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A UNIFORMLY CONVERGENT NUMERICAL METHOD FOR A WEAKLY COUPLED SYSTEM OF SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS WITH BOUNDARY AND WEAK INTERIOR LAYERS†

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Abstract. We consider a weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term. The diffusion term of each equation is associated with a small positive parameter of different magnitude. Presence of discontinuity and different parameters creates boundary and weak interior layers that overlap and interact. A numerical method is constructed for this problem which involves an appropriate piecewise uniform Shishkin mesh. The numerical approximations are proved to converge to the continuous solutions uniformly with respect to the singular perturbation parameters. Numerical results are presented which illustrates the theoretical results.

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1. Introduction

An extensive research had been done on numerical methods for a single singularly perturbed convection-diffusion differential equation [1]-[4], but for system of equations very few works had been done. The classical numerical methods fail to produce good approximations for singularly perturbed problems. Various non-classical approaches produce better approximations and converge uniformly with respect to the small perturbation parameter. In the literature [7]-[14] methods were available to obtain numerical approximation for system of singularly perturbed convection- diffusion differential equations the source term are smooth

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on the whole domain. Farrell et.al [5]-[6] considered scalar singularly perturbed convection-diffusion equation with discontinuous source term. The interior layers in [6] were strong, in the sense that the solution was bounded but the magnitude of the first derivative grew unboundedly as $\varepsilon \to 0$, but in [5] they were weak, in the sense that the solution and the first derivative were bounded but the magnitude of the second derivative grows unboundedly as $\varepsilon \to 0$. In this work, we present a uniformly convergent numerical method for a weakly coupled system of singularly perturbed convection-diffusion equations having discontinuous source term with different diffusion parameters. The solution to such equations has overlapping and interacting boundary and interior layers which makes the construction of numerical methods and analysis quite difficult. Tamilselvan and Ramanujam [15] considered the same problem but with equal diffusion parameters.

Consider a weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term on the unit interval $\Omega = (0, 1)$, having a single discontinuity in the source term at a point $d \in \Omega$. Let $\Omega_1 = (0, d)$ and $\Omega_2 = (d, 1)$. Let the jump in a function ω at a point $d \in \Omega$ given as $[\omega](d) = \omega(d+) - \omega(d-)$. The corresponding boundary value problem is to find $u_1, u_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$, such that

$$\boldsymbol{L}\boldsymbol{u} := -\boldsymbol{E}\boldsymbol{u}^{''} - \boldsymbol{A}\boldsymbol{u}^{'} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{f}, \ x \in \Omega_1 \cup \Omega_2. \tag{1}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{u}(1) = \boldsymbol{u}_1, \tag{2}$$

where $\boldsymbol{E} = \text{diag}(\varepsilon_1, \varepsilon_2)$, the coupling matrix $\boldsymbol{A} = \text{diag}(a_1, a_2)$ and $\boldsymbol{B} = (b_{ij})_{2\times 2}$ with $0 < \varepsilon_1 \le \varepsilon_2 \le 1$, $\boldsymbol{f} = (f_1, f_2)^T$, and $\boldsymbol{u} = (u_1, u_2)^T$. Assume for each $i = 1, 2$ and $x \in \overline{\Omega}$, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy

$$a_i(x) \ge \alpha_i > 0, \tag{3}$$

$$b_{ij}(x) \le 0, \ i \ne j, \ b_{11}(x) + b_{12}(x) \ge 0, b_{21}(x) + b_{22}(x) \ge 0. \tag{4}$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$. Further assume that the source terms f_1, f_2 are sufficiently smooth on $\overline{\Omega} \setminus \{d\}$, and their derivatives have jump discontinuity at the same point.

Notations. Throughout the paper, C denotes a generic positive constant and $\boldsymbol{C} = (C, C)^T$ denotes a generic positive constant vector, both are independent of perturbation parameters $\varepsilon_1, \varepsilon_2$ and the discretization parameter N, but may not be same at each occurrence. Define $\boldsymbol{v} \le \boldsymbol{w}$ if $v_i \le w_i, i = 1, 2$, and $|\boldsymbol{v}| = (|v_1|, |v_2|)^T$. We consider the maximum norm and denote it by $\| \,.\, \|_S$, where S is a closed and bounded subset in $\overline{\Omega}$. For a real valued function $v \in C(S)$ and for a vector valued function $\boldsymbol{v} = (v_1, v_2)^T \in C(S)^2$, we define $\| v \|_S = \max_{x \in S} |v(x)|$ and $\| \boldsymbol{v} \|_S = \max\{\| v_1 \|_S, \| v_2 \|_S\}$. Now let a mesh $\Omega^N = \{x_i\}_{i=0}^N$ be a set of points satisfying $x_0 < x_1 < \cdots < x_N = 1$. A mesh function $V = \{V(x_i)\}_{i=0}^N$ is a real-valued function defined on Ω^N. Define the discrete maximum norm for such functions by $\| V \|_{\Omega^N} = \max_{i=0,1,\ldots,N} \{V(x_i)\}$ and for vector mesh functions

$\boldsymbol{V} = (V_1, V_2)^T = \{V_1(x_i), V_2(x_i)\}_{i=0}^N$ are used and define $\| \boldsymbol{V} \|_{\Omega^N} = \max\{\| V_1 \|_{\Omega^N}, \| V_2 \|_{\Omega^N}\}$.

2. The continuous problem

Theorem 2.1 (Continuous maximum principle). *Suppose* $u_1, u_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$. *Further suppose that* $\boldsymbol{u} = (u_1, u_2)^T$ *satisfies* $\boldsymbol{u}(0) \geq \boldsymbol{0}$, $\boldsymbol{u}(1) \geq \boldsymbol{0}$, $\boldsymbol{Lu}(x) \geq \boldsymbol{0}$ *in* $\Omega_1 \cup \Omega_2$ *and* $[\boldsymbol{u}'](d) \leq \boldsymbol{0}$. *Then* $\boldsymbol{u}(x) \geq \boldsymbol{0}$, *for all* $x \in \overline{\Omega}$.

Proof. Let $\boldsymbol{u} := \begin{cases} e^{\frac{-\alpha(d-x)}{4\varepsilon_2}}\Theta, & x \in \Omega_1, \\ e^{\frac{-\alpha(x-d)}{2\varepsilon_2}}\Theta, & x \in \Omega_2. \end{cases}$

Now let p and q be the points at which $\theta_1(p) := \min_{x\in\overline{\Omega}}\{\theta_1(x)\}$ and $\theta_2(q) := \min_{x\in\overline{\Omega}}\{\theta_2(x)\}$. Assume without loss of generality $\theta_1(p) \leq \theta_2(q)$. If $\theta_1(p) \geq 0$, then there is nothing to prove. Suppose that $\theta_1(p) < 0$, then proof is completed by showing that this leads to a contradiction. Note that $p \neq \{0, 1\}$. So either $p \in \Omega_1 \cup \Omega_2$ or $p = d$.

