J. Appl. Math. & Informatics Vol. **33***(2015), No. 5 - 6, pp. 739 - 748*
http://dx.doi.org/10.14317/jami.2015.739

ON QUASI-LATTICE IMPLICATION ALGEBRAS

YONG HO YON

ABSTRACT. The notion of quasi-lattice implication algebras is a generalization of lattice implication algebras. In this paper, we give an optimized definition of quasi-lattice implication algebra and show that this algebra is a distributive lattice and that this algebra is a lattice implication algebra. Also, we define a congruence relation Φ_F induced by a filter F and show that every congruence relation on a quasi-lattice implication algebra is a congruence relation Φ_F induced by a filter F.

AMS Mathematics Subject Classification : 03G10, 06B10.
Key words and phrases : quasi-lattice implication algebras, lattice implication algebras, filters, congruence relations.

1. Introduction

The notion of lattice implication algebras was introduced in [8] to research a lattice-valued logic, which is a logical system equipped with a logical implication and an involution unary operation on a lattice. This logical system was studied from the algebraic viewpoint in many literature [5, 7, 8, 9], and some operators on this algebra was studied in [4, 11].

A *lattice implication algebra* is a bounded lattice $(L, \wedge, \vee, 0, 1)$ with a binary operation " $\rightarrow$ " and an order-reversing involution " $\prime$ " satisfying the following axioms: for all $x, y, z \in L$,

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $x \rightarrow y = y' \rightarrow x'$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

Received April 21, 2015. Accepted June 29, 2015.

The notion of filters on algebras with implication was introduced and studied in [3, 6]. This filter is known as deductive filter and different from the notion of filters on lattices. This filter was proposed and studied as the notion of filters on lattice implication algebras in [1, 2, 10, 12]. On a lattice implication algebra, the filter of lattice is the generalized concept of the filter.

A quasi-lattice implication algebra was introduced in [5, 7] as an algebraic system $(L, \wedge, \vee, \to, \prime, 0, 1)$ satisfying the axioms (I1)-(I5). This algebra is a generalization of lattice implication algebras and has the binary operation $\to$ and the involution $\prime$ in the axioms for definition.

In this paper, the quasi-lattice implication algebra will be more clearly defined, and we show that this algebra is a distributive lattice, and hence this algebra is a lattice implication algebra. Also, an alternative definition of quasi-lattice implication algebra will be given. In section 3, we define a congruence relation Φ_F induced by a filter F and show that every congruence relation on a quasi-lattice implication algebra is a congruence relation Φ_F induced by a filter F.

2. Quasi-lattice implication algebra

We will define the notion of quasi-lattice implication algebras by the following optimized type, and $x \to y$ will be denoted by xy.

Definition 2.1. A *quasi-lattice implication algebra* is an algebraic system $(L, \cdot, \prime, 1)$ with a binary operation " $\cdot$ ", an involution " $\prime$ " and an element 1 satisfying the following axioms: for all $x, y, z \in L$,

(Q1) $x(yz) = y(xz)$,
(Q2) $xx = 1$,
(Q3) $(xy)y = (yx)x$,
(Q4) $xy = 1$ and $yx = 1$ imply $x = y$,
(Q5) $xy = y'x'$.

In the definition of quasi-lattice implication algebra L, the involution $\prime$ is an unary operation on L such that $x'' = x$ for every $x \in L$.

Lemma 2.2. *Let L be a quasi-lattice implication algebra. Then L satisfies the following: for all $x \in L$,*

(1) $1x = x$,
(2) $x1 = 1$.

Proof. (1) Let $x \in L$. Then $x(1x) = 1(xx) = 11 = 1$ by (Q1) and (Q2). Also, we have $(1x)x = (x1)1 = (x1)(11) = 1((x1)1) = 1((1x)x) = (1x)(1x) = 1$ by (Q3), (Q2) and (Q1). Hence $x(1x) = (1x)x = 1$. This implies $x = 1x$ by (Q4).

(2) Let $x \in L$. Then $x1 = (1x)1 = (1x)(xx) = x((1x)x) = x((x1)1) = (x1)(x1) = 1$ by (1) of this lemma, (Q2), (Q1) and (Q3). $\square$

Lemma 2.3. *Let L be a quasi-lattice implication algebra. If we define a binary relation " $\leq$ " by*

$$x \leq y \iff xy = 1$$

for any $x, y \in L$, then $(L, \leq)$ is a poset with the greatest element 1 and the smallest element $1'$.

