1}{m}}(y,t) + (\varepsilon e^{\gamma t})^{\frac{1}{m}}\Big]dy\big)^m$$
$$\geq \big(\int_\Omega \psi_1(x,y)\alpha^{\frac{1}{m}}(y)\widetilde{w}^{\frac{1}{m}}(y,t)dy\big)^m.$$

Similarly, from (2.3) we have

$$\widetilde{z}(x,t) \geq \big(\int_\Omega \psi_2(x,y)\beta^{\frac{1}{n}}(y)\widetilde{z}^{\frac{1}{n}}(y,t)dy\big)^n. \tag{2.6}$$

Combining (2.2), (2.5) and (2.6), we can get

$$\begin{cases} \widetilde{w}_t - d_1(x,t)\Delta\widetilde{w} \geq \big(\dfrac{d_1(x,t)\Delta\alpha}{\alpha(x)} + c_1\big)\widetilde{w} + \dfrac{c_2\beta(x_0)}{\alpha(x)}\widetilde{z}(x_0,t), & (x,t)\in Q_T,\\ \widetilde{z}_t - d_2(x,t)\Delta\widetilde{z} \geq \big(\dfrac{d_2(x,t)\Delta\beta}{\beta(x)} + c_3\big)\widetilde{z} + \dfrac{c_4\alpha(x_0)}{\beta(x)}\widetilde{w}(x_0,t) & (x,t)\in Q_T,\\ \widetilde{w} \geq \big(\int_\Omega \psi_1(x,y)\alpha^{\frac{1}{m}}(y)\widetilde{w}^{\frac{1}{m}}(y,t)dy\big)^m, & (x,t)\in S_T,\\ \widetilde{z} \geq \big(\int_\Omega \psi_2(x,y)\beta^{\frac{1}{n}}(y)\widetilde{z}^{\frac{1}{n}}(y,t)dy\big)^n, & (x,t)\in S_T,\\ \widetilde{w}(x,0) = w_0(x)+\varepsilon > 0,\ \widetilde{z}(x,0) = z_0(x)+\varepsilon > 0, & x\in\Omega. \end{cases}$$

By Lemma 2.1, we know that $\widetilde{w},\widetilde{z} > 0$, i.e., $w+\varepsilon e^{\gamma t} > 0$, $z+\varepsilon e^{\gamma t} > 0$ on $\overline{Q}_T$. It follows by $\varepsilon \to 0^+$ that $w,z \leq 0$ and hence $u,v\leq 0$. □

Using the scaling transformations (see Section 4):

$$U(x,\tau) = u^m(x,t),\ V(x,\tau) = (n/m)^{n/(n-1)}v^n(x,t),\ \tau = tm,$$

on the basis of the above lemmas, we obtain the following comparison principle for (1.1).

Theorem 2.3. *Let* $(\underline{u},\underline{v})$ *and* $(\overline{u},\overline{v})$ *be a sub-solution and super-solution of problem* (1.1) *on* $\overline{Q}_T$*, respectively. Then* $(\overline{u},\overline{v}) \geq (\underline{u},\underline{v})$ *on* $\overline{Q}_T$*.*

Local in time existence of positive classical solutions of problem (1.1) be obtained by using fixed point theorem [5, 14, 24]. Moreover, the uniqueness of solutions holds if $p_1,q_1,p_2,q_2 \geq 1$. The proof is more or less standard, so it is omitted here. In view of Lemmas 2.1–2.2, we have the following:

Lemma 2.4. *Suppose that* (u_0,v_0) *satisfies* (H1). *Then the solution* (u,v) *of* (1.1) *satisfies* $u_t,v_t\geq 0$ *in any compact subset of* Q_T*.*

3. Global existence and blow-up

Compared with usual homogeneous Dirichlet boundary conditions, due to the boundary functions $f(x,y)$, $g(x,y)$ being nonnegative, satisfying

$$\int_\Omega f(x,y)dy > 0 \text{ and } \int_\Omega g(x,y) > 0$$

for all $x \in \partial\Omega$, the proof of the global existence or global nonexistence results for the system (1.1) would be more difficult. Denote

$$A = \begin{pmatrix} m - p_1 & -q_1 \\ -q_2 & n - p_2 \end{pmatrix}, \quad l = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.$$

Lemma 3.1 (see [9]). *If $m > p_1$, $n > p_2$ and $q_1q_2 < (m-p_1)(n-p_2)$, then there exist two positive constants l_1, l_2, such that $Al = (1,1)^T$. Moreover, $A(cl) > (0,0)^T$ for any constant $c > 0$.*

For convenience, we will denote

$$\Pi_1(u,v) = u_t - \Delta u^m - au^{p_1}v^{q_1}(x_0,t),\ \Pi_2(u,v) = v_t - \Delta v^n - bv^{p_2}u^{q_2}(x_0,t).$$