In the first case for $x \in \Omega_2$,

$(\boldsymbol{Lu})_1(p) = e^{\frac{-\alpha(p-d)}{2\varepsilon_2}}(-\varepsilon_1\theta_1''(p) + (\frac{\alpha\varepsilon_1}{\varepsilon_2} - a_1(p))\theta_1'(p) + \frac{\alpha}{2\varepsilon_2}(a_1(p) - \frac{\alpha\varepsilon_1}{2\varepsilon_2})\theta_1(p)$
$+ (b_{11}(p) + b_{12}(p))\theta_1(p) + b_{12}(p)(\theta_2(p) - \theta_1(p)) < 0.$

In the second case, that is, $p = d$, we have $[\boldsymbol{u}'](d) = [\Theta'](d) - \frac{\alpha}{4\varepsilon_2}\Theta(d)$, and at a negative minimum $[\Theta'](d) \geq 0$, which gives a contradiction. □

Lemma 2.2 (Stability Result). *Let* $\boldsymbol{u} = (u_1, u_2)^T$ *be the solution of* $(1) - (2)$, *then,*

$$\|\boldsymbol{u}\|_{\overline{\Omega}} \leq C \max\{\|\boldsymbol{u}(0)\|, \|\boldsymbol{u}(1)\|, \|\boldsymbol{Lu}\|_{\Omega_1\cup\Omega_2}\}.$$

Proof. Define the function $\Psi^{\pm}(x) := \max\{\|\boldsymbol{u}(0)\|, \|\boldsymbol{u}(1)\|, \|\boldsymbol{Lu}\|_{\Omega_1\cup\Omega_2}\}(2 - x, 2 - x)^T \pm \boldsymbol{u}(x)$. Then $\Psi^{\pm}(0) \geq \boldsymbol{0}$, $\Psi^{\pm}(1) \geq \boldsymbol{0}$, $\boldsymbol{L}\Psi^{\pm}(x) \geq \boldsymbol{0}$ for each $x \in \Omega_1 \cup \Omega_2$, and $[\Psi^{\pm\,\prime}](d) = \pm[\boldsymbol{u}'](d) = \boldsymbol{0}$, since $\boldsymbol{u} \in C^1(\Omega)^2$. It follows from the maximum principle that $\Psi^{\pm}(x) \geq \boldsymbol{0}$ for all $x \in \overline{\Omega}$, which leads to the required bound on $\boldsymbol{u}$. Consequently, the problem $(1) - (2)$ has a unique and stable solution. □

To derive sharper bounds on the derivatives of solution, the solution is decomposed into a sum, composed of a regular component $\boldsymbol{v}$ and a singular component $\boldsymbol{w}$. That is, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$. The regular component $\boldsymbol{v}$, can be written in the form $\boldsymbol{v} = \boldsymbol{v}_0 + \begin{pmatrix} \varepsilon_1\varepsilon_2 & 0 \\ 0 & \varepsilon_1\varepsilon_2 \end{pmatrix} \boldsymbol{v}_1 + \begin{pmatrix} \varepsilon_1^2\varepsilon_2^2 & 0 \\ 0 & \varepsilon_1^2\varepsilon_2^2 \end{pmatrix} \boldsymbol{v}_2$, where $\boldsymbol{v}_0 = (v_{01}, v_{02})^T$, $\boldsymbol{v}_1 = (v_{11}, v_{12})^T$ and $\boldsymbol{v}_2 = (v_{21}, v_{22})^T$ are defined respectively to be the solutions of the problems

$$-\boldsymbol{A}\boldsymbol{v}_0' + \boldsymbol{B}\boldsymbol{v}_0 = \boldsymbol{f}, \;\; \boldsymbol{v}_0(1) = \boldsymbol{u}(1), \;\; x \in \Omega_1 \cup \Omega_2,$$

$$-\boldsymbol{A}\boldsymbol{v}_1' + \boldsymbol{B}\boldsymbol{v}_1 = \begin{pmatrix} \frac{1}{\varepsilon_2} \\ \frac{1}{\varepsilon_1} \end{pmatrix} \boldsymbol{v}_0'', \;\; \boldsymbol{v}_1(1) = \boldsymbol{0}, \;\; x \in \Omega_1 \cup \Omega_2,$$

and

$$\boldsymbol{Lv}_2 = \begin{pmatrix} \frac{1}{\varepsilon_2} \\ \frac{1}{\varepsilon_1} \end{pmatrix} \boldsymbol{v}_1'', \ x \neq d \ \boldsymbol{v}_2(0) = \boldsymbol{v}_2(d) = \boldsymbol{v}_2(1) = \boldsymbol{0}, \ x \in \Omega_1 \cup \Omega_2.$$

Thus the regular component $\boldsymbol{v}$ is the solution of

$$\boldsymbol{Lv} = \boldsymbol{f}, \ x \in \Omega_1 \cup \Omega_2, \boldsymbol{v}(0) = \boldsymbol{v}_0(0) + \varepsilon_1\varepsilon_2\boldsymbol{v}_1(0), \boldsymbol{v}(d) = \boldsymbol{v}_0(d) + \varepsilon_1\varepsilon_2\boldsymbol{v}_1(d), \boldsymbol{v}(1) = \boldsymbol{u}(1).$$

Further we decompose $\boldsymbol{w}$ as $\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ where $\boldsymbol{w}_1 = (w_{11}, w_{12})^T$, $\boldsymbol{w}_2 = (w_{21}, w_{22})^T$. Thus $w_1 = w_{11} + w_{21}$ and $w_2 = w_{12} + w_{22}$, where $\boldsymbol{w}_1$ is the solution of

$$\boldsymbol{Lw}_1 = \boldsymbol{0}, \ x \in \Omega, \ \boldsymbol{w}_1(0) = \boldsymbol{u}(0) - \boldsymbol{v}(0), \ \boldsymbol{w}_1(1) = \boldsymbol{0},$$

and $\boldsymbol{w}_2$ is the solution of

$$\boldsymbol{Lw}_2 = \boldsymbol{0}, \ x \in \Omega_1 \cup \Omega_2, \ \boldsymbol{w}_2(0) = \boldsymbol{0}, \ \boldsymbol{w}_2(1) = \boldsymbol{0}, \ [\boldsymbol{w}_2'](d) = -[\boldsymbol{v}'](d).$$

Lemma 2.3. *For each integer k, satisfying $0 \leq k \leq 3$, the regular component $\boldsymbol{v}$ and its derivatives satisfy the bounds given by*

$$\|\boldsymbol{v}^{(k)}\|_{\Omega_1 \cup \Omega_2} \leq C.$$

Proof. The proof follows from [7] and [2]. □

Lemma 2.4. *For each integer k, satisfying $0 \leq k \leq 3$, the singular component $\boldsymbol{w}_1$ and its derivatives satisfy the bounds given by*

$$|w_{11}(x)| \leq C \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big), \ |w_{12}(x)| \leq C \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big),$$

$$|w_{11}'(x)| \leq C\Big(\varepsilon_1^{-1} \exp\Big(-\frac{\alpha x}{\varepsilon_1}\Big) + \varepsilon_2^{-1} \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big)\Big), \ |w_{12}'(x)| \leq C\Big(\varepsilon_2^{-1} \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big)\Big),$$

$$|w_{11}''(x)| \leq C\Big(\varepsilon_1^{-2} \exp\Big(-\frac{\alpha x}{\varepsilon_1}\Big) + \varepsilon_2^{-2} \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big)\Big), \ |w_{12}''(x)| \leq C\Big(\varepsilon_2^{-2} \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big)\Big),$$

$$|w_{11}'''(x)| \leq C\Big(\varepsilon_1^{-3} \exp\Big(-\frac{\alpha x}{\varepsilon_1}\Big) + \varepsilon_2^{-3} \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big)\Big),$$

$$|w_{12}'''(x)| \leq C\varepsilon_2^{-1}\Big(\varepsilon_1^{-2} \exp\Big(-\frac{\alpha x}{\varepsilon_1}\Big) + \varepsilon_2^{-2} \exp\Big(-\frac{\alpha x}{\varepsilon_2}\Big)\Big).$$

