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UPPER AND LOWER BOUNDS FOR THE POWER OF EIGENVALUES IN SEIDEL MATRIX

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Abstract. In this paper, we generalize the concept of the energy of Seidel matrix $S(G)$ which denoted by $S^{\alpha}(G)$ and obtain some results related to this matrix. Also, we obtain an upper and lower bound for $S^{\alpha}(G)$ related to all of graphs with $|detS(G)| \geq (n-1), n \geq 3$.

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1. Introduction

All of graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set V(G) and edge set E(G). Let A(G) be the (0,1)-adjacency matrix of G. In 1966, Van Lint and Seidel in [13] introduced a symmetric (0,-1,1)-adjacency matrix for a graph G, called the **Seidel matrix** of G as $S(G) = J - 2A(G) - I$, where J is a square matrix which all of entries are equal to 1. Thus S(G) has 0 on the diagonal and ± 1 off diagonal, where -1 indicates adjacency, unless is equal to 1. It is obvious that -S(G) is the Seidel matrix of the complement of G. Haemers in [10], similar to the normal energy, defined the Seidel energy $E_s(G)$ of G which is the sum of the absolute values of the eigenvalues of the Seidel matrix. For example consider the complete graph K_n, its Seidel matrix is $I - J$. Hence the eigenvalues of $S(K_n)$ are (1-n) and 1 with multiplicity (n-1). So $E_s(K_n) = 2n - 2$. The Seidel matrix of a graph can be interpreted as the incidence matrix of a design, or as the generator matrix of an alternative binary code. We refer the reader to [5, 6, 9] for more information related to eigenvalue and adjacency matrix and their properties.
In section 2, we proceed with the study of generalization of Seidel matrix and define $S^{\alpha}(G)$ and we obtain a lower bound for $S^{\alpha}(G)$, $\alpha \geq 2$.

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Section 3 contains a brief summary of KKT method and establishes the relation between nonlinear programming and upper bound.
In [7], Ghorbani obtained a lower bound for $S^{\alpha}(G)$, $0 \leq \alpha \leq 2$. In this section we obtain an upper bound for $S^{\alpha}(G)$, $\alpha \geq 2$. As for prerequisites, the reader is expected to be familiar with nonlinear programming. Undefined notations and terminology from nonlinear programming, can be found in [2, 12].

2. The generalization of Seidel matrix

At first, we define the concept of generalized Seidel matrix and then we obtain some results related to this concept.

Definition 2.1. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the Seidel matrix S(G). The **power of eigenvalues** of Seidel matrix S(G) is denoted by $S^{\alpha}(G)$ and define as follows:

$$S^{\alpha}(G) = \sum_{i=1}^{n} |\lambda_i|^{\alpha}.$$

Remark 2.1. *If $\alpha = 1$, then $S^{\alpha}(G) = E_s(G)$; i.e., $S^{\alpha}(G)$ is a generalization of Seidel energy of G.*

For the proof of the next theorem, we need the concept of conference graph.

Definition 2.2 (Conference matrix [10])**.** A conference matrix is a square matrix C of order n with zero diagonal and ± 1 off diagonal, such that $CC^T = (n-1)I$. If C is symmetric, then C is the Seidel matrix of a graph and this graph is called a **conference graph**.

Conference matrices are a class of Hadamard matrices and its have the Hadamard properties. For more details we refer the reader to [3, 10]. The remainder of this section will be devoted to the proof of a lower bound for $S^{\alpha}(G)$ for all $\alpha \geq 2$.

Definition 2.3 (Hölder's Inequalities [8])**.** Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Then Hölder's inequality for the n-dimensional Euclidean space, when the set S is $\{1, ..., n\}$ with the counting measure, we have

$$\sum_{k=1}^{n} |a_k b_k| \leq (\sum_{k=1}^{n} |a_k|^p)^{1/p} (\sum_{k=1}^{n} |b_k|^q)^{1/q}$$

for all $X = (x_1, x_2, ..., x_n)$, $Y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ with equality when $|b_k| = c|a_k|^{p-1}$. If $p = q = 2$, this inequality becomes Cauchy's inequality.