Proof of Theorem 1.1. It is easy to prove that there exists a positive function $\phi \in C^2(\overline{\Omega})$ such that

$$\varepsilon\phi(x) \geq \max\{\Lambda_1(x),\ \Lambda_2(x)\}, \quad \text{for } x \in \partial\Omega,$$

where

$$\Lambda_1(x) = (\int_\Omega f^{\frac{m}{m-1}}(x,y)dy)^{m-1}\int_\Omega(\varepsilon\phi(y)+\varphi(y)+1)dy - 1, \quad x \in \partial\Omega,$$

and

$$\Lambda_2(x) = (\int_\Omega g^{\frac{n}{n-1}}(x,y)dy)^{n-1}\int_\Omega(\varepsilon\phi(y)+\varphi(y)+1)dy - 1, \quad x \in \partial\Omega,$$

$0 < \varepsilon \leq \max_{\overline{\Omega}} 1/(2\mid\Delta\phi\mid)$ is a constant and φ is the solution of the following elliptic problem:

$$-\Delta\varphi = 1, \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial\Omega.$$

Let $C_1 = \max_{x\in\overline{\Omega}}\varphi(x)$, $C_2 = \max_{x\in\overline{\Omega}}\varphi(x)$. We construct a super-solution which exists global for any $T > 0$ as

$$\overline{u} = \alpha e^{l_1 t}(\varepsilon\phi(x)+\varphi(x)+1)^{1/m}, \quad \overline{v} = \beta e^{l_2 t}(\varepsilon\phi(x)+\varphi(x)+1)^{1/n}, \tag{3.1}$$

where $0 < l_1, l_2 < 1$ satisfy $ml_1, nl_2 < 1$ and $\alpha, \beta > 0$ are to be chosen later. Clearly, $(\overline{u},\overline{v})$ is bounded for any $t > 0$ and $\overline{u} \geq \alpha$, $\overline{v} \geq \beta$. The direct computation gives

$$\begin{cases} \Pi_1(\overline{u},\overline{v}) = \alpha l_1 e^{l_1 t}(\varepsilon\phi(x)+\varphi(x)+1)^{1/m} + \alpha^m e^{l_1 mt} - \alpha^m e^{l_1 mt}\varepsilon\Delta\phi(x) \\ \qquad -a\alpha^{p_1}\beta^{q_1}e^{l_1p_1t+q_1l_2t}(\varepsilon\phi(x)+\varphi(x)+1)^{p_1/m}(\varepsilon\phi(x_0)+\varphi(x_0)+1)^{q_1/n}, \\ \qquad \geq \frac{1}{2}\alpha^m e^{l_1mt} - a\alpha^{p_1}\beta^{q_1}e^{l_1p_1t+q_1l_2t}(\varepsilon C_1+C_2+1)^{p_1/m+q_1/n}, \\ \Pi_2(\overline{u},\overline{v}) \geq \frac{1}{2}\beta^n e^{l_2nt} - b\beta^{p_2}\alpha^{q_2}e^{l_2p_2t+q_2l_1t}(\varepsilon C_1+C_2+1)^{p_2/n+q_2/m}. \end{cases}$$

If $m > p_1$, $n > p_2$ and $q_1q_2 < (m-p_1)(n-p_2)$, by Lemma 3.1, there exist positive constants l_1, l_2 such that

$$ml_1 > p_1l_1 + q_1l_2, \quad nl_2 > p_2l_2 + q_2l_1, \quad \text{and } ml_1,\ nl_2 < 1.$$

Therefore, we can choose α, β sufficiently large that

$$\alpha \geq \max\big\{2^{\frac{n-p_2+q_1}{D}}a^{\frac{q_1}{D}}b^{\frac{n-p_2}{D}}(\varepsilon C_1+C_2+1)^{\frac{q_1q_2-p_1p_2+p_1n+q_1m}{mD}},\ \max_{x\in\overline{\Omega}}u_0(x)\big\},$$

$$\beta \geq \max\left\{2^{\frac{m-p_1+q_2}{D}} a^{\frac{q_2}{D}} b^{\frac{m-p_1}{D}} (\varepsilon C_1 + C_2 + 1)^{\frac{q_1q_2-p_1p_2+p_2m+q_2n}{nD}},\ \max_{x\in\overline{\Omega}} v_0(x)\right\},$$

where $D = (m - p_1)(n - p_2) - q_1q_2$, then

$$\Pi_1(\overline{u}, \overline{v}) \geq 0,\ \Pi_2(\overline{u}, \overline{v}) \geq 0,\ \text{and } \overline{u} \geq u_0(x),\ \overline{v} \geq v_0(x).$$

Also, for $(x, t) \in S_T$, we have

$$\begin{aligned}\overline{u}(x,t) &= \alpha e^{l_1 t}(\varepsilon\phi(x) + 1)^{1/m} \\ &\geq \alpha e^{l_1 t}\Big(\int_\Omega f^{\frac{m}{m-1}}(x,y)dy\Big)^{m-1/m}\Big(\int_\Omega(\varepsilon\phi(y) + \varphi(y) + 1)dy\Big)^{1/m} \\ &\geq \alpha e^{l_1 t}\int_\Omega f(x,y)(\varepsilon\phi(y) + \varphi(y) + 1)^{1/m}dy = \int_\Omega f(x,y)\overline{u}(y,t)dy.\end{aligned}$$