Proof. The proof follows from [7] and [2]. □

Lemma 2.5. *For each integer k, satisfying $0 \leq k \leq 3$, the singular component $\boldsymbol{w}_2$ and its derivatives satisfy the bounds given by*

$$|w_{21}(x)| \leq \begin{cases} C\varepsilon_2 \exp(-\frac{\alpha x}{\varepsilon_2}), & x \in \Omega_1 \\ C\varepsilon_2 \exp(-\frac{\alpha(x-d)}{\varepsilon_2}), & x \in \Omega_2, \end{cases}$$

$$|w_{22}(x)| \leq \begin{cases} C\varepsilon_2 \exp(-\frac{\alpha x}{\varepsilon_2}), & x \in \Omega_1 \\ C\varepsilon_2 \exp(-\frac{\alpha(x-d)}{\varepsilon_2}), & x \in \Omega_2, \end{cases}$$

$$|w_{21}'(x)| \leq \begin{cases} C(\exp(-\frac{\alpha x}{\varepsilon_1}) + \exp(-\frac{\alpha x}{\varepsilon_2})), & x \in \Omega_1 \\ C(\exp(-\frac{\alpha(x-d)}{\varepsilon_1}) + \exp(-\frac{\alpha(x-d)}{\varepsilon_2})), & x \in \Omega_2, \end{cases}$$

$$|w_{22}'(x)| \leq \begin{cases} C\exp(-\frac{\alpha x}{\varepsilon_2}), & x \in \Omega_1 \\ C\exp(-\frac{\alpha(x-d)}{\varepsilon_2}), & x \in \Omega_2, \end{cases}$$

$$|w_{21}''(x)| \leq \begin{cases} C(\varepsilon_1^{-1}\exp(-\frac{\alpha x}{\varepsilon_1}) + \varepsilon_2^{-1}\exp(-\frac{\alpha x}{\varepsilon_2})), & x \in \Omega_1 \\ C(\varepsilon_1^{-1}\exp(-\frac{\alpha(x-d)}{\varepsilon_1}) + \varepsilon_2^{-1}\exp(-\frac{\alpha(x-d)}{\varepsilon_2})), & x \in \Omega_2, \end{cases}$$

$$|w_{22}''(x)| \leq \begin{cases} C(\varepsilon_2^{-1}\exp(-\frac{\alpha x}{\varepsilon_2})), & x \in \Omega_1 \\ C(\varepsilon_2^{-1}\exp(-\frac{\alpha(x-d)}{\varepsilon_2})), & x \in \Omega_2, \end{cases}$$

$$|w_{21}'''(x)| \leq \begin{cases} C(\varepsilon_1^{-2}\exp(-\frac{\alpha x}{\varepsilon_1}) + \varepsilon_2^{-2}\exp(-\frac{\alpha x}{\varepsilon_2})), & x \in \Omega_1 \\ C(\varepsilon_1^{-2}\exp(-\frac{\alpha(x-d)}{\varepsilon_1}) + \varepsilon_2^{-2}\exp(-\frac{\alpha(x-d)}{\varepsilon_2})), & x \in \Omega_2, \end{cases}$$

$$|w_{22}'''(x)| \leq \begin{cases} C\varepsilon_2^{-1}(\varepsilon_1^{-1}\exp(-\frac{\alpha x}{\varepsilon_1}) + \varepsilon_2^{-1}\exp(-\frac{\alpha x}{\varepsilon_2})), & x \in \Omega_1 \\ C\varepsilon_2^{-1}(\varepsilon_1^{-1}\exp(-\frac{\alpha(x-d)}{\varepsilon_1}) + \varepsilon_2^{-1}\exp(-\frac{\alpha(x-d)}{\varepsilon_2})), & x \in \Omega_2. \end{cases}$$

Proof. Consider the barrier function $\phi(1,1)^T \pm \boldsymbol{w}_2$, where

$$\phi(x) := \begin{cases} \frac{\varepsilon_2 A}{\alpha}, & x \in \Omega_1 \\ \frac{\varepsilon_2 A}{\alpha}\exp(\frac{-\alpha(x-d)}{\varepsilon_2}), & x \in \Omega_2, \end{cases}$$

to bound $\boldsymbol{w}_2$. To bound derivatives of $\boldsymbol{w}_2$, use the technique used in [7] and bound on $\boldsymbol{w}_2$ on the domain Ω_1 and Ω_2. □

3. Discretization of the Problem

We use piecewise uniform Shishkin mesh which uses these transition parameters:

$$\sigma_{\varepsilon_{l_2}} := \min\left\{\frac{d}{2}, \frac{\varepsilon_2}{\alpha}\ln N\right\},\ \sigma_{\varepsilon_{r_2}} := \min\left\{\frac{(1-d)}{2}, \frac{\varepsilon_2}{\alpha}\ln N\right\},$$

$$\sigma_{\varepsilon_{l_1}} := \min\left\{\frac{\sigma_{\varepsilon_{l_2}}}{2}, \frac{\varepsilon_1}{\alpha}\ln N\right\},\ \sigma_{\varepsilon_{r_1}} := \min\left\{\frac{\sigma_{\varepsilon_{r_2}}}{2}, \frac{\varepsilon_1}{\alpha}\ln N\right\}.$$

The interior points of the mesh are denoted by
$\Omega^N = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\} = \Omega_1^N \cup \Omega_2^N$.
Let $h_i = x_i - x_{i-1}$ be the i^{th} mesh step and $\hbar_i = \frac{h_i + h_{i+1}}{2}$, clearly $x_{\frac{N}{2}} = d$ and $\overline{\Omega}^N = \{x_i : i = 0, 1, \ldots, N\}$. Let $N = 2^l$, $l \geq 5$ be any positive integer.
We divide $\overline{\Omega}_1^N$ into three sub-intervals $[0, \sigma_{\varepsilon_{l_1}}]$, $[\sigma_{\varepsilon_{l_1}}, \sigma_{\varepsilon_{l_2}}]$ and $[\sigma_{\varepsilon_{l_2}}, d]$ for some $0 < \sigma_{\varepsilon_{l_1}} \leq \sigma_{\varepsilon_{l_2}} \leq \frac{d}{2}$. The sub-intervals $[0, \sigma_{\varepsilon_{l_1}}]$ and $[\sigma_{\varepsilon_{l_1}}, \sigma_{\varepsilon_{l_2}}]$ are divided into $N/8$ equidistant elements and the sub-interval $[\sigma_{\varepsilon_{l_2}}, d]$ is divided into $N/4$ equidistant elements. Similarly, in $\overline{\Omega}_2^N$ the sub-intervals $[d, d + \sigma_{\varepsilon_{r_1}}]$ and $[d + \sigma_{\varepsilon_{r_1}}, d + \sigma_{\varepsilon_{r_2}}]$ are divided into $N/8$ equidistant elements and the sub-interval $[d + \sigma_{\varepsilon_{r_2}}, 1]$ is divided into $N/4$ equidistant elements, for some $0 < \sigma_{\varepsilon_{r_1}} \leq \sigma_{\varepsilon_{r_2}} \leq \frac{1-d}{2}$.
Define the discrete finite difference operator $\boldsymbol{L}^N$ as follows

$$\boldsymbol{L}^N \boldsymbol{U} = \boldsymbol{f} \text{ for all } x_i \in \Omega^N, \tag{5}$$

with boundary conditions

$$\boldsymbol{U}(x_0) = \boldsymbol{u}_0, \;\; \boldsymbol{U}(x_N) = \boldsymbol{u}_1, \tag{6}$$

where

$$\boldsymbol{L}^N = -\boldsymbol{E}\delta^2 - \boldsymbol{A}D^+ + \boldsymbol{B}$$

and at $x_{N/2} = d$ the scheme is given by

$$D^+\boldsymbol{U}(x_{\frac{N}{2}}) = D^-\boldsymbol{U}(x_{\frac{N}{2}}) \tag{7}$$

where
$\delta^2 Z(x_i) = (D^+Z(x_i) - D^-Z(x_i))\frac{1}{\bar{h}_i}$, $D^+Z(x_i) = \frac{Z(x_{i+1})-Z(x_i)}{h_{i+1}}$, $D^-Z(x_i) = \frac{Z(x_i)-Z(x_{i-1})}{h_i}$.