Proof. For every $x \in L$, $x \leq x$ by (Q2), and for any $x, y \in L$, $x \leq y$ and $y \leq x$ imply $x = y$ by (Q4).

To show the transitivity, let $x \leq y$ and $y \leq z$. Then $xy = 1$ and $yz = 1$, and we have $xz = x(1z) = x((yz)z) = x((zy)y) = (zy)(xy) = (zy)1 = 1$ by 2.2, (Q3) and (Q1). This implies $x \leq z$. Hence $(L, \leq)$ is a poset.

Also, 1 is the greatest element in L by 2.2(2). Let $x \in L$. Then $1 = x'1 = 1'x'' = 1'x$ by 2.2(2), (Q5) and the definition of involution $'$. This implies $1' \leq x$ for every $x \in L$. Hence $1'$ is the smallest element in L. □

We will denote the smallest element $1'$ in L by 0.

Lemma 2.4. *Let L be a quasi-lattice implication algebra. Then L satisfies the following: for every $x, y, z \in L$,*

(1) $1' = 0$ *and* $0' = 1$,
(2) $x' = x0$,
(3) $x \leq yz$ *implies* $y \leq xz$,
(4) $(xy) \leq (yz)(xz)$,
(5) $(xy) \leq (zx)(zy)$,
(6) $x \leq y$ *implies* $yz \leq xz$ *and* $zx \leq zy$,
(7) $x \leq (xy)y$,
(8) $y \leq xy$,
(9) $x \leq y$ *implies* $y' \leq x'$,
(10) $((xy)y)y = xy$.

Proof. (1) It is clear that $1' = 0$ by 2.3, and $0' = 1'' = 1$.

(2) For every $x \in L$, $x' = 1x' = x''1' = x0$ by 2.2(1), (Q5) and (1) of this lemma.

(3) Let $x \leq yz$. Then $y(xz) = x(yz) = 1$. Hence $y \leq xz$.

(4) Let $x, y \in L$. Then we have

$$\begin{aligned}
(xy)((yz)(xz)) &= (xy)(x((yz)z)) && \text{(by (Q1))}\\
&= (xy)(x((zy)y)) && \text{(by (Q3))}\\
&= (xy)((zy)(xy)) && \text{(by (Q1))}\\
&= (zy)((xy)(xy)) && \text{(by (Q1))}\\
&= (zy)1\\
&= 1.
\end{aligned}$$

Hence $xy \leq (yz)(xz)$.

(5) Let $x, y, z \in L$. Then $zx \leq (xy)(zy)$ by (4) of this lemma. Hence $xy \leq (zx)(zy)$ by (3) of this lemma.

(6) Let $x \leq y$. Then $1 = xy \leq (yz)(xz)$ by (4) of this lemma. This implies $(yz)(xz) = 1$. Hence $yz \leq xz$. Also, $1 = xy \leq (zx)(zy)$ by (5) of this lemma. This implies $(zx)(zy) = 1$, and hence $zx \leq zy$.

(7) Let $x, y \in L$. Then $x((xy)y) = (xy)(xy) = 1$. Hence $x \leq (xy)y$.

(8) Let $x, y \in L$. Then $y(xy) = x(yy) = x1 = 1$. Hence $y \leq xy$.

(9) Let $x \leq y$, Then $1 = xy = y'x'$. Hence $y' \leq x'$.

(10) Let $x, y \in L$. Then $xy \leq ((xy)y)y$ by (7) of this lemma. Also, since $x \leq (xy)y$, $((xy)y)y \leq xy$ by (6) of this lemma. Thus $xy = ((xy)y)y$. □

Theorem 2.5. *A quasi-lattice implication algebra L is a lattice with*

$$x \vee y := \sup\{x, y\} = (xy)y \quad \textit{and} \quad x \wedge y := \inf\{x, y\} = (x' \vee y')'$$

for every $x, y \in L$.