Similarly, for $(x, t) \in S_T$, we have

$$\overline{v}(x,t) \geq \int_\Omega f(x,y)\overline{v}(y,t)dy.$$

Now, $(\overline{u}, \overline{v})$ defined by (3.1) is a positive super-solution of (1.1). By Theorem 2.3, we conclude $(u, v) \leq (\overline{u}, \overline{v})$, which implies (u, v) exists globally. □

Proof of Theorem 1.2. Case 1: Assume $m < p_1$. Define

$$\max\Big\{\max_{x\in\partial\Omega}\int_\Omega f(x,y)dy,\ \max_{x\in\partial\Omega}\int_\Omega g(x,y)dy\Big\} = \rho \in (0, 1).$$

Let w be the unique solution of the elliptic problem

$$-\Delta w = 1,\ x \in \Omega;\quad w(x) = C_0,\ x \in \partial\Omega.$$

The $C_0 \leq w \leq C_0 + M$ for some $M > 0$ independent of C_0. Let C_0 be so large such that

$$\frac{1 + C_0}{1 + C_0 + M} \geq \max\{\rho^m,\ \rho^n\}.$$

Due to $m < p_1$ and $q_2 > 0$, it is easy to verify that for fixed positive constants C_0, M and K_2, there exists $K_1 > 0$ small such that

$$K_1 \geq aK_1^{\frac{p_1}{m}}K_2^{\frac{q_1}{n}}(1+C_0+M)^{\frac{p_1}{m}+\frac{q_1}{n}},\ K_2 \geq bK_1^{\frac{q_2}{m}}K_2^{\frac{p_2}{n}}(1+C_0+M)^{\frac{q_2}{m}+\frac{p_2}{n}}. \tag{3.2}$$

Set $\overline{u}(x,t) = \big(K_1(1 + w(x))\big)^{1/m}$, $\overline{u}(x,t) = \big(K_2(1 + w(x))\big)^{1/n}$. We have

$$\begin{cases}\Pi_1(\overline{u}, \overline{v}) \geq K_1 - aK_1^{\frac{p_1}{m}}K_2^{\frac{q_1}{n}}(1 + C_0 + M)^{\frac{p_1}{m}+\frac{q_1}{n}} \geq 0, \\ \Pi_2(\overline{u}, \overline{v}) \geq K_2 - bK_1^{\frac{q_2}{m}}K_2^{\frac{p_2}{n}}(1 + C_0 + M)^{\frac{q_2}{m}+\frac{p_2}{n}} \geq 0.\end{cases}$$

On the other hand, we have on the boundary that

$$\begin{aligned}\overline{u}(x,t) &= \big(K_1(1 + C_0)\big)^{\frac{1}{m}} \geq \big(K_1\rho^m(1 + C_0 + M)\big)^{\frac{1}{m}} \\ &\geq \int_\Omega \big(K_1(1 + C_0 + M)\big)^{\frac{1}{m}} f(x,y)dy \\ &\geq \int_\Omega f(x,y)\overline{u}(y,t)dy,\ x \in \partial\Omega,\ t > 0.\end{aligned}$$

Similarly,

$$\overline{v}(x,t) \geq \int_\Omega g(x,y)\overline{v}(y,t)dy,\ x \in \partial\Omega,\ t > 0.$$

By Theorem 2.3, $(\overline{u}, \overline{v})$ is a global super-solution of (1.1) provided the initial data are so mall such that $u_0(x) \leq \big(K_1(1+w(x))\big)^{\frac{1}{m}}$, $v_0(x) \leq \big(K_2(1+w(x))\big)^{\frac{1}{n}}$ for $x \in \Omega$.

Case 2: Assume $n < p_2$. The case can be treated by exchanging the roles of u and v in the case 1.

Case 3: Assume $q_1q_2 > (m-p_1)(n-p_2)$. For the case, we only need to prove the case of $m \geq p_1$ and $n \geq p_2$. We claim that (3.2) holds with sufficiently small K_1 and K_2. In fact, in the special case of $m = p_1$, the first inequality in (3.2) is trivial with small K_2 independent of K_1, and then the second one in (3.2) is true also provided K_1 is small. The same argument admits $n = p_2$. If $m > p_1$, $n > p_2$ with $q_1q_2 > (m-p_1)(n-p_2)$, then $0 < n-p_2 < q_1q_2/(m-p_1)$, and hence

$$\begin{aligned} K_2^{1-\frac{p_2}{n}} &\geq bK_1^{\frac{q_2}{m}}(1+C_0+M)^{\frac{p_2}{n}+\frac{q_2}{m}} \\ &\geq ba^{\frac{q_2}{m-p_1}}K_2^{\frac{q_1q_2}{n(m-p_1)}}(1+C_0+M)^{\frac{p_2}{n}+\frac{q_2}{m}+(\frac{p_1}{m}+\frac{q_1}{n})\frac{q_2}{m-p_1}} \end{aligned} \tag{3.3}$$

for K_1 and K_2 small enough. Clearly, (3.3) is equivalent to (3.2). Like for the proof for the case 1, we know that the solution of (1.1) for small initial data $u_0(x) \leq \big(K_1(1+w(x))\big)^{\frac{1}{m}}$, $v_0(x) \leq \big(K_2(1+w(x))\big)^{\frac{1}{n}}$ for $x \in \Omega$. □