Theorem 2.4. *Let G be a graph with n vertices, then for all $\alpha \geq 2$, $n \geq 3$, $S^{\alpha}(G) \geq n(\sqrt{n-1})^{\alpha}$ and equality holds if and only if G is a conference graph.*

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the Seidel matrix S(G), then the trace of $S^2(G)$ is equal $\sum_{i=1}^{n} \lambda_i^2 = n(n-1)$. Let $X = (\lambda_1^2, \lambda_2^2, ..., \lambda_n^2)$ and $Y = (1, 1, ..., 1)$. By the Hölder's Inequalities, we have $|X^T Y| \leq ||X||_p ||Y||_q$, $\frac{1}{p} + \frac{1}{q} = 1$; i.e., $\sum_{i=1}^{n} \lambda_i^2 \leq (\sum_{i=1}^{n} (\lambda_i^2)^p)^{1/p} (\sum_{i=1}^{n} 1^q)^{1/q}$. But since $\sum_{i=1}^{n} \lambda_i^2 = n(n-1)$, we have $n(n-1) \leq (\sum \lambda_i^{2p})^{1/p} n^{1-\frac{1}{p}}$, and hence $(\sum_{i=1}^{n} \lambda^{2p})^{\frac{1}{p}} \geq \frac{n(n-1)}{n^{1-\frac{1}{p}}} = n^{\frac{1}{p}}(n-1)$, therefore $\sum_{i=1}^{n} \lambda_i^{2p} \geq n(n-1)^p$. We assume that $2p = \alpha$, then we get $S^{\alpha}(G) = \sum |\lambda_i|^{\alpha} \geq n(\sqrt{n-1})^{\alpha}$ with equality if and only if $|\lambda_i| = \sqrt{n-1}$ for i = 1, ..., n. Moreover, if each eigenvalue equal to $\pm\sqrt{n-1}$, then $S(G)S^T(G) = S^2(G) = (n-1)I$, which means that the Seidel matrix S(G) is a symmetric and hence the graph G is a conference graph. □

3. Computation of upper bound of $S^{\alpha}(G)$ by using KKT method

In nonlinear programming, the Karush-Kuhn-Tucker (KKT) conditions are **necessary** for a local solution to a maximization problem provided that some regularity conditions are satisfied. Allowing inequality constraints, the KKT approach to nonlinear programming generalizes the method of Lagrange multipliers, which allows only equality constraints. This construction is adapted from [1, 2, 12]. We consider the following nonlinear optimization problem:

$$\begin{aligned} &Maximize\ f(X) \\ &subject\ to: \\ &\qquad g_i(X) \leq 0,\ i \in I \\ &\qquad h_j(X) = 0,\ j \in J \\ &\qquad X \in \mathbb{R}^n \end{aligned} \tag{1}$$

where I and J are finite sets of indices. Suppose that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the constraint functions $g_i : \mathbb{R}^n \to \mathbb{R},\ i \in I$ and $h_j : \mathbb{R}^n \to \mathbb{R},\ j \in J$ are continuously differentiable at a point X^*.

Definition 3.1. Let X^* be a point satisfying the constraints:

$$h_j(X^*) = 0\ ,\ g_i(X^*) \leq 0\ ,\ j \in J\ ,\ i \in I \tag{2}$$

and let I_* be the set of indices i for which $g_i(X^*) = 0$. Then X^* is said to be a **regular point** of the constraints (3.2), if the gradient vectors $\nabla h_j\ ,\ j \in J\ ,\ \nabla g_i(X^*),\ i \in I_*$ are linearly independent.