Proof. Let $x, y \in L$. Then $(xy)y$ is an upper bound of x and y by (7) and (8) of 2.4. Suppose that u is an upper bound of x and y. Then $x \leq u$ and $y \leq u$, and $yu = 1$. This implies $uy \leq xy$, and

$$(xy)y \leq (uy)y = (yu)u = 1u = u$$

by 2.4(6) and (Q3). Hence $(xy)y$ is the least upper bound of x and y, and $x \vee y = (xy)y$.

Also, since $x' \leq x' \vee y'$ and $y' \leq x' \vee y'$, $(x' \vee y')' \leq x'' = x$ and $(x' \vee y')' \leq y'' = y$ by 2.4(9). Hence $(x' \vee y')'$ is a lower bound of x and y. Suppose that $l \leq x$ and $l \leq y$. Then $x' \leq l'$ and $y' \leq l'$. This implies $x' \vee y' \leq l'$. Hence $l = l'' \leq (x' \vee y')'$. This means $(x' \vee y')'$ is the greatest lower bound of x and y, and $x \wedge y = (x' \vee y')'$. □

Lemma 2.6. *Let L be a quasi-lattice implication algebra. Then L satisfies the following: for every $x, y, z \in L$,*

(1) $(x \vee y)' = x' \wedge y'$,
(2) $(x \wedge y)' = x' \vee y'$,
(3) $(x \vee y)z = (xz) \wedge (yz)$,
(4) $z(x \wedge y) = (zx) \wedge (zy)$.

Proof. (1) Let $x, y \in L$. Then $x' \wedge y' = (x'' \vee y'')' = (x \vee y)'$ by 2.5.

(2) Let $x, y \in L$. Then $(x \wedge y)' = (x' \vee y')'' = x' \vee y'$ by 2.5.

(3) Let $x, y, z \in L$. Then $x \leq x \vee y$ and $y \leq x \vee y$. This implies $(x \vee y)z \leq xz$ and $(x \vee y)z \leq yz$ by 2.4(6). Hence $(x \vee y)z \leq (xz) \wedge (yz)$. Also, since $(xz) \wedge (yz) \leq xz$ and $(xz) \wedge (yz) \leq yz$, we have

$$x \leq ((xz) \wedge (yz))z \quad \text{and} \quad y \leq ((xz) \wedge (yz))z$$

by 2.4(3). This implies $x \vee y \leq ((xz) \wedge (yz))z$, and $(xz) \wedge (yz) \leq (x \vee y)z$ by 2.4(3). Hence $(x \vee y)z = (xz) \wedge (yz)$.

(4) Let $x, y, z \in L$. Then we have $z(x \wedge y) = (x \wedge y)'z' = (x' \vee y')z' = (x'z') \wedge (y'z') = (zx) \wedge (zy)$ by (Q5) and (3) of this lemma. □

Theorem 2.7. *Let L be a quasi-lattice implication algebra. Then L is distributive.*

Proof. Let $x, y, z \in L$. Then it is clear that $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ since $x \wedge y \leq x \wedge (y \vee z)$ and $x \wedge z \leq x \wedge (y \vee z)$. Conversely, we have

$$\begin{aligned} x' \vee (y \vee z)' &= (x'(y \vee z)')(y \vee z)' \\ &= (x'(y' \wedge z'))(y \vee z)' \qquad \text{(by 2.6(1))} \\ &= ((x'y') \wedge (x'z'))(y \vee z)' \qquad \text{(by 2.6(4))} \\ &= ((x'y') \wedge (x'z'))(y' \wedge z') \\ &= (((x'y') \wedge (x'z'))y') \wedge (((x'y') \wedge (x'z'))z') \\ &\geq ((x'y')y') \wedge ((x'z')z') \qquad \text{(by 2.4(6))} \\ &= (x' \vee y') \wedge (x' \vee z') \end{aligned}$$

since $(x'y') \wedge (x'z') \leq x'y'$ and $(x'y') \wedge (x'z') \leq x'z'$. This implies

$$\begin{aligned} x \wedge (y \vee z) &= (x' \vee (y \vee z)')' \\ &\leq ((x' \vee y') \wedge (x' \vee z'))' \qquad \text{(by 2.4(9))} \\ &= (x' \vee y')' \vee (x' \vee z')' \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

Hence $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. □

In a lattice L, it is well known that the following are equivalent:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $x, y, z \in L$,
(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for every $x, y, z \in L$.