Proof of Theorem 1.3. Denote

$$\rho_0 = \max_{x\in\partial\Omega}\Big\{\int_\Omega f(x,y)dy,\ \int_\Omega g(x,y)dy\Big\} < 1.$$

Let $\psi(x)$ be the unique solution of the following elliptic problem:

$$-\Delta\psi(x) = \varepsilon_0, \quad x \in \Omega; \quad \psi(x) = \rho_0, \quad x \in \partial\Omega, \tag{3.4}$$

where ε_0 is a positive constant such that $0 \leq \psi(x) \leq 1$ (as $\rho_0 < 1$, there exists such ε_0). Set $\max_{x\in\overline{\Omega}}\psi(x) = K$. Let

$$w_1(x,t) = L\psi^{\frac{1}{m}}(x), \quad w_2(x,t) = L\psi^{\frac{1}{n}}(x),$$

where L is a constant to be determined later. A series of computations yields

$$\begin{cases} \Pi_1(w_1,w_2) = L^m\varepsilon_0 - aL^{p_1+q_1}\psi^{\frac{p_1}{m}}\psi^{\frac{q_1}{n}}(x_0) \geq L^m\varepsilon_0 - aL^{p_1+q_1}K^{\frac{p_1}{m}+\frac{q_1}{n}}, \\ \Pi_2(w_1,w_2) = L^m\varepsilon_0 - bL^{p_2+q_2}\psi^{\frac{p_2}{n}}\psi^{\frac{q_2}{m}}(x_0) \geq L^m\varepsilon_0 - bL^{p_2+q_2}K^{\frac{p_2}{n}+\frac{q_2}{m}}. \end{cases}$$

We choose $a \leq \varepsilon_0K^{-p_1/m-q_1/n}$, $b \leq \varepsilon_0K^{-p_2/n-q_2/m}$. Then

$$\Pi_1(w_1,w_2) \geq 0, \quad \Pi_2(w_1,w_2) \geq 0.$$

On the other hand, we have

$$w_1(x,t) = L\rho_0^{\frac{1}{m}} \geq L(\int_\Omega f(x,y)dy)^{\frac{1}{m}} \geq L\int_\Omega f(x,y)dy$$

$$\geq L\int_\Omega f(x,y)\psi^{\frac{1}{m}}dy=\int_\Omega f(x,y)w_1(y,t)dy,\quad \text{for } x\in\partial\Omega,\ t>0,$$

$$w_2(x,t)\geq\int_\Omega g(x,y)w_2(y,t)dy,\quad \text{for } x\in\partial\Omega,\ t>0.$$

Here we used $\int_\Omega f(x,y)dy<1$, $\int_\Omega g(x,y)dy<1$ and $0\leq\psi(x)\leq 1$. Therefore, (w_1,w_2) is an upper solution of (1.1). By Theorem 2.3, $w_1(x,t)\geq u(x,t)$, $w_2(x,t)\geq v(x,t)$. Thus, (u,v) exists globally. □

Next prove the blow-up conclusions with or without large initial data (Theorems 1.4–1.6).

Proof of Theorem 1.4. We consider the following well-known degenerate reaction-diffusion system with nonlinear localized sources (see [27]):

(3.5)

$$\begin{cases}\underline{u}_t=\Delta\underline{u}^m+a\underline{u}^{p_1}\underline{v}^{q_1}(x_0,t),\ \underline{v}_t=\Delta\underline{v}^n+b\underline{v}^{p_2}\underline{u}^{q_2}(x_0,t), & (x,t)\in\Omega\times(0,T),\\ \underline{u}=0,\ \underline{v}=0, & (x,t)\in\partial\Omega\times(0,T),\\ \underline{u}(x,0)=u_0(x),\underline{v}(x,0)=v_0(x), & x\in\Omega.\end{cases}$$

Let $(\underline{u},\underline{v})$ be the solution of the system. It is obviously that $(\underline{u},\underline{v})$ is a subsolution of (1.1). It is known to all that the nonnegative solution of (3.5) blows up in finite time for sufficiently large initial values provided $m<p_1$ or $n<p_2$ or $q_1q_2>(m-p_1)(n-p_2)$. By Theorem 2.3, the solution of (1.1) blows up in finite time for sufficiently large initial values. □