Definition 3.2 (Karush-Kuhn-Tucker Conditions(KKT) [1, 12])**.** Let X^* be a relative maximum point for the problem (3.1) and suppose X^* is a regular point for the constraints, then there exist constants $\mu_j\ (j \in J),\ \lambda_i\ (i \in I)$, which these constants are called KKT multipliers, such that the following conditions are hold:
Stationarity

$$\nabla f(X^*) + \sum_{j \in J} \mu_j \nabla h_j(X^*) - \sum_{i \in I} \lambda_i \nabla g_i(X^*) = 0 \tag{3}$$

Primal feasibility

$$\begin{aligned} g_i(X^*) \leq 0,\ for\ all\ i \in\ I \\ h_j(X^*) = 0,\ for\ all\ j \in\ J \end{aligned}$$

Dual feasibility

$$\lambda_i \geq 0, \ for\ all\ i \in \ I$$

Complementary slackness

$$\lambda_i g_i(X^*) = 0$$

In the particular case, set I is empty, i.e., when there are no inequality constraints, the KKT conditions turn into the Lagrange conditions, and the KKT multipliers are called Lagrange multipliers.

Definition 3.3 (Linear Independence Constraint Qualification [2, 12])**.** Given the point X^* is feasible and the active set $I_* = \{i|g_i(X^*) = 0, i \in I\}$ defined in (3.2), we say that the linear independence constraint qualification **(LICQ)** holds if the set of active constraint gradients $\{\nabla g_i(X^*), i \in I_*)\}$ is linearly independent.

Theorem 3.4 (LICQ and Multipliers [12])**.** *Given a point X^*, that satisfies the KKT conditions, along with an active set $A(X^*) \equiv I_* \bigcup J$ with multipliers $\mu_j^*,\ \lambda_i^*$, if LICQ holds at X^*, then the multipliers are unique.*

Definition 3.5 (Mangasarian-Fromovitz Constraint Qualification [11])**.** Given X^* is a local solution of (3.1), and active set is $A(X^*)$. The Mangasarian-Fromovitz Constraint Qualification **MFCQ** is defined by linear independence of the equality constraint gradients and the existence of a search direction **d** such that $\nabla g_i(X^*)^T d < 0, \nabla h_j(X^*)^T d = 0$, for all $i,\ j$ in $A(X^*)$.

The MFCQ is always satisfied if the LICQ is satisfied. Also, satisfaction of the MFCQ leads to bounded multipliers, μ^*, λ^*, although they are not necessarily unique.

Theorem 3.6 ([11])**.** *If a local maximum X^* of the function $f(X)$ subject to the constraints $g_i(X^*) = 0,\ i \in I_*,\ h_j(X^*) = 0,\ j \in J$, satisfies MFCQ, then it satisfies the KKT conditions.*

Now we continue by recalling the relevant upper bound of eigenvalue Seidel matrix. Let $G = (V, E)$ be a simple, undirected graph on vertex set $V = \{v_1, ..., v_n\}$ and let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the set of all eigenvalues of $S(G)$. We formulate this as an optimization problem. For doing this case, we need to come up with appropriate constraints. The following assumption will be needed throughout the paper. The main constraint is made by the assumption $|detS(G)| \geq n-1$. The other ones are obtained by the following straightforward lemma.

Lemma 3.7. *For any graph G with n vertices, we have:*

(i) $S^2(G) = \sum_{i=1}^n \lambda_i^2 = (n-1)^2 + (n-1)$

(ii) $S^4(G) = \sum_{i=1}^n \lambda_i^4 \leq (n-1)^4 + (n-1)$

(iii) $S^4(G) = \sum_{i=1}^{n} \lambda_i^4 \geq n(n-1)^2$

(iv) $Max\lambda_i^2 \leq (n-1)^2$

Lemma 3.8 ([4]). *Suppose α, β, ν, ω, a, b, c, d are positive numbers and that*

$$\alpha + \beta = \nu + \omega,\ a\alpha + b\beta = c\nu + d\omega,\ max\{a, b\} \leq max\{c, d\},\ a^{\alpha} b^{\beta} \geq c^{\nu} d^{\omega}.$$

Then the inequality $\alpha a^p + \beta b^p \leq \nu c^p + \omega d^p$ holds for $p \geq 1$.