This mean that a lattice L is a distributive if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for every $x, y, z \in L$.

Theorem 2.8. *Every quasi-lattice implication algebra is a lattice implication algebra.*

Proof. Let $(L, \rightarrow, ', 1)$ be a quasi-lattice implication algebra. Then it satisfies the axioms (I1)-(I5) from the definition. Also L is a lattice and satisfies the axiom (L1) by 2.6(3). So we need only to show that it satisfies the axiom (L2).

Let $x, y, z \in L$. Then we have

$$\begin{aligned} ((x \wedge y)z)z &= z \vee (x \wedge y) \qquad \text{(by 2.5)} \\ &= (z \vee x) \wedge (z \vee y) \qquad \text{(by 2.7)} \\ &= ((xz)z) \wedge ((yz)z) \qquad \text{(by 2.5)} \\ &= ((xz) \vee (yz))z \qquad \text{(by 2.6(3))}. \end{aligned}$$

This implies

$$\begin{aligned}(x \wedge y)z &= (((x \wedge y)z)z)z \qquad \text{(by 2.4(10))}\\ &= (((xz) \vee (yz))z)z\\ &= ((xz) \vee (yz)) \vee z \qquad \text{(by 2.5)}\\ &= (xz) \vee (yz),\end{aligned}$$

since $z \leq (xz) \vee (yz)$. Thus L satisfies the axiom (L2) and so it is a lattice implication algebra. □

It is clear that a lattice implication algebra is a quasi-lattice implication algebra. From the above theorem, the notion of quasi-lattice implication algebras is equivalent to that of lattice implication algebras.

Theorem 2.9. *A set L is a quasi-lattice implication algebra if and only if there are a binary operation $\cdot$ on L and two elements $0, 1$ in L satisfying the following: for every $x, y, z \in L$,*

(Q1) $x(yz) = y(xz)$,
(Q2) $xx = 1$,
(Q3) $(xy)y = (yx)x$,
(Q4) $xy = 1$ *and* $yx = 1$ *imply* $x = y$,
(B) $0x = 1$.

Proof. Let L be a quasi-lattice implication algebra. Then it satisfies the properties (Q1)-(Q4) and (B) by the definition of quasi-lattice implication algebra and 2.3.

Conversely, suppose that L be a set with a binary operation $\cdot$ and two elements $0, 1$ satisfying the properties (Q1)-(Q4) and (B). Then we need to show that there is an involution $'$ on L satisfying (Q5) : $xy = y'x'$ for every $x, y \in L$.

It can be proved that $1x = x$ for every $x \in L$ in the same way as the proof of 2.2(1). Let $x' = x0$. Then we have $x'' = (x0)0 = (0x)x = 1x = x$ by (Q3) and (B). So $'$ is an involution, and it satisfies the following.

$$y'x' = (y0)(x0) = x((y0)0) = xy'' = xy$$

by (Q1). Hence $(L, \cdot, ', 1)$ is a quasi-lattice implication algebra. □

3. Congruence relations on quasi-lattice implication algebras

A subset F of a quasi-lattice implication algebra L is called a *filter* of L if it satisfies the following: for any $x, y \in L$,

(F1) $1 \in F$,
(F2) $x \in F$ and $xy \in F$ imply $y \in F$.

Lemma 3.1. *If F is a filter of a quasi-lattice implication algebra L, then F is a lattice filter of lattice L, i.e., F satisfies the following:*

(1) $x \in F$ *and* $x \leq y$ *imply* $y \in F$,
(2) $x, y \in F$ *implies* $x \wedge y \in F$.

Proof. (1) Let $x \in F$ and $x \leq y$. Then $xy = 1 \in F$. Hence $y \in F$ since $x \in F$.

(2) Let $x, y \in F$. Then $y \leq xy$. This implies $xy \in F$ by (1) of this lemma, and $x(x \wedge y) = (xx) \wedge (xy) = 1 \wedge (xy) = xy \in F$ by 2.6(4). Since $x(x \wedge y) \in F$ and $x \in F$, $x \wedge y \in F$. □

The converse of Lemma 3.1 is not true in general, as the following example shows.

