Intuitionistic Fuzzy Rough Approximation Operators

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Abstract

Since upper and lower approximations could be induced from the rough set structures, rough sets are considered as approximations. The concept of fuzzy rough sets was proposed by replacing crisp binary relations with fuzzy relations by Dubois and Prade. In this paper, we introduce and investigate some properties of intuitionistic fuzzy rough approximation operators and intuitionistic fuzzy relations by means of topology.

Keywords: Intuitionistic fuzzy topology, Intuitionistic fuzzy approximation space

1. Introduction

A Chang's fuzzy topology [1] is a crisp subfamily of fuzzy sets, and hence fuzziness in the notion of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. In order to give fuzziness of the fuzzy sets, Çoker [2] introduced intuitionistic fuzzy topological spaces using the idea of intuitionistic fuzzy sets which was proposed by Atanassov [3]. Also Çoker and Demirci [4] defined intuitionistic fuzzy topological spaces in Šostak's sense as a generalization of smooth topological spaces and intuitionistic fuzzy topological spaces. Since then, many researchers [5–9] investigated such intuitionistic fuzzy topological spaces.

On the other hand, the theory of rough sets was proposed by Z. Pawlak [10]. It is a new mathematical tool for the data reasoning, and it is an extension of set theory for the research of intelligent systems characterized by insufficient and incomplete informations. The fundamental structure of rough set theory is an approximation space. Based on rough set theory, upper and lower approximations could be induced. By using these approximations, knowledge hidden in information systems may be exposed and expressed in the form of decision rules(see [10, 11]). The concept of fuzzy rough sets was proposed by replacing crisp binary relations with fuzzy relations by Dubois and Prade [12]. The relations between fuzzy rough sets and fuzzy topological spaces have been studied in some papers [13–15].

The main interest of this paper is to investigate characteristic properties of intuitionistic fuzzy rough approximation operators and intuitionistic fuzzy relations by means of topology. We prove that the upper approximation of a set is the set itself if and only if the set is a lower set whenever the intuitionistic fuzzy relation is reflexive. Also we have the result that if an intuitionistic fuzzy upper approximation operator is a closure operator or an intuitionistic fuzzy lower approximation operator is an interior operator in the intuitionistic fuzzy topology, then the order is a preorder.

2. Preliminaries

Let X be a nonempty set. An *intuitionistic fuzzy set* A is an ordered pair

$$A = (\mu_A, \nu_A)$$

where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively and $\mu_A + \nu_A \leq 1$(see [3]). Obviously, every fuzzy set μ in X is an intuitionistic fuzzy set of the form $(\mu, \tilde{1} - \mu)$.

Throughout this paper, 'IF' stands for 'intuitionistic fuzzy.' $I \otimes I$ denotes the family of all intuitionistic fuzzy numbers (a, b) such that $a, b \in [0, 1]$ and $a + b \leq 1$, with the order relation defined by

$$(a, b) \leq (c, d) \text{ iff } a \leq c \text{ and } b \geq d.$$

For any IF set $A = (\mu_A, \nu_A)$ of X, the value

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is called an *indeterminancy degree*(or *hesitancy degree*) of x to A(see [3]). Szmidt and Kacprzyk call $\pi_A(x)$ an *intuitionistic index* of x in A(see [16]). Obviously

$$0 \leq \pi_A(x) \leq 1, \quad \forall x \in X.$$

Note $\pi_A(x) = 0$ iff $\nu_A(x) = 1 - \mu_A(x)$. Hence any fuzzy set μ_A can be regarded as an IF set (μ_A, ν_A) with $\pi_A = 0$.

IF(X) denotes the family of all intuitionistic fuzzy sets in X, and cIF(X) denotes the family of all intuitionistic fuzzy sets in X with constant hesitancy degree, i.e., if $A \in$ cIF(X), then $\pi_A = c$ for some constant $c \in [0, 1)$. When we process basic operations on IF(X), we do as in [3].