Proof of Theorem 1.5. Since u_0, $v_0>0$ for $x\in\Omega$, $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy>0$ for $x\in\partial\Omega$, and

$$u_0(x)=\int_\Omega f(x,y)u_0(y)dy,\quad v_0(x)=\int_\Omega g(x,y)v_0(y)dy,\quad x\in\partial\Omega,$$

by the compatibility conditions, we have $u_0,v_0>0$ for $x\in\partial\Omega$. Denote by ϵ the positive constant such that $u_0,v_0\geq\epsilon$ for $x\in\overline{\Omega}$. By Lemma 2.4, we have $u,v\geq\epsilon$ for $(x,t)\in\overline{\Omega}\times[0,T)$. Furthermore, $u(x,t)$ satisfies

$$\begin{cases}u_t\geq\Delta u^m+a\epsilon^{q_1}u^{p_1}(x,t), & (x,t)\in\Omega\times(0,T),\\ u=\int_\Omega f(x,y)u(y,t)dy, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega.\end{cases}$$

Let $\underline{u}(x,t)\equiv s(t)$ be the unique solution of the ODE problem

$$\begin{cases}s'(t)=a\epsilon^{q_1}s^{p_1}(t),\\ s(0)=\frac{1}{2}\epsilon.\end{cases}$$

Then $\underline{u}(x,t)$ blows up in finite time since $p_1>1$. Clearly,

$$\underline{u}_t=\Delta\underline{u}^m+a\epsilon^{q_1}\underline{u}^{p_1},\quad \underline{u}(x,0)\leq u_0(x).$$

Furthermore, the assumption $\int_\Omega f(x,y)dy \geq 1$ implies

$$\underline{u}(x,t) \leq \underline{u}\int_\Omega f(x,y)dy = s(t)\int_\Omega f(x,y)dy = \int_\Omega f(x,y)\underline{u}(y,t)dy,\ (x,t)\in S_T.$$

By Theorem 2.3, $u(x,t) \geq \underline{u}(x,t)$ as long as both $u(x,t)$ and $\underline{u}(x,t)$ exist, and thus $u(x,t)$ blows up in finite time for any positive initial data. □

Proof of Theorem 1.6. We know from the proof of Theorem 1.5 that $u, v \geq \epsilon$ for $(x,t)\in\overline{\Omega}\times[0,T)$. Let $(\underline{u}(x,t),\underline{v}(x,t)) \equiv (\omega(t),\mu(t))$ be the unique solution of the ODE problem

$$\begin{cases}\omega'(t) = a\omega^{p_1}(t)\mu^{q_1}(t), \quad \mu'(t) = b\mu^{p_2}(t)\omega^{q_2}(t),\\ \omega(0) = \frac{1}{2}\epsilon, \quad \mu(0) = \frac{1}{2}\epsilon.\end{cases}$$

We know with $q_1q_1 \geq (1-p_1)(1-p_2)$ that $(\underline{u}(x,t),\underline{v}(x,t))$ blows up in finite time (see [27]). Similarly the proof of Theorem 1.5, $(\underline{u}(x,t),\underline{v}(x,t))$ satisfies

$$\begin{cases}\underline{u}_t = \Delta\underline{u}^m + a\underline{u}^{p_1}\underline{v}^{q_1}(x_0,t),\ \underline{v}_t = \Delta\underline{v}^n + b\underline{v}^{p_2}\underline{u}^{q_2}(x_0,t), & (x,t)\in\Omega\times(0,T),\\ \underline{u} \leq \int_\Omega f(x,y)\underline{u}(y,t)dy,\ \underline{v} \leq \int_\Omega g(x,y)\underline{v}(y,t)dy, & (x,t)\in\partial\Omega\times(0,T),\\ \underline{u}(x,0) \leq u_0(x),\ \underline{v}(x,0)\leq v_0(x), & x\in\Omega.\end{cases}$$

In view of Theorem 2.3, we have $(u,v) \geq (\underline{u}(x,t),\underline{v}(x,t))$ for their common existence time. Thus $u(x,t)$ blows up in finite time for any positive initial data. □

4. Blow-up rate estimates

In this section, we will show the blow-up rate of solution to problem (1.1). We also assume that the solution (U,V) blows up in finite time T^*. To obtain the estimate, we first introduce some transformations. Let $U(x,\tau) = u^m(x,t),\ V(x,\tau) = (n/m)^{n/(n-1)}v^n(x,t),\ \tau = tm$, then (1.1) becomes the following system not in divergence form:

$$\begin{cases}U_\tau = U^{r_1}\big(\Delta U + a_1U^{p_3}V^{q_3}(x_0,\tau)\big), & x\in\Omega,\ \tau>0,\\ V_\tau = V^{r_2}\big(\Delta V + b_1V^{p_4}U^{q_4}(x_0,\tau)\big), & x\in\Omega,\ \tau>0,\\ U = \big(\int_\Omega f(x,y)U^{\frac{1}{m}}(y,\tau)dy\big)^m, & x\in\partial\Omega,\ \tau>0,\\ V = \big(\int_\Omega g(x,y)V^{\frac{1}{n}}(y,\tau)dy\big)^n, & x\in\partial\Omega,\ \tau>0,\\ U(x,0) = U_0(x),\ V(x,0) = V_0(x), & x\in\Omega,\end{cases} \tag{4.1}$$