Example 3.2. Let $Q = \{0, a, b, c, d, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

·	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	d	1
d	a	c	d	c	1	1
1	0	a	b	c	d	1

If we define $x' = x0$ for every $x \in Q$, then $(Q, \cdot, ', 1)$ is a quasi-lattice implication algebra. Also, it is a lattice with $x \vee y = (xy)y$ and $x \wedge y = (x' \vee y')' = ((x'y')y')'$. This lattice is depicted by Hasse diagram of Figure 1. Let $F = \{a, c, d, 1\}$. Then

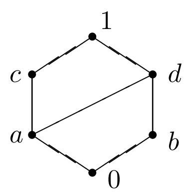

FIGURE 1. Hasse diagram of a lattice Q

F is a lattice filter of lattice Q, but it is not filter of Q, because $a \in F$ and $ab = d \in F$ but $b \notin F$.

For any filter F of a quasi-lattice implication algebra L, we can define a binary relation Φ_F on L by

$$x\Phi_F y \iff xy \in F \text{ and } yx \in F$$

for any $x, y \in L$.

Lemma 3.3. *Let F be a filter of L. Then Φ_F is a congruence relation.*

Proof. For any $x \in L$, it is clear that $x\Phi_F x$, since $xx = 1 \in F$, and that $x\Phi_F y$ implies $y\Phi_F x$.

To show the transitivity of Φ_F, let $x\Phi_F y$ and $y\Phi_F z$. Then $xy, yx \in F$ and $yz, zy \in F$. Since $xy \leq (yz)(xz)$ by 2.4(4), $(yz)(xz) \in F$ by 3.1(1). This implies $xz \in F$ since $yz \in F$. Also, we can see $zx \in F$ in the similar way. Hence $x\Phi_F z$. Thus Φ_F is an equivalence relation in L.

To show that Φ_F is a congruence relation on L, let $x\Phi_F y$ and $z \in L$. Then $xy, yx \in F$. Since $xy \in F$ and $xy \leq (zx)(zy)$ by 2.4(5), $(zx)(zy) \in F$. Also, since $yx \in F$ and $yx \leq (zy)(zx)$, $(zy)(zx) \in F$. Thus $(zx)\Phi_F(zy)$, and Φ_F is left compatible. In the similar way, we can show $(xz)(yz), (yz)(xz) \in F$ by 2.4(4). That is $(xz)\Phi_F(yz)$, and Φ_F is right compatible. Hence Φ_F is a congruence relation on L. □

We will call this relation Φ_F as a *congruence relation induced by a filter* F.

For any equivalence relation Θ on a quasi-lattice implication algebra L, we will write $[x]_\Theta$ for the equivalence classes.

Lemma 3.4. *Let L be a quasi-lattice implication algebra. If Θ is a congruence relation on L, then $[1]_\Theta$ is a filter of L.*

Proof. Let Θ be a congruence relation on L. Then it is trivial that $1 \in [1]_\Theta$. If $x \in [1]_\Theta$ and $xy \in [1]_\Theta$, then $x\Theta 1$ and $xy\Theta 1$ imply $xy\Theta 1y$ and $xy\Theta 1$, hence $y\Theta 1$ since $1y = y$. This implies $y \in [1]_\Theta$. Thus $[1]_\Theta$ is a filter of L. □

Theorem 3.5. *Every congruence relation on a quasi-lattice implication algebra L is a congruence relation induced by a filter.*

Proof. Suppose that Θ is a congruence relation on L. Then $[1]_\Theta$ is a filter by 3.4. Set $F = [1]_\Theta$, and we will show that $\Theta = \Phi_F$.

Let $x\Theta y$. Then $xy\Theta yy$ and $yx\Theta yy$, since Θ is a congruence relation. This implies $xy\Theta 1$ and $yx\Theta 1$, and $xy, yx \in [1]_\Theta = F$. Thus $x\Phi_F y$. Also, let $x\Phi_F y$. Then $xy, yx \in F = [1]_\Theta$. This implies $xy\Theta 1$ and $yx\Theta 1$, and $(xy)y\Theta 1y$ and $(yx)x\Theta 1x$, i.e., $(xy)y\Theta y$ and $(yx)x\Theta x$. Hence $x\Theta y$ since $(xy)y = (yx)x$.