Definition 2.1. ([2, 17]) Any subfamily $\mathcal{T}$ of IF(X) is called an *intuitionistic fuzzy topology* on X in the sense of Lowen([18]), if

(1) for each $(a, b) \in I \otimes I$, $\widetilde{(a, b)} \in \mathcal{T}$,

(2) $A, B \in \mathcal{T}$ implies $A \cap B \in \mathcal{T}$,

(3) $\{A_j \mid j \in J\} \subseteq \mathcal{T}$ implies $\bigcup_{j \in J} A_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called an *intuitionistic fuzzy topological space*. Every member of $\mathcal{T}$ is called an *intuitionistic fuzzy open set* in X. Its complement is called an *intuitionistic fuzzy closed set* in X. We denote $\mathcal{T}^C = \{A \in$ IF$(X) \mid A^C \in \mathcal{T}\}$. The interior and closure of A denoted by int(A) and cl(A)

respectively for each $A \in$ IF(X) are defined as follows:

$$\text{int}(A) \text{ or } \text{int}_{\mathcal{T}}(A) = \bigcup \{B \in \mathcal{T} \mid B \subseteq A\},$$

$$\text{cl}(A) \text{ or } \text{cl}_{\mathcal{T}}(A) = \bigcap \{B \in \mathcal{T}^C \mid A \subseteq B\}.$$

An IF topology $\mathcal{T}$ is called an *Alexandrov topology* [19] if (2) in Definition 2.1 is replaced by

(2)$'$
$$\{A_j \mid j \in J\} \subseteq \mathcal{T} \text{ implies } \bigcap_{j \in J} A_j \in \mathcal{T}.$$

Definition 2.2. ([20]) An IF set R on $X \times X$ is called an *intuitionistic fuzzy relation* on X. Moreover, R is called

(i) *reflexive* if $R(x, x) = (1, 0)$ for all $x \in X$,

(ii) *symmetric* if $R(x, y) = R(y, x)$ for all $x, y \in X$,

(iii) *transitive* if $R(x, y) \wedge R(y, z) \leq R(x, z)$ for all $x, y, z \in X$,

A reflexive and transitive IF relation is called an *intuitionistic fuzzy preorder*. A symmetric IF preorder is called an *intuitionistic fuzzy equivalence*. An IF preorder on X is called an *intuitionistic fuzzy partial order* if for any $x, y \in X$, $R(x, y) = R(y, x) = (1, 0)$ implies that $x = y$.

Let R be an IF relation on X. R^{-1} is called the *inverse relation* of R if $R^{-1}(x, y) = R(y, x)$ for any $x, y \in X$. Also, R^C is called the *complement* of R if $R^C(x, y) = \left(\nu_{R(x,y)}, \mu_{R(x,y)}\right)$ for any $x, y \in X$ when $R(x, y) = \left(\mu_{R(x,y)}, \nu_{R(x,y)}\right)$. It is obvious that $R^{-1} \neq R^C$.

Definition 2.3. ([21]) Let R be an IF relation on X. The pair (X, R) is called an *intuitionistic fuzzy approximation space*. The *intuitionistic fuzzy lower approximation* of $A \in$ IF(X) with respect to (X, R), denoted by $\underline{R}(A)$, is defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in X} (R^C(x, y) \vee A(y)).$$

Similarly, the *intuitionistic fuzzy upper approximation* of $A \in$ IF(X) with respect to (X, R), denoted by $\overline{R}(A)$, is defined as follows:

$$\overline{R}(A)(x) = \bigvee_{y \in X} (R(x, y) \wedge A(y)).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the *intuitionistic fuzzy rough set* of A with respect to (X, R).

$\underline{R} :$ IF$(X) \to$ IF(X) and $\overline{R} :$ IF$(X) \to$ IF(X) are called the *intuitionistic fuzzy lower approximation operator* and the

intuitionistic fuzzy upper approximation operator, respectively. In general, we refer to $\underline{R}$ and $\overline{R}$ as the *intuitionistic fuzzy rough approximation operators*.