Lemma 3.1. *Suppose that a mesh function $\boldsymbol{Z}(x_i)$ satisfies $\boldsymbol{Z}(x_0) \geq \boldsymbol{0}, \boldsymbol{Z}(x_N) \geq \boldsymbol{0}$, $\boldsymbol{L}^N\boldsymbol{Z}(x_i) \geq \boldsymbol{0}$ for all $x_i \in \Omega^N$ and $D^+\boldsymbol{Z}(x_{\frac{N}{2}}) - D^-\boldsymbol{Z}(x_{\frac{N}{2}}) \leq \boldsymbol{0}$, then $\boldsymbol{Z}(x_i) \geq \boldsymbol{0}$ for all $x_i \in \overline{\Omega}^N$.*

Lemma 3.2. *If $\boldsymbol{Z}(x_i)$ is any mesh function, then,*

$$\|\boldsymbol{Z}\|_{\overline{\Omega}^N} \leq \max\left\{\|\boldsymbol{Z}(0)\|, \|\boldsymbol{Z}(1)\|, \|\boldsymbol{L}^N\boldsymbol{Z}\|_{\Omega_1^N \cup \Omega_2^N}\right\}.$$

The discrete solution $\boldsymbol{U}$ can be decomposed into the sum $\boldsymbol{U} = \boldsymbol{V} + \boldsymbol{W}$. The function $\boldsymbol{V}$, is defined as the solution of the following problem:

$$\boldsymbol{L}^N\boldsymbol{V}(x_i) = \boldsymbol{f}(x_i), \;\text{ for all } x_i \in \Omega^N \setminus \{d\}, \tag{8}$$

$$\boldsymbol{V}(0) = \boldsymbol{v}(0), \;\; \boldsymbol{V}(d) = \boldsymbol{v}(d), \;\; \boldsymbol{V}(1) = \boldsymbol{v}(1). \tag{9}$$

The function $\boldsymbol{W}$, is defined as the solution of the following problem:

$$\boldsymbol{L}^N\boldsymbol{W}(x_i) = \boldsymbol{0}, \;\text{ for all } x_i \in \Omega^N \setminus \{d\}, \tag{10}$$

$$\boldsymbol{W}(0) = \boldsymbol{w}(0), \;\; \boldsymbol{W}(1) = \boldsymbol{w}(1), \;\; [D\boldsymbol{W}](d) = -[D\boldsymbol{V}](d), \tag{11}$$

where the jump in the discrete derivative of a mesh function $\boldsymbol{Z}$ at the point $x_i = d$ is given by:

$$[D\boldsymbol{Z}](d) = D^+\boldsymbol{Z}(d) - D^-\boldsymbol{Z}(d).$$

Further decompose $\boldsymbol{W}$ as $\boldsymbol{W} = \boldsymbol{W}_1 + \boldsymbol{W}_2$, where the function $\boldsymbol{W}_1$ is defined as the solution of the following problem:

$$\boldsymbol{L}^N\boldsymbol{W}_1(x_i) = \boldsymbol{0}, \;\text{ for all } x_i \in \Omega^N \cup \{d\}, \tag{12}$$

$$\boldsymbol{W}_1(0) = \boldsymbol{w}(0), \;\; \boldsymbol{W}_1(1) = \boldsymbol{0}, \tag{13}$$

and the function $\boldsymbol{W}_2$ is defined as the solution of the following problem:

$$\boldsymbol{L}^N\boldsymbol{W}_2(x_i) = \boldsymbol{0}, \;\text{ for all } x_i \in \Omega^N \setminus \{d\}, \tag{14}$$

$$\boldsymbol{W}_2(0) = \boldsymbol{0}, \;\; \boldsymbol{W}_2(1) = \boldsymbol{0}, \;\; [D\boldsymbol{W}_2](d) = -[D\boldsymbol{V}](d) - [D\boldsymbol{W}_1](d). \tag{15}$$

4. Convergence analysis

By Taylor's expansion and bounds on regular components defined in lemma 2.3 gives

$$|(\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{v}(x_i)| \leq \left(\begin{array}{c} \frac{\varepsilon_1}{3}(x_{i+1}-x_{i-1})\|v_1'''\| + \frac{a_1(x_i)}{2}(x_i-x_{i-1})\|v_1''\| \\ \frac{\varepsilon_2}{3}(x_{i+1}-x_{i-1})\|v_2'''\| + \frac{a_2(x_i)}{2}(x_i-x_{i-1})\|v_2''\| \end{array}\right)$$
$$\leq C\left(\begin{array}{c} N^{-1} \\ N^{-1} \end{array}\right).$$

Define the mesh function $\Psi^{\pm}(x_i)$ as

$$\Psi^{\pm}(x_i) := \left(\begin{array}{c} \left(\begin{array}{c} CN^{-1}(d-x_i) \\ CN^{-1}(d-x_i) \end{array}\right) \pm (\boldsymbol{V}-\boldsymbol{v})(x_i) \text{ for } x_i \in \Omega_1^N \\ \left(\begin{array}{c} CN^{-1}(1-x_i) \\ CN^{-1}(1-x_i) \end{array}\right) \pm (\boldsymbol{V}-\boldsymbol{v})(x_i) \text{ for } x_i \in \Omega_2^N \end{array}\right).$$

Using discrete maximum principle, the error of the regular component satisfies the estimate

$$|(\boldsymbol{V}-\boldsymbol{v})(x_i)| \leq \left(\begin{array}{c} \left(\begin{array}{c} CN^{-1}(d-x_i) \\ CN^{-1}(d-x_i) \end{array}\right) \text{ for } x_i \in \Omega_1^N \\ \left(\begin{array}{c} CN^{-1}(1-x_i) \\ CN^{-1}(1-x_i) \end{array}\right) \text{ for } x_i \in \Omega_2^N \end{array}\right). \tag{16}$$

As in [7], the error of the singular component satisfies the estimate

$$|(\boldsymbol{W}_1-\boldsymbol{w}_1)(x_i)| \leq C\left(\begin{array}{c} N^{-1}\ln N \\ N^{-1}\ln N \end{array}\right). \tag{17}$$

Lemma 4.1. *The following $\varepsilon_1, \varepsilon_2-$ uniform bound*

$$|[D\,\boldsymbol{W}_2](d)| \leq \left(\begin{array}{c} C(1+\varepsilon_1^{-1}N^{-1}) \\ C(1+\varepsilon_2^{-1}N^{-1}) \end{array}\right),$$

where $\boldsymbol{W}_2$ is the solution of (14) – (15).