This mean $\Theta = \Phi_F$, and Θ is a congruence relation induced by the filter $F = [1]_\Theta$. □

Let L be a quasi-lattice implication algebra. Then the family $Fil(L)$ (resp. $Con(L)$) of all filters of L (resp. all congruence relations on L) is partially ordered by set inclusion, and it is a complete lattice with

$$\bigwedge_{\alpha\in\Lambda} F_\alpha = \bigcap_{\alpha\in\Lambda} F_\alpha \quad \text{and} \quad \bigvee_{\alpha\in\Lambda} F_\alpha = \left\langle \bigcup_{\alpha\in\Lambda} F_\alpha \right\rangle$$

$$(\text{resp. } \bigwedge_{\alpha\in\Lambda} \Theta_\alpha = \bigcap_{\alpha\in\Lambda} \Theta_\alpha \quad \text{and } \bigvee_{\alpha\in\Lambda} \Theta_\alpha = \left\langle \bigcup_{\alpha\in\Lambda} \Theta_\alpha \right\rangle)$$

for arbitrary subset $\{F_\alpha \mid \alpha \in \Lambda\}$ of $Fil(L)$ (resp. $\{\Theta_\alpha \mid \alpha \in \Lambda\}$ of $Con(L)$), where $\langle X \rangle$ is the filter (resp. the congruence relation) generated by a subset X of L (resp. of $L \times L$).

Lemma 3.6. *Let L be a quasi-lattice implication algebra. Then it satisfies the following:*

(1) $[1]_{\Phi_F} = F$ *for every* $F \in Fil(L)$,
(2) $\Phi_{[1]_\Theta} = \Theta$ *for every* $\Theta \in Con(L)$,
(3) $F \subseteq G$ *if and only if* $\Phi_F \subseteq \Phi_G$ *for any* $F, G \in FIl(L)$,
(4) $\Theta \subseteq \Psi$ *if and only if* $[1]_\Theta \subseteq [1]_\Psi$ *for any* $\Theta, \Psi \in Con(L)$.

Proof. (1) Let $F \in Fil(L)$. Then we have

$$x \in [1]_{\Phi_F} \iff x\Phi_F 1 \iff (x1 = 1 \in F \text{ and } 1x = x \in F) \iff x \in F.$$

Hence $[1]_{\Phi_F} = F$.

(2) It was proved in the proof of 3.5.

(3) Let $F \subseteq G$ in $Fil(L)$ and $x\Phi_F y$. Then $xy, yx \in F$, and $xy, yx \in G$ since $F \subseteq G$. This implies $x\Phi_G y$. Hence $\Phi_F \subseteq \Phi_G$. Conversely, let $\Phi_F \subseteq \Phi_G$ and $x \in F$. Then $x1 = 1 \in F$, since F is a filter, and $1x = x \in F$. This implies $x\Phi_F 1$, and $x\Phi_G 1$ since $\Phi_F \subseteq \Phi_G$. This implies $1x = x \in G$. Hence $F \subseteq G$.

(4) Let $\Theta, \Psi \in Con(L)$. Then since $\Theta = \Phi_{[1]_\Theta}$ and $\Psi = \Phi_{[1]_\Psi}$ by (2) of this lemma, we have

$$\Theta \subseteq \Psi \iff \Phi_{[1]_\Theta} \subseteq \Phi_{[1]_\Psi} \iff [1]_\Theta \subseteq [1]_\Psi$$

by (3) of this lemma. □

Theorem 3.7. *Let L be a quasi-lattice implication algebra and $\phi : Fil(L) \to Con(L)$ a map defined by $\phi(F) = \Phi_F$ for every $F \in Fil(L)$. Then it satisfies the following:*

(1) *ϕ is order-isomorphism,*
(2) *ϕ preserves arbitrary join and arbitrary meet.*

Proof. (1) Let $\phi : Fil(L) \to Con(L)$ be a map defined by $\phi(F) = \Phi_F$ for every $F \in Fil(L)$. Then $F \subseteq G$ if and only if $\phi(F) = \Phi_F \subseteq \Phi_G = \phi(G)$ by 3.6(3). Hence ϕ is order-embedding. Also let $\Theta \in Con(L)$. Then there exists $F = [1]_\Theta \in Fil(L)$ such that $\phi(F) = \Phi_{[1]_\Theta} = \Theta$ by 3.6(2). Hence ϕ is onto.