Proposition 2.4. ([17, 21]) Let (X, R) be an IF approximation space. Then for any $A, B \in \mathrm{IF}(X)$, $\{A_j \mid j \in J\} \subseteq \mathrm{IF}(X)$ and $(a, b) \in I \otimes I$,

(1) $\underline{R}(\widetilde{(1,0)}) = \widetilde{(1,0)}$, $\overline{R}(\widetilde{(0,1)}) = \widetilde{(0,1)}$,

(2) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$,

(3) $\underline{R}(A^C) = (\overline{R}(A))^C$, $\overline{R}(A^C) = (\underline{R}(A))^C$,

(4) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$,

(5) $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} (\underline{R}(A_j))$,
$\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} (\overline{R}(A_j))$,

(6) $\underline{R}(\widetilde{(a,b)} \cup A) = \widetilde{(a,b)} \cup \underline{R}(A)$, $\overline{R}(\widetilde{(a,b)} \cap A) = \widetilde{(a,b)} \cap \overline{R}(A)$.

Remark 2.5. Let (X, R) be an IF approximation space. Then

$$\overline{R}(x_{(1,0)})(y) = \bigvee_{z \in X} (R(y,z) \wedge x_{(1,0)}(z)) = R(y,x),$$

$$\underline{R}(x_{(1,0)}^C)(y) = \bigwedge_{z \in X} (R^C(y,z) \vee x_{(1,0)}^C(z)) = R^C(y,x).$$

Let (X, R) be an IF approximation space. (X, R) is called a *reflexive*(resp., *preordered*) intuitionistic fuzzy approximation space, if R is a reflexive intuitionistic fuzzy relation(resp., an intuitionistic fuzzy preorder). If R is an intuitionistic fuzzy partial order, then (X, R) is called a *partially ordered* intuitionistic fuzzy approximation space. An intuitionistic fuzzy preorder R is called an *intuitionistic fuzzy equality*, if R is both an intuitionistic fuzzy equivalence and an intuitionistic fuzzy partial order.

Theorem 2.6. ([17, 21]) Let (X, R) be an IF approximation space. Then

(1) R is reflexive

$$\Leftrightarrow \forall A \in \mathrm{IF}(X), \ \underline{R}(A) \subseteq A$$
$$\Leftrightarrow \forall A \in \mathrm{IF}(X), \ A \subseteq \overline{R}(A).$$

(2) R is transitive

$$\Leftrightarrow \forall A \in \mathrm{IF}(X), \ \underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$$
$$\Leftrightarrow \forall A \in \mathrm{IF}(X), \ \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A).$$

3. IF Rough Approximation Operator

Definition 3.1. ([22]) Let (X, R) be an IF approximation space. Then $A \in \mathrm{IF}(X)$ is called an *intuitionistic fuzzy upper set* in (X, R) if

$$A(x) \wedge R(x,y) \leq A(y), \ \forall x, y \in X.$$

Dually, A is called an *intuitionistic fuzzy lower set* in (X, R) if $A(y) \wedge R(x,y) \leq A(x)$ for all $x, y \in X$.

Let R be an IF preorder on X. For $x, y \in X$, the real number $R(x, y)$ can be interpreted as the degree to which '$x \leq y$' holds true. The condition $A(x) \wedge R(x, y) \leq A(y)$ can be interpreted as the statement that if x is in A and $x \leq y$, then y is in A. Particularly, if R is an IF equivalence, then an IF set A is an upper set in (X, R) if and only if it is a lower set in (X, R).

The classical preorder $x \leq y$ can be naturally extended to $R(x, y) = (1, 0)$ in an IF preorder. Obviously, the notion of IF upper sets and IF lower sets agrees with that of upper sets and lower sets in classical preordered space.

Proposition 3.2. Let (X, R) be an IF approximation space and $A \in \mathrm{IF}(X)$. Then the following are equivalent:

(1) $\overline{R}(A) \subseteq A$.

(2) A is a lower set in (X, R).

(3) A is an upper set in (X, R^{-1}).

Proof. (1) $\Rightarrow$ (2). Suppose that $\overline{R}(A) \subseteq A$. Since for each $x \in X$,

$$\bigvee_{y \in X} (A(y) \wedge R(x,y)) = \overline{R}(A)(x) \leq A(x),$$

we have

$$A(y) \wedge R(x,y) \leq A(x).$$

Thus A is a lower set in (X, R).