where $0 < r_1 = (m-1)/m$, $0 < r_2 = (m-1)/n$, $p_3 = p_1/m$, $q_3 = q_1/n$, $p_4 = p_2/n$, $q_4 = q_2/m$, $a_1 = a(m/n)^{q_1/(n-1)}$, $b_1 = b(m/n)^{(p_2-n/(n-1)}$, $U_0(x) = u_0^m(x)$, $V_0(x) = (n/m)^{n/(n-1)}v_0^n(x)$. By the conditions $q_2 > p_1-1$ and $q_1 > p_2-1$, we have $q_4-p_3-r_1+1>0$ and $q_3-p_4-r_2+1>0$.

Under this transformation, the assumptions (H1)–(H2) become

(H1′) $\Delta U_0(x) + a_1U_0^{p_3}(x)V^{q_3}(x_0) \geq 0$, $\Delta V_0(x) + b_1V_0^{p_4}(x)U^{q_4}(x_0) \geq 0$ for $x\in\Omega$;

(H2′) there exists a constant $\delta \geq \delta_0 > 0$ such that

$$\Delta U_0(x) + a_1 U_0^{p_3}(x) V^{q_3}(x_0) - \delta U_0^{k_1+1-r_1}(x) \geq 0,$$

$$\Delta V_0(x) + b_1 V_0^{p_4}(x) U^{q_4}(x_0) - \delta v_0^{k_2+1-r_2}(x) \geq 0,$$

where δ_0, k_1, k_2 will be given later.

Denote $M_1(\tau) = \max_{x\in\overline{\Omega}} U(x,\tau)$, $M_2(\tau) = \max_{x\in\overline{\Omega}} V(x,\tau)$. We can obtain the blow-up rate from the following lemmas.

Lemma 4.1. *Assume that* $U_0(x)$, $V_0(x)$ *satisfy* (H1′)–(H2′). *If* $q_4 - p_3 - r_1 + 1 > 0$ *and* $q_3 - p_4 - r_2 + 1 > 0$, *then there exists a positive constant* C_5 *such that*

(4.2)
$$M_1^{q_4-p_3-r_1+1}(\tau) + M_2^{q_3-p_4-r_2+1}(\tau) \geq C_5 (T^* - \tau)^{-\frac{(q_4-p_3-r_1+1)(q_3-p_4-r_2+1)}{q_3 q_4 - (1-p_3-r_1)(1-p_4-r_2)}}.$$

Proof. From (4.1), we have (see [16], Theorem 4.5)

$$M_1'(\tau) \leq a_1 M_1^{p_3+r_1} M_2^{q_3}, \quad M_2'(\tau) \leq b_1 M_1^{q_4} M_2^{p_4+r_2}, \quad \text{a.e. } t \in (0, T^*).$$

Noticing that $q_4 - p_3 - r_1 + 1 > 0$ and $q_3 - p_4 - r_2 + 1 > 0$, hence we have

$$\begin{aligned} &\big(M_1^{q_4-p_3-r_1+1}(\tau) + M_2^{q_3-p_4-r_2+1}(\tau)\big)' \\ &\leq \big((q_4 - p_3 - r_1 + 1)a_1 + (q_3 - p_4 - r_2 + 1)b_1\big) M_1^{q_4}(\tau) M_2^{q_3}(\tau) \\ (4.3) \quad &\leq C_6 \big(M_1^{q_4-p_3-r_1+1}(\tau) + M_2^{q_3-p_4-r_2+1}(\tau)\big)^{\frac{q_3(q_4-p_3-r_1+1)+q_4(q_3-p_4-r_2+1)}{(q_4-p_3-r_1+1)(q_3-p_4-r_2+1)}}, \end{aligned}$$

by virtue of Young's inequality. Integrating (4.3) from τ to T^*, we draw the conclusion. □

Lemma 4.2. *Assume that* $U_0(x)$, $V_0(x)$ *satisfy* (H1′)–(H2′),

$$\int_\Omega f(x,y)dy, \int_\Omega g(x,t)dy \leq 1$$

for all $x \in \partial\Omega$. *If* $q_4 - p_3 - r_1 + 1 > 0$ *and* $q_3 - p_4 - r_2 + 1 > 0$, *then*