(2) Let $\{F_\alpha \mid \alpha \in \Lambda\}$ be an arbitrary subset of $Fil(L)$. Then $\phi(F_\beta) \subseteq \phi(\bigvee_{\alpha\in\Lambda} F_\alpha)$ for every $\beta \in \Lambda$, since ϕ is order-preserving by (1) of this theorem. Hence $\phi(\bigvee_{\alpha\in\Lambda} F_\alpha)$ is an upper bound of the set $\{\phi(F_\alpha) \mid \alpha \in \Lambda\}$

Suppose that $\phi(F_\beta) = \Phi_{F_\beta} \subseteq \Theta$ for every $\beta \in \Lambda$. Then by (1) and (4) of 3.6, $F_\beta = [1]_{\Phi_{F_\beta}} \subseteq [1]_\Theta$ for every $\beta \in \Lambda$. This implies $\bigvee_{\alpha\in\Lambda} F_\alpha \subseteq [1]_\Theta$, and hence

$$\phi(\textstyle\bigvee_{\alpha\in\Lambda} F_\alpha) \subseteq \phi([1]_\Theta) = \Phi_{[1]_\Theta} = \Theta.$$

Thus $\phi(\bigvee_{\alpha\in\Lambda} F_\alpha)$ is the least upper bound of the set $\{\phi(F_\alpha) \mid \alpha \in \Lambda\}$. Hence $\phi(\bigvee_{\alpha\in\Lambda} F_\alpha) = \bigvee_{\alpha\in\Lambda} \phi(F_\alpha)$. Also, we can show $\phi(\bigwedge_{\alpha\in\Lambda} F_\alpha) = \bigwedge_{\alpha\in\Lambda} \phi(F_\alpha)$ in the similar way. □

From the above theorem, $Fil(L)$ and $Con(L)$ of a quasi-lattice implication algebra L have the same structure as complete lattices.

References

1. R.A. Borzooei and S.F. Hosseiny, *Finite lattice implication algebras*, Jordan J. Math. Stat. **6** (2013), 265–283.
2. Y.B. Jun, *Fantastic filters of lattice implication algebras*, Int. J. Math. Sci **24** (2000), 277–281.
3. S.E. Kang and S.S. Ahn, *Rough filters of BE-algebras*, J. Appl. Math. & Informatics **30** (2012), 1023–1030.

4. S.D. Lee and K.H. Kim, *On derivations of lattice implication algebras*, Ars Combin. **108** (2013), 279-288.
5. J. Liu and Y. Xu, *Lattice implication algebras and MV-algebras*, Chinese Quart. J. Math. **14** (1999), 17–23.
6. E. L. Marsden, *A note on implicative models*, Notre Dame J. Form. Log. **XIV(1)** (1973), 139–144.
7. E.H. Roh, S.Y. Kim and Y.B. Jun, *Some operations on lattice implication algebras*, Int. J. Math. Sci. **27** (2001), 45–52.
8. Y. Xu, *Lattice implication algebras*, J. Southwest Jiaotong Univ. **1** (1993), 20–27.
9. Y. Xu and K.Y. Qin, *Lattice H implication algebras and lattice implication algebra classes*, J. Hebei Mining and Civil Engineering Institute **3** (1992), 139–143.
10. Y. Xu and K.Y. Qin, *On filters of lattice implication algebras*, J. Fuzzy Math. **1** (1993), 251–260.
11. Y.H. Yon and K.H. Kim, *On f-derivations of lattice implication algebras*, Ars Combin. **110** (2013), 205–215.
12. H. Zhu, J.B. Zhao and Y. Xu, *The n-fold prime filter of residuated lattice implication algebra*, J. Zhengzhou Univ. Nat. Sci. Ed. **40** (2008), 19–22.

Yong Ho Yon received M.Sc. from Chungbuk National University, and Ph.D. from Chungbuk National University. He is currently a professor at Mokwon University since 2011. His research interests are lattice theory and algebras with implication.

Division of Information and Communication Convergence Engineering, Mokwon University, Daejeon 302-729, Korea.
e-mail: yhyon@mokwon.ac.kr