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). Suppose that A is an upper set in (X, R^{-1}). Then for any $x, y \in X$, $A(x) \wedge R^{-1}(x,y) \leq A(y)$. So $A(x) \wedge R(y,x) \leq A(y)$. Thus

$$\overline{R}(A)(y) = \bigvee_{x \in X} (A(x) \wedge R(y,x)) \leq A(y).$$

Hence $\overline{R}(A) \subseteq A$.

Corollary 3.3. Let (X, R) be an IF approximation space and $A \in \text{IF}(X)$. If R is reflexive, then the following are equivalent:

(1) $\overline{R}(A) = A$.

(2) A is a lower set in (X, R).

(3) A is an upper set in (X, R^{-1}).

Proof. This holds by Theorem 2.6 and Proposition 3.2.

Proposition 3.4. Let (X, R) be an IF approximation space and $A \in \text{IF}(X)$. Then the following are equivalent:

(1) $\underline{R}(A) \supseteq A$.

(2) A^C is a lower set in (X, R).

(3) A^C is an upper set in (X, R^{-1}).

Proof. (1) $\Rightarrow$ (2). Suppose that $\underline{R}(A) \supseteq A$. Since for each $x \in X$,

$$\bigwedge_{y \in X} \big(A(y) \vee R^C(x, y) \big) = \underline{R}(A)(x) \geq A(x),$$

we have

$$A(y) \vee R^C(x, y) \geq A(x),$$
$$A^C(y) \wedge R(x, y) \leq A^C(x).$$

Thus A^C is a lower set in (X, R).

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). Suppose that A^C is an upper set in (X, R^{-1}). Then for any $x, y \in X$, $A^C(x) \wedge R^{-1}(x, y) \leq A^C(y)$. So $A^C(x) \wedge R(y, x) \leq A^C(y)$. Thus

$$A(x) \vee R^C(y, x) \geq A(y), \forall x, y \in X.$$

So

$$\underline{R}(A)(y) = \bigwedge_{x \in X} \big(A(x) \vee R^C(y, x) \big) \geq A(y).$$

Hence $\underline{R}(A) \supseteq A$.

Corollary 3.5. Let (X, R) be an IF approximation space and $A \in \text{IF}(X)$. If R is reflexive, then the following are equivalent:

(1) $\underline{R}(A) = A$.

(2) A^C is a lower set in (X, R).

(3) A^C is an upper set in (X, R^{-1}).

Proof. This holds by Theorem 2.6 and the above proposition.

For each $z \in X$, we define IF sets $[z]^R : X \to I \otimes I$ by $[z]^R(x) = R(z, x)$, and $[z]_R : X \to I \otimes I$ by $[z]_R(x) = R(x, z)$.

Theorem 3.6. Let (X, R) be an IF approximation space. Then

(1) R is reflexive

$$\Leftrightarrow \forall x \in X, \ [x]_R(x) = (1, 0)$$
$$\Leftrightarrow \forall x \in X, \ [x]^R(x) = (1, 0).$$

(2) R is symmetric

$$\Leftrightarrow \forall x \in X, \ [x]_R = [x]^R$$
$$\Leftrightarrow \forall A \in \text{IF}(X), \ A \text{ is a lower set iff } A \text{ is an upper set.}$$

(3) R is transitive

$$\Leftrightarrow \forall x \in X, \ [x]_R \text{ is a lower set in } (X, R)$$
$$\Leftrightarrow \forall x \in X, \ [x]^R \text{ is an upper set in } (X, R)$$
$$\Leftrightarrow \forall A \in \text{IF}(X), \ \overline{R}(A) \text{ is a lower set in } (X, R).$$

Proof. (1) and (2) are obvious. (3) By Proposition 3.2,

$\forall A \in \text{IF}(X), \ \overline{R}(A)$ is a lower set

$$\Leftrightarrow \forall A \in \text{IF}(X), \ \overline{R}\big(\overline{R}(A)\big) \subseteq \overline{R}(A)$$
$$\Leftrightarrow R \text{ is transitive}$$
$$\Leftrightarrow \forall x, y, z \in X, \ R(x, y) \wedge R(y, z) \leq R(x, z)$$
$$\Leftrightarrow \forall x, y, z \in X, \ R(x, y) \wedge [z]_R(y) \leq [z]_R(x)$$
$$\Leftrightarrow \forall x \in X, \ [x]_R \text{ is a lower set.}$$