(4.4)
$$U_\tau(x,\tau) - \delta U^{k_1+1}(x,\tau) \geq 0, \; V_\tau(x,\tau) - \delta V^{k_2+1}(x,\tau) \geq 0, \; (x,\tau) \in \Omega \times (0, T^*),$$

where

$$k_1 = \frac{q_3 q_4 - (1-p_3-r_1)(1-p_4-r_2)}{q_3 - p_4 - r_2 + 1}, \; k_2 = \frac{q_3 q_4 - (1-p_3-r_1)(1-p_4-r_2)}{q_4 - p_3 - r_1 + 1},$$

$$\delta_1 = \frac{a_1 k_1 (1 + k_1 - p_3)}{r_1(2k_1 + 1 - r_1 - p_3)} \Big(\frac{1 + k_1 - p_3}{q_3 + k_2}\Big)^{\frac{q_3(2k_1+1-r_1-p_3)}{(q_3+k_2)k_1}},$$

$$\delta_1 = \frac{b_1 k_2 (1 + k_2 - p_4)}{r_2(2k_2 + 1 - r_2 - p_4)} \Big(\frac{1 + k_2 - p_4}{q_4 + k_1}\Big)^{\frac{q_4(2k_2+1-r_2-p_4)}{(q_4+k_1)k_2}},$$

$$\delta > \delta_0 = \max\{|\,\delta_1\,|, \; |\,\delta_2\,|\}.$$

Proof. Let $J_1(x,\tau)=U_\tau-\delta U^{k_1+1}$, $J_2=V_\tau-\delta V^{k_2+1}$. A series of computations yields

$$\begin{aligned}&J_{1\tau}-U^{r_1}\Delta J_1-\big(2\delta r_1U^{k_1}+a_1p_3U^{r_1+p_3-1}V^{q_3}(x_0,\tau)\big)J_1\\&\quad-a_1q_3U^{r_1+p_3}V^{q_3-1}(x_0,\tau)J_2(x_0,\tau)\\&=r_1U^{-1}J_1^2+\delta k_1(k_1+1)U^{k_1+r_1-1}\mid\nabla U\mid^2+r_1\delta^2U^{2k_1+1}\\&\quad+a_1q_3\delta U^{r_1+p_3}V^{q_3+k_2}(x_0,\tau)-a_1\delta(1+k_1-p_3)U^{k_1+r_1+p_3}V^{q_3}(x_0,\tau)\\&\geq r_1\delta^2U^{2k_1+1}+a_1q_3\delta U^{r_1+p_3}V^{q_3+k_2}(x_0,\tau)\\&\quad-a_1\delta(1+k_1-p_3)U^{k_1+r_1+p_3}V^{q_3}(x_0,\tau).\end{aligned}$$

If $1+k_1\leq p_3$, obviously we have

$$\begin{aligned}&J_{1\tau}-U^{r_1}\Delta J_1-\big(2\delta r_1U^{k_1}+a_1p_3U^{r_1+p_3-1}V^{q_3}(x_0,\tau)\big)J_1\\&-a_1q_3U^{r_1+p_3}V^{q_3-1}(x_0,\tau)J_2(x_0,\tau)\geq 0.\end{aligned}\tag{4.5}$$

Otherwise, noticing that $k_1/(2k_1+1-r_1-p_3)+q_3/(q_3+k_2)=1$, by virtue of Young's inequality

$$U^{k_1}V^{q_3}(x_0,\tau)\leq\frac{k_1}{2k_1+1-r_1-p_3}(\theta U^{k_1})^{\frac{2k_1+1-r_1-p_3}{k_1}}+\frac{q_3}{q_3+k_2}\Big(\frac{V^{q_3}(x_0,\tau)}{\theta}\Big)^{\frac{q_3+k_2}{q_3}},$$

where $\theta=(\frac{k_1+1-p_3}{q_3+k_2})^{q_3/(q_3+k_2)}$. Thus, we have

$$\begin{aligned}&J_{1\tau}-U^{r_1}\Delta J_1-\big(2\delta r_1U^{k_1}+a_1p_3U^{r_1+p_3-1}V^{q_3}(x_0,\tau)\big)J_1\\&\quad-a_1q_3U^{r_1+p_3}V^{q_3-1}(x_0,\tau)J_2(x_0,\tau)\\&\geq r_1\delta^2U^{2k_1+1}+a_1q_3\delta U^{r_1+p_3}V^{q_3+k_2}(x_0,\tau)\\&\quad-a_1\delta(1+k_1-p_3)U^{k_1+r_1+p_3}V^{q_3}(x_0,\tau)\\&\geq r_1\delta(\delta-\delta_1)U^{2k_1+1}\geq 0.\end{aligned}\tag{4.6}$$

Similarly, we also have

$$\begin{aligned}&J_{2\tau}-V^{r_2}\Delta J_2-\big(2\delta r_2V^{k_2}+b_1p_4V^{r_2+p_4-1}U^{q_4}(x_0,\tau)\big)J_2\\&-b_1q_4V^{r_2+p_4}V^{q_4-1}(x_0,\tau)J_1(x_0,\tau)\geq 0.\end{aligned}\tag{4.7}$$

Fix $(x,t)\in\partial\Omega\times(0,T^*)$, we have

$$\begin{aligned}J_1(x,\tau)&=U_\tau-\delta U^{k_1+1}\\&=\Big(\int_\Omega f(x,y)u(y,\tau)dy\Big)^{m-1}\\&\quad\Big(m\int_\Omega f(x,y)u_\tau(y,\tau)dy-\delta\Big(\int_\Omega f(x,y)u(y,\tau)dy\Big)^{k_1m+1}\Big).\end{aligned}$$