Proof. At the point $x=d$ we know that

$$[D\,\boldsymbol{W}_2](d) = -[D\,\boldsymbol{V}](d) - [D\,\boldsymbol{W}_1](d).$$

First consider

$$D^-\,\boldsymbol{V}(d) = D^-(\boldsymbol{V}-\boldsymbol{v})(d) + D^-\boldsymbol{v}(d).$$

From lemma 2.3 we have

$$\|\boldsymbol{v}'\|_{\Omega_1} \leq \left(\begin{array}{c} C \\ C \end{array}\right).$$

$$|D^-\boldsymbol{v}(d)| \leq \left(\begin{array}{c} C \\ C \end{array}\right) \text{ and } |D^-(\boldsymbol{V}-\boldsymbol{v})(d)| \leq \left(\begin{array}{c} CN^{-1} \\ CN^{-1} \end{array}\right).$$

Therefore, $|D^- \boldsymbol{V}(d)| \le \begin{pmatrix} C(1+N^{-1}) \\ C(1+N^{-1}) \end{pmatrix}$.
Similarly, consider

$$D^+ \boldsymbol{V}(d) = D^+(\boldsymbol{V} - \boldsymbol{v})(d) + D^+\boldsymbol{v}(d).$$

Again from lemma 2.3 we have

$$\|\boldsymbol{v'}\|_{\Omega_2} \le \begin{pmatrix} C \\ C \end{pmatrix} \text{ and } \begin{pmatrix} |\varepsilon_1 D^+(V_1 - v_1)(d)| \\ |\varepsilon_2 D^+(V_2 - v_2)(d)| \end{pmatrix} \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

Therefore, $|D^+ \boldsymbol{V}(d)| \le \begin{pmatrix} C(1+\varepsilon_1^{-1}N^{-1}) \\ C(1+\varepsilon_2^{-1}N^{-1}) \end{pmatrix}$.
On Ω_1, $|\boldsymbol{W}_1(x_i)| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}$ implies that $|D^- \boldsymbol{W}_1(d)| \le \begin{pmatrix} C \\ C \end{pmatrix}$. On Ω_2, $D^+ \boldsymbol{W}_1(d) = D^+(\boldsymbol{W}_1 - \boldsymbol{w}_1)(d) + D^+\boldsymbol{w}_1(d)$. From lemma 2.4 $\|\boldsymbol{w}_1'\| \le \begin{pmatrix} C \\ C \end{pmatrix}$ and $|D^+(\boldsymbol{W}_1 - \boldsymbol{w}_1)(x_i)| \le \begin{pmatrix} C \\ C \end{pmatrix}$.
Therefore,

$$|[D\,\boldsymbol{W}_2](d)| \le \begin{pmatrix} C(1+\varepsilon_1^{-1}N^{-1}) \\ C(1+\varepsilon_2^{-1}N^{-1}) \end{pmatrix}.$$

□

Lemma 4.2. *The following $\varepsilon_1, \varepsilon_2-$ uniform bound*

$$|\boldsymbol{W}_2(x_i)| \le \begin{pmatrix} C\varepsilon_1 |[DW_{21}](d)| \\ C\varepsilon_2 |[DW_{22}](d)| \end{pmatrix}$$

is valid, where $\boldsymbol{W}_2$ is the solution of (14) – (15).

Proof. Consider the following function $\phi_j^{\pm}, j = 1, 2$ where

$$\phi_j^{\pm}(x_i) := \tfrac{C\varepsilon_j |[DW_{2j}](d)|}{\alpha} \begin{cases} 1, & x_i \le d \\ \psi_j(x_i), & x_i \ge d \end{cases} \pm W_{2j}(x_i)$$

where $\Psi = (\psi_1, \psi_2)^T$ is the solution of

$$-\varepsilon_j \delta^2 \psi_j(x_i) - \alpha D^+ \psi_j(x_i) = 0, \; x_i \in \Omega^N \cap \Omega_2^N,$$

$$\psi_j(d) = 1, \; \psi_j(1) = 0 \text{ and } D^+\psi_j(x_i) < 0, \; x_i \ge d.$$

Using the discrete maximum principle we get the required result. □

Lemma 4.3. *The error of the singular component satisfies the estimate*

$$|(\boldsymbol{W}_2 - \boldsymbol{w}_2)(x_i)| \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Proof.

$$\begin{aligned}[D(\boldsymbol{W}_2-\boldsymbol{w}_2)](d) &= [D\boldsymbol{W}_2](d)-[D\boldsymbol{w}_2](d)\\ &= [\boldsymbol{v}'](d)-[D\boldsymbol{V}](d)+[\boldsymbol{w}_2'](d)-[D\boldsymbol{w}_2](d)-[D\boldsymbol{W}_1](d).\end{aligned}$$

Now

$[\boldsymbol{v}'](d)-[D\boldsymbol{V}](d)=\boldsymbol{v}'(d^+)-D^+\boldsymbol{v}(d)-\boldsymbol{v}'(d^-)+D^-\boldsymbol{v}(d)+[D(\boldsymbol{v}-\boldsymbol{V})](d).$

From lemma 4.1 we have

$$|[D(\boldsymbol{V}-\boldsymbol{v})](d)|\le\begin{pmatrix}C\varepsilon_1^{-1}N^{-1}\\ C\varepsilon_2^{-1}N^{-1}\end{pmatrix}$$

and

$$|[\boldsymbol{v}'](d)-[D\boldsymbol{v}](d)|\le\begin{pmatrix}CN^{-1}\\ CN^{-1}\end{pmatrix}.$$

Hence

$$|[\boldsymbol{v}'](d)-[D\boldsymbol{V}](d)|\le\begin{pmatrix}C\varepsilon_1^{-1}N^{-1}\\ C\varepsilon_2^{-1}N^{-1}\end{pmatrix}.$$

Likewise,

$$\begin{aligned}|[\boldsymbol{w}_2'](d)-[D\boldsymbol{w}_2](d)| &\le |D^+w_2(d)-w_2'(d+)|+|D^-w_2(d)-w_2'(d-)|\\ &\le\begin{pmatrix}Ch_{\varepsilon_{r_1}}|w_{21}^{(2)}(d+)|+CH_1|w_{21}^{(2)}(d-)|\\ Ch_{\varepsilon_{r_1}}|w_{22}^{(2)}(d+)|+CH_1|w_{22}^{(2)}(d-)|\end{pmatrix}.\end{aligned}$$

Using the bounds on derivatives of $\boldsymbol{w}_2$ given in lemma 2.5, we have

$$|[\boldsymbol{w}_2'](d)-[D\boldsymbol{w}_2](d)|\le\begin{pmatrix}CN^{-1}\ln N\\ CN^{-1}\ln N\end{pmatrix}.$$

$$|[D\boldsymbol{w}_1](d)|\le\begin{pmatrix}C(h_{\varepsilon_{r_1}}+H_1)|w_{11}^{(2)}(d-H_1)|\\ C(h_{\varepsilon_{r_1}}+H_1)|w_{12}^{(2)}(d-H_1)|\end{pmatrix}.$$

Using the bounds on derivatives of $\boldsymbol{w}_1$ given in lemma 2.4, we have

$$|[D\boldsymbol{w}_1](d)|\le\begin{pmatrix}CN^{-1}\ln N\\ CN^{-1}\ln N\end{pmatrix}.$$

Also,

$$|[D(\boldsymbol{W}_1-\boldsymbol{w}_1)](d)|\le\begin{pmatrix}C\frac{N^{-1}\ln N}{\varepsilon_1}\\ C\frac{N^{-1}\ln N}{\varepsilon_2}\end{pmatrix}.$$