Also,

$\overline{R}(A)$ is a lower set

$$\Leftrightarrow \forall x, y, z \in X, \ R(x, y) \wedge R(y, z) \leq R(x, z)$$
$$\Leftrightarrow \forall x, y, z \in X, \ [x]^R(y) \wedge R(y, z) \leq [x]^R(z)$$
$$\Leftrightarrow \forall x \in X, \ [x]^R \text{ is an upper set.}$$

Proposition 3.7. Let (X, R) be an IF approximation space. Then

R is symmetric

$$\Leftrightarrow \forall (x, y) \in X \times X, \ \underline{R}(x^C_{(1,0)})(y) = \underline{R}(y^C_{(1,0)})(x)$$

$$\Leftrightarrow \forall (x,y) \in X \times X, \ \overline{R}(x_{(1,0)})(y) = \overline{R}(y_{(1,0)})(x).$$

Proof. By Remark 2.5, $\underline{R}(x_{(1,0)}^C)(y) = R^C(y,x) = R^C(x,y) = \underline{R}(y_{(1,0)}^C)(x)$, because R is symmetric. Similarly we have that $\overline{R}(x_{(1,0)})(y) = R(y,x) = R(x,y) = \overline{R}(y_{(1,0)})(x)$.

Theorem 3.8. Let R be an IF relation on X and let $\mathcal{T}$ be an IF topology on X. If one of the following conditions is satisfied, then R is an IF preorder.

(1) $\overline{R}$ is a closure operator of $\mathcal{T}$.

(2) $\underline{R}$ is an interior operator of $\mathcal{T}$.

Proof. Suppose that $\mathcal{T}$ satisfies (1). By Remark 2.5, $\overline{R}(x_{(1,0)})(y) = R(y,x)$ for each $x \in X$. Since $\overline{R}$ is a closure operator of $\mathcal{T}$, for each $x \in X$,

$$R(x,x) = \overline{R}(x_{(1,0)})(x) = \mathrm{cl}_{\mathcal{T}}(x_{(1,0)})(x)$$
$$\geq (x_{(1,0)})(x) = (1,0).$$

Thus R is reflexive. For any $x,y,z \in X$, let $\mathrm{cl}_{\mathcal{T}}(z_{(1,0)})(y) = (a,b)$. Then by Remark 2.5 and Proposition 2.4,

$$R(x,y) \wedge R(y,z)$$
$$= \overline{R}(y_{(1,0)})(x) \wedge \overline{R}(z_{(1,0)})(y) = \overline{R}(y_{(1,0)})(x) \wedge \mathrm{cl}_{\mathcal{T}}(z_{(1,0)})(y)$$
$$= \overline{R}(y_{(1,0)})(x) \wedge (a,b) = \overline{R}((a,b) \wedge y_{(1,0)})(x)$$
$$= \mathrm{cl}_{\mathcal{T}}((a,b) \wedge y_{(1,0)})(x) = \mathrm{cl}_{\mathcal{T}}(\mathrm{cl}_{\mathcal{T}}(z_{(1,0)})(y) \wedge y_{(1,0)})(x)$$
$$\leq \mathrm{cl}_{\mathcal{T}}\left(\bigcup_{y \in X} [\mathrm{cl}_{\mathcal{T}}(z_{(1,0)})(y) \wedge y_{(1,0)}] \right)(x) = \mathrm{cl}_{\mathcal{T}}(\mathrm{cl}_{\mathcal{T}}(z_{(1,0)}))(x)$$
$$= \mathrm{cl}_{\mathcal{T}}(z_{(1,0)})(x) = R(x,z).$$

Hence R is transitive. Therefore R is an IF preorder.
Similarly we can prove for the case of (2).