Since $U_\tau(y,\tau)=J_1(y,\tau)+\delta U^{k_1+1}(y,\tau)$, we have

$$m\int_\Omega f(x,y)u_\tau(y,\tau)dy-\delta\Big(\int_\Omega f(x,y)u(y,\tau)dy\Big)^{k_1m+1}$$

$$= \int_\Omega f(x,y)U^{\frac{1-m}{m}}(y,\tau)J_1(y,\tau)dy + \delta\Big(\int_\Omega f(x,y)U^{\frac{k_1m+1}{m}}(y,\tau)dy - \big(\int_\Omega f(x,y)U^{\frac{1}{m}}(y,\tau)dy\big)^{k_1m+1}\Big).$$

Noticing that $0 < F(x) = \int_\Omega f(x,y)dy \le 1,\ x \in \partial\Omega$, we can apply Jensen's inequality to the last integral in the above inequality,

$$\int_\Omega f(x,y)U^{\frac{k_1m+1}{m}}(y,\tau)dy - \big(\int_\Omega f(x,y)U^{\frac{1}{m}}(y,\tau)dy\big)^{k_1m+1}$$
$$\ge F(x)\Big(\int_\Omega f(x,y)U^{\frac{1}{m}}(y,\tau)dy/F(x)\Big)^{k_1m+1} - \Big(\int_\Omega f(x,y)U^{\frac{1}{m}}(y,\tau)dy\Big)^{k_1m+1}$$
$$\ge 0.$$

Here we use $mk_1+1>1$ and $0<F(x)\le 1$ in the last inequality. Hence for $(x,t)\in\partial\Omega\times(0,T^*)$, we have

$$J_1(x,\tau) \ge \Big(\int_\Omega f(x,y)U^{\frac{1}{m}}(y,\tau)dy\Big)^{m-1}\int_\Omega f(x,y)U^{\frac{1-m}{m}}(y,\tau)J_1(y,\tau)dy. \tag{4.8}$$

Similarly, we also have

$$J_2(x,\tau) \ge \Big(\int_\Omega f(x,y)V^{\frac{1}{n}}(y,\tau)dy\Big)^{n-1}\int_\Omega f(x,y)U^{\frac{1-n}{n}}(y,\tau)J_2(y,\tau)dy. \tag{4.9}$$

On the other hand, (H1′)–(H2′) imply that

$$J_1(x,0)\ge 0,\quad J_2(x,0)\ge 0. \tag{4.10}$$

By (4.5)–(4.10), Lemma 2.2 implies that $J_1, J_2 \ge 0$ for $(x,t)\in\Omega\times(0,T^*)$. That is (4.4) holds. □

Integrating (4.4) from τ to T^*, we conclude that

$$\begin{cases} M_1(\tau) \le C_7(T^*-\tau)^{-(q_3-p_4-r_2+1)/(q_3q_4-(1-p_3-r_1)(1-p_4-r_2))}, \\ M_2(\tau) \le C_8(T^*-\tau)^{-(q_4-p_3-r_1+1)/(q_3q_4-(1-p_3-r_1)(1-p_4-r_2))}, \end{cases} \tag{4.11}$$

where C_7, C_8 are positive constants independent of τ. It follows from Lemma 4.1 and (4.11), we have the following lemma.

Lemma 4.3. *Assume that $U_0(x)$, $V_0(x)$ satisfy* (H1′)–(H2′),

$$\int_\Omega f(x,y)dy, \int_\Omega g(x,t)dy \le 1$$

for all $x\in\partial\Omega$, $q_4-p_3-r_1+1>0$ and $q_3-p_4-r_2+1>0$. If (U,V) blows up in finite time T^*, then there exists a positive constant $C_i'(i=1,2,3,4)$ such that

(4.12)
$$\begin{cases} C_1' \le \max_{x\in\overline{\Omega}} U(x,\tau)(T^*-\tau)^{-(q_3-p_4-r_2+1)/(q_3q_4-(1-p_3-r_1)(1-p_4-r_2))} \le C_2', \\ C_3' \le \max_{x\in\overline{\Omega}} V(x,\tau)(T^*-\tau)^{-(q_4-p_3-r_1+1)/(q_3q_4-(1-p_3-r_1)(1-p_4-r_2))} \le C_4'. \end{cases}$$

According the transform and Lemma 4.3, we can obtain Theorem 1.7 immediately.

Remark 4.1. From Theorem 1.7, we know that in the case of $\int_\Omega f(x,y)dy \leq 1$ and $\int_\Omega g(x,t)dy \leq 1$ for all $x \in \partial\Omega$, the blow-up rate of degenerate parabolic system with coupled nonlinear localized sources subject to weighted nonlocal Dirichlet boundary conditions is the same as that of general degenerate parabolic system with nonlinear localized sources subject to null Dirichlet boundary conditions.

Acknowledgements. The authors are indebted to the referee for giving some important suggestions which improved the presentations of this paper.

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