Collecting all the previous inequalities we get that

$$|[D(\boldsymbol{W}_2-\boldsymbol{w}_2)](d)|\le\begin{pmatrix}C\frac{N^{-1}\ln N}{\varepsilon_1}\\ C\frac{N^{-1}\ln N}{\varepsilon_2}\end{pmatrix}.$$

By the Taylor's expansion and bounds on the derivatives of $\boldsymbol{w}_2$, given in lemma 2.5 we have

Case (i) For $x_i \in [d+\sigma_{\varepsilon_{r_2}}, 1]$.

$$|\boldsymbol{L}^N(\boldsymbol{W}_2 - \boldsymbol{w}_2)(x_i)| \leq \begin{pmatrix} C\varepsilon_1\|w''_{21}\|_{(x_{i-1},x_{i+1})} + C\|w'_{21}\|_{[x_i,x_{i+1}]} \\ C\varepsilon_2\|w''_{22}\|_{(x_{i-1},x_{i+1})} + C\|w'_{22}\|_{[x_i,x_{i+1}]} \end{pmatrix} \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

Similar arguments prove a similar result for the sub-interval $[\sigma_{\varepsilon_{l_1}}, d)$.

Case (ii) For $x_i \in (0, \sigma_{\varepsilon_{l_1}})$.

$$\begin{aligned} &|(\boldsymbol{L}^N(\boldsymbol{W}_2 - \boldsymbol{w}_2))_1(x_i)| \\ \leq& C\int_{x_{i-1}}^{x_{i+1}} \varepsilon_1^{-1}\exp\Big(-\frac{\alpha t}{\varepsilon_1}\Big) + \varepsilon_1\varepsilon_2^{-2}\exp\Big(-\frac{\alpha t}{\varepsilon_2}\Big)dt \\ \leq& C\Big(\exp\Big(-\frac{\alpha x_{i-1}}{\varepsilon_1}\Big) - \exp\Big(-\frac{\alpha x_{i+1}}{\varepsilon_1}\Big) + \varepsilon_1\varepsilon_2^{-1}\Big(\exp\Big(-\frac{\alpha x_{i-1}}{\varepsilon_2}\Big) - \exp\Big(-\frac{\alpha x_{i+1}}{\varepsilon_2}\Big)\Big)\Big) \\ \leq& C\Big(\exp\Big(-\frac{\alpha x_i}{\varepsilon_1}\Big)\sinh\Big(\frac{\alpha h_{\varepsilon_{l_1}}}{\varepsilon_1}\Big) + \varepsilon_1\varepsilon_2^{-1}\exp\Big(-\frac{\alpha x_i}{\varepsilon_2}\Big)\sinh\Big(\frac{\alpha h_{\varepsilon_{l_1}}}{\varepsilon_2}\Big)\Big) \\ \leq& CN^{-1}\ln N, \text{ since } \sinh t \leq Ct \text{ for } 0 \leq t \leq 1. \end{aligned}$$

$$\begin{aligned} &|(\boldsymbol{L}^N(\boldsymbol{W}_2 - \boldsymbol{w}_2))_2(x_i)| \\ \leq& C\int_{x_{i-1}}^{x_{i+1}} \varepsilon_1^{-1}\exp\Big(-\frac{\alpha t}{\varepsilon_1}\Big) + \varepsilon_2^{-1}\exp\Big(-\frac{\alpha t}{\varepsilon_2}\Big)dt \\ \leq& C\Big(\exp\Big(-\frac{\alpha x_{i-1}}{\varepsilon_1}\Big) - \exp\Big(-\frac{\alpha x_{i+1}}{\varepsilon_1}\Big) + \exp\Big(-\frac{\alpha x_{i-1}}{\varepsilon_2}\Big) - \exp\Big(-\frac{\alpha x_{i+1}}{\varepsilon_2}\Big)\Big) \\ \leq& C\Big(\exp\Big(-\frac{\alpha x_i}{\varepsilon_1}\Big)\sinh\Big(\frac{\alpha h_{\varepsilon_{l_1}}}{\varepsilon_1}\Big) + \exp\Big(-\frac{\alpha x_i}{\varepsilon_2}\Big)\sinh\Big(\frac{\alpha h_{\varepsilon_{l_1}}}{\varepsilon_2}\Big)\Big) \\ \leq& CN^{-1}\ln N. \end{aligned}$$

Similar arguments prove a similar result for the sub-interval $(d, d+\sigma_{\varepsilon_{r_1}})$.

Case (iii) $[\sigma_{\varepsilon_{l_1}}, \sigma_{\varepsilon_{l_2}})$.

$$|(\boldsymbol{L}^N(\boldsymbol{W}_2 - \boldsymbol{w}_2))_1(x_i)| \leq C\int_{x_{i-1}}^{x_{i+1}} \varepsilon_1^{-1}\exp\Big(-\frac{\alpha t}{\varepsilon_1}\Big) + \varepsilon_1\varepsilon_2^{-2}\exp\Big(-\frac{\alpha t}{\varepsilon_2}\Big)dt.$$

Using the inequality

$$\varepsilon_1^{-1}\exp\Big(-\frac{\alpha t}{\varepsilon_1}\Big) \leq \varepsilon_2^{-1}\exp\Big(-\frac{\alpha t}{\varepsilon_2}\Big) \text{ for } t > \frac{2\varepsilon_1}{\alpha},$$

$$\begin{aligned} |(\boldsymbol{L}^N(\boldsymbol{W}_2 - \boldsymbol{w}_2))_1(x_i)| &\leq C\varepsilon_1\varepsilon_2^{-1}\Big(\exp\Big(-\frac{\alpha x_{i-1}}{\varepsilon_2}\Big) - \exp\Big(-\frac{\alpha x_{i+1}}{\varepsilon_2}\Big)\Big) \\ &\leq C\varepsilon_1\varepsilon_2^{-1}\exp\Big(-\frac{\alpha x_i}{\varepsilon_2}\Big)\sinh\Big(\frac{\alpha h_{\varepsilon_{l_2}}}{\varepsilon_2}\Big) \\ &\leq CN^{-1}\ln N, \text{ since } \sinh t \leq Ct \text{ for } 0 \leq t \leq 1. \end{aligned}$$

Likewise, $|(\boldsymbol{L}^N(\boldsymbol{W}_2 - \boldsymbol{w}_2))_2(x_i)| \leq CN^{-1}\ln N$.
Similar arguments prove a similar result for the sub-interval $[d+\sigma_{\varepsilon_{r_1}}, d+\sigma_{\varepsilon_{r_2}})$.
Combining all these gives,

$$|\boldsymbol{L}^N(\mathbf{W}_2 - \mathbf{w}_2)(x_i)| \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix},$$

$$|[D(\boldsymbol{W}_2 - \boldsymbol{w}_2)](d)| \leq \begin{pmatrix} C\frac{N^{-1}\ln N}{\varepsilon_1} \\ C\frac{N^{-1}\ln N}{\varepsilon_2} \end{pmatrix}.$$

Consider the following function $\phi_j^{\pm}, j = 1, 2$ where

$$\phi_j^{\pm}(x_i) := CN^{-1}\ln N \begin{cases} 1, & x_i \leq d \\ \psi_j(x_i), & x_i \geq d \end{cases} + CN^{-1}\ln N(1 - x_i)$$

where $\Psi = (\psi_1, \psi_2)^T$ is the solution of

$$-\varepsilon_j\delta^2\psi_j(x_i) - \alpha D^+\psi_j(x_i) = 0,\ x_i \in \Omega^N \cap \Omega_2^N,$$

$$\psi_j(d) = 1,\ \psi_j(1) = 0 \text{ and } D^+\psi_j(x_i) < 0,\ x_i \geq d.$$

Using the discrete maximum principle we get the required result. □

Theorem 4.4. *Let* $\boldsymbol{u}$ *be the solution of given problem* (1) − (2) *and* $\mathbf{U}$ *is the solution of discrete problem on the Shishkin mesh defined in section 3, then*

$$\|\boldsymbol{U} - \boldsymbol{u}\|_{\overline{\Omega}^N} \leq CN^{-1}\ln N.$$

Proof. Using the equation (16), (17) and lemma 4.3 we get the required result. □

5. Numerical Results

To illustrate the theoretical results the scheme in Section 3 is implemented on these test examples.