Definition 3.9. For each $A \in \mathrm{IF}(X)$, we define

$$R_A = \{(x,y) \in X \times X \mid A(x) > A(y)\}.$$

Obviously, $R_A = \emptyset$ iff $A = \widetilde{(a,b)}$ for some $(a,b) \in I \otimes I$ or $A(x)$ and $A(y)$ are non-comparable for all $x,y \in X$.

Proposition 3.10. Let (X,R) be an IF approximation space. Let A be an IF set with constant hesitancy degree, i.e., $A \in \mathrm{cIF}(X)$ with $R_A \neq \emptyset$. Then we have

(1) $\underline{R}(A) \supseteq A \Leftrightarrow R^C(x,y) \geq A(x) \vee A(y)$ for all $(x,y) \in R_A$,

(2) $\overline{R}(A) \subseteq A \Leftrightarrow R(y,x) \leq A(x) \wedge A(y)$ for all $(x,y) \in R_A$.

Proof. (1) ($\Rightarrow$) Suppose that $\underline{R}(A) \supseteq A$. Note that for each $x \in X$,

$$\bigwedge_{y \in X} (A(y) \vee R^C(x,y)) = \underline{R}(A)(x) \geq A(x).$$

Then $A(y) \vee R^C(x,y) \geq A(x)$ for any $x,y \in X$. Since $A(x) > A(y)$ for each $(x,y) \in R_A$, we have

$$R^C(x,y) \geq A(x) = A(x) \vee A(y) \text{ for all } (x,y) \in R_A.$$

($\Leftarrow$) Suppose that for each $(x,y) \in R_A, R^C(x,y) \geq A(x) \vee A(y)$. Let $z \in X$.
(i) If $A(z) > A(y)$, then

$$A(y) \vee R^C(z,y) \geq A(y) \vee (A(z) \vee A(y)) \geq A(z).$$

(ii) If $A(z) \leq A(y)$, then

$$A(y) \vee R^C(z,y) \geq A(y) \vee (A(z) \vee A(y)) \geq A(y) \geq A(z).$$

Hence $\underline{R}(A)(z) = \bigwedge_{y \in X} (A(y) \vee R^C(z,y)) \geq A(z)$ for any $z \in X$. Thus $\underline{R}(A) \supseteq A$.

(2) ($\Rightarrow$) Suppose that $\overline{R}(A) \subseteq A$. Note that for each $y \in X$,

$$\bigvee_{x \in X} (A(x) \wedge R(y,x)) = \overline{R}(A)(y) \leq A(y).$$

Then $A(x) \wedge R(y,x) \leq A(y)$ for any $x,y \in X$. Since $A(x) > A(y)$ for each $(x,y) \in R_A$, we have

$$R(y,x) \leq A(y) = A(x) \wedge A(y).$$

($\Leftarrow$) Suppose that for any $(x,y) \in R_A, R(y,x) \leq A(x) \wedge A(y)$. Let $z \in X$.
(i) If $A(x) > A(z)$, then

$$A(x) \wedge R(z,x) \leq A(x) \wedge (A(x) \wedge A(z)) \leq A(z).$$

(ii) If $A(x) \leq A(z)$, then

$$A(x) \wedge R(z,x) \leq A(x) \wedge (A(x) \wedge A(z)) \leq A(x) \leq A(z).$$

Thus $\overline{R}(A)(z) = \bigvee_{x \in X} (A(x) \wedge R(z,x)) \leq A(z)$. Hence $\overline{R}(A) \subseteq A$.

Corollary 3.11. Let (X,R) be a reflexive IF approximation space. Then for each $A \in \mathrm{cIF}(X)$ with $R_A \neq \emptyset$,

(1) $\underline{R}(A) = A \Leftrightarrow R^C(x,y) \geq A(x) \vee A(y)$ for all $(x,y) \in$

R_A,

(2) $\overline{R}(A) = A \Leftrightarrow R(y,x) \le A(x) \wedge A(y)$ for all $(x,y) \in R_A$.

Proof. By the above proposition and the reflexivity of R, it can be easily proved.

Let R_1 and R_2 be two IF relations on X. We denote $R_1 \subseteq R_2$ if $R_1(x,y) \le R_2(x,y)$ for any $x, y \in X$. And $R_1 = R_2$ if $R_1 \subseteq R_2$ and $R_2 \subseteq R_1$.

Proposition 3.12. Let (X, R_1) and (X, R_2) be two IF approximation spaces. Then for each $A \in \mathrm{IF}(X)$,

(1) $R_1 \subseteq R_2 \quad \Rightarrow \quad \overline{R_1}(A) \subseteq \overline{R_2}(A)$ and $\underline{R_1}(A) \supseteq \underline{R_2}(A)$.

(2) $\overline{(R_1 \cup R_2)}(A) = \overline{R_1}(A) \cup \overline{R_2}(A), \quad \underline{(R_1 \cup R_2)}(A) = \underline{R_1}(A) \cap \underline{R_2}(A)$.

Proof. (1) For each $x \in X$,

$$\overline{R_1}(A)(x) = \bigvee_{y \in X} \big(A(y) \wedge (R_1)(x,y)\big)$$
$$\le \bigvee_{y \in X} \big(A(y) \wedge (R_2)(x,y)\big) = \overline{R_2}(A)(x).$$

Thus we have $\overline{R_1}(A) \subseteq \overline{R_2}(A)$. Dually,

$$\overline{R_1}(A^C) \subseteq \overline{R_2}(A^C) \Leftrightarrow \big(\overline{R_1}(A^C)\big)^C \supseteq \big(\overline{R_2}(A^C)\big)^C$$
$$\Leftrightarrow \underline{R_1}(A) \supseteq \underline{R_2}(A).$$

(2) For each $x \in X$,

$$\overline{(R_1 \cup R_2)}(A)(x)$$
$$= \bigvee_{y \in X} \big(A(y) \wedge (R_1 \cup R_2)(x,y)\big)$$
$$= \bigvee_{y \in X} \big(A(y) \wedge (R_1(x,y) \vee R_2(x,y))\big)$$
$$= \bigvee_{y \in X} \big((A(y) \wedge R_1(x,y)) \vee (A(y) \wedge R_2(x,y))\big)$$
$$\le \big(\bigvee_{y \in X} (A(y) \wedge R_1(x,y))\big) \vee \big(\bigvee_{y \in X} (A(y) \wedge R_2(x,y))\big)$$
$$= \overline{R_1}(A)(x) \vee \overline{R_2}(A)(x)$$
$$= \big(\overline{R_1}(A) \cup \overline{R_2}(A)\big)(x).$$

Thus we have $\overline{(R_1 \cup R_2)}(A) \subseteq \overline{R_1}(A) \cup \overline{R_2}(A)$. Moreover, since $R_1 \subseteq R_1 \cup R_2$ and $R_2 \subseteq R_1 \cup R_2$, we have $\overline{R_1}(A) \subseteq \overline{(R_1 \cup R_2)}(A)$ and $\overline{R_1}(A) \subseteq \overline{(R_1 \cup R_2)}(A)$. Thus $\overline{R_1}(A) \cup \overline{R_2}(A) \subseteq \overline{(R_1 \cup R_2)}(A)$. Hence we have $\overline{(R_1 \cup R_2)}(A) = $

$\overline{R_1}(A) \cup \overline{R_2}(A)$. By Proposition 2.4,

$$\underline{R_1}(A) \cap \underline{R_2}(A) = (\overline{R_1}(A^C))^C \cap (\overline{R_2}(A^C))^C$$
$$= (\overline{R_1}(A^C) \cup \overline{R_2}(A^C))^C = (\overline{(R_1 \cup R_2)}(A^C))^C = \underline{(R_1 \cup R_2)}(A).$$

Proposition 3.13. Let (X, R_1) and (X, R_2) be two reflexive IF approximation spaces. Then for each $A \in \mathrm{IF}(X)$,

(1) $\underline{R_2}\big(\underline{R_1}(A)\big) \subseteq \underline{(R_1 \cup R_2)}(A)$ and $\underline{R_1}\big(\underline{R_2}(A)\big) \subseteq \underline{(R_1 \cup R_2)}(A)$.