Example 5.1 Consider the following singularly perturbed convection-diffusion problem with discontinuous source term:

$$-\varepsilon_1 u_1''(x) - 0.8u_1'(x) + 3u_1(x) - u_2(x) = f_1(x),\ x \in \Omega_1 \cup \Omega_2$$
$$-\varepsilon_2 u_2''(x) - u_2'(x) - u_1(x) + 3u_2(x) = f_2(x),\ x \in \Omega_1 \cup \Omega_2$$
$$u_1(0) = 0,\ u_1(1) = 2,\ u_2(0) = 0,\ u_2(1) = 2.$$

where

$$f_1(x) = \begin{cases} 2 & \text{for } 0 \leq x < 0.5 \\ -1 & \text{for } 0.5 \leq x \leq 1 \end{cases} \text{ and } f_2(x) = \begin{cases} 1.8 & \text{for } 0 \leq x < 0.5 \\ -0.8 & \text{for } 0.5 \leq x \leq 1. \end{cases}$$

For the construction of piecewise-uniform Shishkin mesh $\overline{\Omega}_N$, we take $\alpha = 0.8$. The Exact solution of the examples are not known. Therefore we estimate the error for $\boldsymbol{U}$ by comparing it to the numerical solution $\widetilde{\boldsymbol{U}}$ obtained on the mesh $\widetilde{x}_j$ that contains the mesh points of the original mesh and their midpoints, that is, $\widetilde{x}_{2j} = x_j$, j=0,...,N, $\widetilde{x}_{2j+1} = (x_j + x_{j+1})/2$, j=0,...,N-1.
For different values of N and $\varepsilon_1, \varepsilon_2$, we compute

$$D^N_{\varepsilon_1,\varepsilon_2} := \|(\boldsymbol{U} - \widetilde{\boldsymbol{U}})(x_j)\|_{\overline{\Omega}^N}.$$

TABLE 1. Maximum point-wise errors $D^N_{\varepsilon_1}$, D^N and $\varepsilon_1, \varepsilon_2$−uniform rate of convergence p^N for Example 5.1.

$\varepsilon_1 = 10^{-j}$	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048
$j = 0$	9.51E-04	4.84E-04	2.44E-04	1.22E-04	6.16E-05	3.08E-05	1.54E-05
$j = 1$	3.03E-02	1.56E-02	7.95E-03	4.01E-03	2.01E-03	1.00E-03	5.04E-04
$j = 2$	4.08E-02	2.82E-02	1.76E-02	1.06E-02	6.17E-03	3.49E-03	1.94E-03
$j = 3$	4.99E-02	3.33E-02	2.11E-02	1.25E-02	7.16E-03	3.99E-03	2.20E-03
$j = 4$	5.12E-02	3.35E-02	2.12E-02	1.25E-02	7.20E-03	4.08E-03	2.29E-03
$j = 5$	5.14E-02	3.35E-02	2.12E-02	1.25E-02	7.34E-03	4.21E-03	2.36E-03
$j = 6$	5.15E-02	3.35E-02	2.12E-02	1.26E-02	7.44E-03	4.27E-03	2.39E-03
$j = 7$	5.15E-02	3.35E-02	2.12E-02	1.26E-02	7.45E-03	4.28E-03	2.40E-03
$j = 8$	5.15E-02	3.35E-02	2.12E-02	1.26E-02	7.46E-03	4.28E-03	2.40E-03
D^N	5.15E-02	3.35E-02	2.12E-02	1.26E-02	7.46E-03	4.28E-03	2.40E-03
p^N	8.38E-01	8.52E-01	9.22E-01	9.17E-01	9.44E-01	9.68E-01	

If $\varepsilon_1 = 10^{-j}$ for some non-negative integer j , set

$$D^N_{\varepsilon_1} := \max\{D^N_{\varepsilon_1,1}, D^N_{\varepsilon_1,10^{-1}}, D^N_{\varepsilon_1,10^{-2}}, \ldots, D^N_{\varepsilon_1,10^{-j}}\}.$$

Then the parameter-uniform error is computed as $D^N := \max\{D^N_1, D^N_{10^{-1}}, \ldots, D^N_{10^{-8}}\}$, and the order of convergence is calculated using the formula $p^N := \dfrac{\ln D^N - \ln D^{2N}}{\ln(2\ln N) - \ln(\ln(2N))}$.

Finally, we want to show that similar results can be obtained for coupled system of $M(> 2)$ singularly perturbed convection diffusion problem with discontinuous source term. Letting $N = 2^M \times \tau$, where τ is some positive power of 2, the mesh is defined using the following transition points

$$\sigma_{\varepsilon_{l_M}} := \min\left\{\frac{d}{2}, \frac{\varepsilon_M}{\alpha}\ln N\right\}, \quad \sigma_{\varepsilon_{r_M}} := \min\left\{\frac{(1-d)}{2}, \frac{\varepsilon_M}{\alpha}\ln N\right\},$$

$$\sigma_{\varepsilon_{l_k}} := \min\left\{\frac{\sigma_{\varepsilon_{l_{k+1}}}}{2}, \frac{\varepsilon_k}{\alpha}\ln N\right\},$$

$$\sigma_{\varepsilon_{r_k}} := \min\left\{\frac{\sigma_{\varepsilon_{r_{k+1}}}}{2}, \frac{\varepsilon_k}{\alpha}\ln N\right\}, \quad k = M-1, \ldots, 1.$$