(2) $\overline{R_2}\big(\overline{R_1}(A)\big) \supseteq \overline{(R_1 \cup R_2)}(A)$ and $\overline{R_1}\big(\overline{R_2}(A)\big) \supseteq \overline{(R_1 \cup R_2)}(A)$.

Proof. (1) By Theorem 2.6, $\underline{R_2}\big(\underline{R_1}(A)\big) \subseteq \underline{R_2}(A)$ and $\underline{R_2}\big(\underline{R_1}(A)\big) \subseteq \underline{R_1}(A)$. Thus we have

$$\underline{R_2}\big(\underline{R_1}(A)\big) \subseteq \underline{R_1}(A) \cap \underline{R_2}(A) \subseteq \underline{(R_1 \cup R_2)}(A).$$

Similarly, we can prove that $\underline{R_1}\big(\underline{R_2}(A)\big) \subseteq \underline{(R_1 \cup R_2)}(A)$.

(2) The proof is similar to (1).

Proposition 3.14. Let (X, R_1) and (X, R_2) be two IF approximation spaces. If R_1 is reflexive, R_2 is transitive and $R_1 \subseteq R_2$, then

$$\underline{R_1}\big(\underline{R_2}(A)\big) = \underline{R_2}(A) \text{ and } \overline{R_1}\big(\overline{R_2}(A)\big) = \overline{R_2}(A).$$

Proof. By Theorem 2.6, $\overline{R_1}\big(\overline{R_2}(A)\big) \supseteq \overline{R_2}(A)$. For each $x \in X$, by $R_1 \subseteq R_2$ and the transitivity of R_2, we have

$$\overline{R_1}\big(\overline{R_2}(A)\big)(x) = \bigvee_{y \in X} \big(\overline{R_2}(A)(y) \wedge R_1(x,y)\big)$$
$$= \bigvee_{y \in X} \big(\big(\bigvee_{z \in X} (A(z) \wedge R_2(y,z))\big) \wedge R_1(x,y)\big)$$
$$= \bigvee_{y \in X} \big(\bigvee_{z \in X} \big((A(z) \wedge R_2(y,z)) \wedge R_1(x,y)\big)\big)$$
$$= \bigvee_{y \in X} \big(\bigvee_{z \in X} \big(A(z) \wedge \big(R_2(y,z) \wedge R_1(x,y)\big)\big)\big)$$
$$\le \bigvee_{y \in X} \big(\bigvee_{z \in X} \big(A(z) \wedge \big(R_2(y,z) \wedge R_2(x,y)\big)\big)\big)$$
$$\le \bigvee_{y \in X} \big(\bigvee_{z \in X} \big(A(z) \wedge R_2(x,z)\big)\big)$$
$$= \bigvee_{z \in X} \big(A(z) \wedge R_2(x,z)\big) = \overline{R_2}(A)(x).$$

Thus $\overline{R_1}\big(\overline{R_2}(A)\big) \subseteq \overline{R_2}(A)$. So $\overline{R_1}\big(\overline{R_2}(A)\big) = \overline{R_2}(A)$. By

Proposition 2.4,

$$\underline{R_1}\big(\underline{R_2}(A)\big) = \underline{R_1}\big((\overline{R_2}(A^C))^C\big) = \big(\overline{R_1}\big(\overline{R_2}(A^C)\big)\big)^C$$
$$= (\overline{R_2}(A^C))^C = \underline{R_2}(A).$$

4. Conclusion

We obtained characteristic properties of intuitionistic fuzzy rough approximation operator and intuitionistic fuzzy relation by means of topology. Particularly, we proved that the upper approximation of a set is the set itself if and only if the set is a lower set whenever the intuitionistic fuzzy relation is reflexive. Also we had the result that if an intuitionistic fuzzy upper approximation operator is a closure operator or an intuitionistic fuzzy lower approximation operator is an interior operator in the intuitionistic fuzzy topology, then the order is an preorder.

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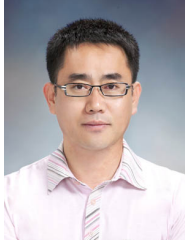
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