Then we divide the interval $[0, d]$ into M+1 subintervals $[0, \sigma_{\varepsilon_{l_1}}], [\sigma_{\varepsilon_{l_1}}, \sigma_{\varepsilon_{l_2}}], \ldots,$ $[\sigma_{\varepsilon_{l_{M-1}}}, \sigma_{\varepsilon_{l_M}}], [\sigma_{\varepsilon_{l_M}}, d]$. On the subinterval $[0, \sigma_{\varepsilon_{l_1}}]$ a uniform mesh of $N/2^{M+1}$ mesh intervals, on $[\sigma_{\varepsilon_{l_k}}, \sigma_{\varepsilon_{l_{k+1}}}], 1 \le k \le M-1$, a uniform mesh of $N/2^{M-k+2}$ mesh intervals, and on $[\sigma_{\varepsilon_{l_M}}, d]$ a uniform mesh of N/4 mesh intervals are placed. Similarly, we divide the interval $[d, 1]$ into subintervals $[d, d+\sigma_{\varepsilon_{r_1}}], [d+\sigma_{\varepsilon_{r_1}}, d+\sigma_{\varepsilon_{r_2}}], \ldots, [d+\sigma_{\varepsilon_{r_{M-1}}}, d+\sigma_{\varepsilon_{r_M}}], [d+\sigma_{\varepsilon_{r_M}}, 1]$. On the subinterval $[d, d+\sigma_{\varepsilon_{r_1}}]$ a uniform mesh of $N/2^{M+1}$ mesh intervals, on $[d+\sigma_{\varepsilon_{r_k}}, d+\sigma_{\varepsilon_{r_{k+1}}}], 1 \le k \le M-1$, a uniform mesh of $N/2^{M-k+2}$ mesh intervals, and on $[d+\sigma_{\varepsilon_{r_M}}, 1]$ a uniform mesh of $N/4$ mesh intervals are placed. Let $h_{\varepsilon_{l_1}}$ and $h_{\varepsilon_{r_1}}$ be the mesh lengths on $[0, \sigma_{\varepsilon_{l_1}}]$ and on $[d, d+\sigma_{\varepsilon_{r_1}}]$ respectively. Let H_1 and H_2 be the mesh lengths on $[\sigma_{\varepsilon_{l_M}}, d]$ and on $[d+\sigma_{\varepsilon_{r_M}}, 1]$ respectively; $h_{\varepsilon_{l_k}}$ and $h_{\varepsilon_{r_k}}$ be the mesh lengths on $[\sigma_{\varepsilon_{l_k}}, \sigma_{\varepsilon_{l_{k+1}}}]$ and on $[d+\sigma_{\varepsilon_{r_k}}, d+\sigma_{\varepsilon_{r_{k+1}}}], k = 2, \ldots, M$ respectively.

In this case also, we obtain the scheme similar to (3.1), with $\boldsymbol{u} = (u_1, u_2, \ldots, u_M)^T \in C^0(\overline{\Omega})^M \cap C^1(\Omega)^M \cap C^2(\Omega_1 \cup \Omega_2)^M$ and also expect the error bound $\| \boldsymbol{U} - \boldsymbol{u} \|_{\overline{\Omega}^N} \leq C(N^{-1} \ln N)$ to hold, although attempts of the authors have failed so far to provide a proof. To illustrate the order of uniform convergence of this method we consider the following test example.

Example 5.2 Consider the following singularly perturbed convection-diffusion problem with discontinuous source term:

$$
\begin{array}{ll}
-\varepsilon_1 u_1''(x) - (2x+1)u_1'(x) + 3xu_1 - xu_2(x) - xu_3(x) = f_1(x), & x \in \Omega_1 \cup \Omega_2 \\
-\varepsilon_2 u_2''(x) - 3u_2'(x) - u_1(x) + 4u_2(x) - u_3(x) = f_2(x), & x \in \Omega_1 \cup \Omega_2 \\
-\varepsilon_3 u_3''(x) - (2-x)u_3'(x) - x^2 u_1(x) + (1+x)u_3(x) = f_3(x), & x \in \Omega_1 \cup \Omega_2
\end{array}
$$

$$\boldsymbol{u}(0) = \boldsymbol{0}, \ \boldsymbol{u}(1) = \boldsymbol{0},$$

where $f_1(x) = \begin{cases} \exp(x) & \text{for } 0 \leq x \leq 0.5 \\ 1 & \text{for } 0.5 < x \leq 1, \end{cases}$ $f_2(x) = \begin{cases} \cos(x) & \text{for } 0 \leq x \leq 0.5 \\ 4 & \text{for } 0.5 < x \leq 1 \end{cases}$

and

$$f_3(x) = \begin{cases} \sinh(x) & \text{for } 0 \leq x \leq 0.5 \\ 2 & \text{for } 0.5 < x \leq 1. \end{cases}$$

TABLE 2. Maximum point-wise errors $D^N_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$, D^N with $\varepsilon_2 = 10^{-4}$, $\varepsilon_3 = 10^{-1}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$−uniform rate of convergence $p^N$for Example 5.2.

$\varepsilon_1 = 10^{-j}$	N=64	N=128	N=256	N=512	N=1024	N=2048
4	3.95E-02	2.67E-02	1.86E-02	1.17E-02	6.97E-03	3.99E-03
5	4.29E-02	2.93E-02	1.81E-02	1.08E-02	6.27E-03	3.53E-03
6	4.52E-02	3.89E-02	2.83E-02	1.92E-02	1.21E-02	7.21E-03
7	4.76E-02	4.11E-02	2.92E-02	2.02E-02	1.26E-02	7.52E-03
8	4.79E-02	4.14E-02	2.93E-02	2.03E-02	1.26E-02	7.55E-03
9	4.79E-02	4.14E-02	2.93E-02	2.03E-02	1.26E-02	7.55E-03
10	4.79E-02	4.14E-02	2.93E-02	2.03E-02	1.26E-02	7.58E-03
D^N	4.79E-02	4.14E-02	2.93E-02	2.03E-02	1.26E-02	7.58E-03
p^N	2.70E-01	6.18E-01	6.38E-01	8.11E-01	8.50E-01	

REFERENCES

1. J.J.H. Miller, E.O. Riordan and G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions*, World Scientific, Singapore, 1996.
2. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E.O. Riordan and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman & Hall, CRC Press, Boca Raton, Florida, 2000.
3. H.G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer, Berlin, 1996.

4. H.G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Springer, Berlin, 2008.
5. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E.O Riordan and G.I. Shishkin, *Singularly perturbed convection-diffusion problems with boundary and weak interior layers*, J. Comput. Appl. Math. **166** (2004), 133-151.
6. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E.O. Riordan, and G.I. Shishkin, *Global Maximum Norm Parameter-Uniform Numerical Method for a Singularly Perturbed Convection-Diffusion Problem with Discontinuous Convection Coefficient*, Math. Comput. Modelling **40** (2004), 1375-1392.
7. Z. Cen, *Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations*, Intern. J. Comput. Math. **82** (2005), 177-192.
8. T. Linss, *Analysis of a system of singularly perturbed convection-diffusion equations with strong coupling*, SIAM J. Numer. Anal. **47** (2009), 1847-1862.
9. C. Clavero, J.L. Gracia and F. Lisbona, *High order methods on Shishkin meshes for singular perturbation problems of convection-diffusion type*, Numer. Algorithms **22** (1999), 73-97.
10. E.O. Riordan and M. Stynes, *Numerical analysis of a strongly coupled system of two singularly preturbed convection-diffusion problems*, Adv. Comput. Math. **30** (2009), 101-121.
11. S. Bellew and E.O. Riordan, *A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations*, Appl. Numer. Math. **51** (2004), 171-186.
12. J.B. Munyakazi, *A uniformly convergent nonstandard finite difference scheme for a system of convection-diffusion equations*, Comp. Appl. Math. doi:10.1007/s40314-014-0171-6, (2014).
13. R.M. Priyadharshini and N. Ramanaujam, *Uniformly-convergent numerical methods for a system of coupled singularly perturbed convection-diffusion equations with mixed type boundary conditions*, Math. Model. Anal. **18** (2013), 577-598.
14. A. Tamilselvan, N. Ramanujam, R.M. Priyadharshini and T. Valanarasu, *Parameter - uniform numerical method for a system of coupled singularly perturbed convection-diffusion equations with mixed type boundary conditions*, J. Appl. Math. Informatics **28** (2010), 109-130.
15. A. Tamilselvan and N. Ramanujam, *A numerical method for singularly perturbed system of second order ordinary differential equations of convection diffusion type with a discontinuous source term*, J. Appl. Math. Informatics **27** (2009), 1279-1292.

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