The reminder of this section will be devoted to the proof of Theorem 3.9.

Theorem 3.9. *Let G be a graph with $n \geq 3$ vertices and let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of S(G). If $|detS(G)| \geq n-1$, then the following condition is hold:*

$$for\ all\ \alpha \geq 2,\ S^{\alpha}(G) \leq (n-1)^{\alpha} + (n-1).$$

Proof. We prove this theorem by using of **KKT** method in nonlinear programming. Now, we can describe our problem as the maximization of the function $f(X)$ with assume $|\lambda_i|^2 = x_i$. Hence, we have:

$$Max\ f(X) := x_1^p + x_2^p + ... + x_n^p,\ p \geq 1$$

$$s.t.$$

$$g(X) = x_1 + x_2 + ... + x_n - n(n-1) = 0 \quad (4)$$

$$h(X) = x_1^2 + x_2^2 + ... + x_n^2 - (n-1)^4 - (n-1) \leq 0 \quad (5)$$

$$k(X) = n(n-1)^2 - (x_1^2 + x_2^2 + ... + x_n^2) \leq 0 \quad (6)$$

$$l(X) = (n-1)^2 - \prod_{i=1}^{n} x_i \leq 0 \quad (7)$$

$$m_i(X) = x_i - (n-1)^2 \leq 0\ ,\ for\ i = 1, 2, ..., n \quad (8)$$

$$n_i(X) = \delta - x_i \leq 0\ ,\ for\ i = 1, 2, ..., n \quad (9)$$

Since $|detS(G)| \geq n-1$, we have $\lambda_i > 0, 1 \leq i \leq n$ and hence δ must be non zero and non-negative, as a fixed number. Also, for all i, we must have $x_i \neq \delta$, because, if for some i, $x_i = \delta$, then $\prod_{i=1}^{n} x_i < (n-1)^2$ which is contradiction with (3.7). In continue, we need prove the following claim:

Claim 3.10. Let λ be a local maximum of $f(X)$ according to the constraints (3.4)-(3.9). Then λ satisfies MFCQ.

Proof: Let $\lambda = (\lambda_1, ..., \lambda_n)$. Without loss of generality, we can assume $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. If $\lambda_1 = \lambda_n$, then in view of (3.4), all of λ_i are equal to $n-1$. In this case, in the above formulas (3.5) till (3.9), we have the equality only in (3.6) and hence λ is not a local maximum where $f(\lambda) = n(n-1)^p < (n-1)^{2p} + (n-1),\ p > 1,\ n \geq 3$, and therefore λ is not satisfies MFCQ. If $\lambda_1 > \lambda_n$, then MFCQ is fulfilled by setting $d = (1, 0, ..., 0, -1)$. □

Now, we continue the proof of Theorem 3.9. We show that the maximum value of $f(X)$ according to conditions (3.4)-(3.9) is equal to $(n-1)^{2p} + (n-1)$.

So assume that $X = (x_1, x_2, ..., x_n)$ is a local maximum of $f(X)$ subject to the constraints (3.4) - (3.9). With no loss of generality suppose that $x_1 \geq x_2 \geq ... \geq x_n$. By Theorem (3.6) and Lemma (3.7), X satisfies KKT conditions, namely:

$$\nabla f(X) + \mu_1 \nabla g(X) - \mu_2 \nabla h(X) - \mu_3 \nabla k(X) - \mu_4 \nabla l(X) \qquad (10)$$

$$- \sum_{i=1}^{n} (\rho_i \nabla m_i(X) + \gamma_i \nabla n_i(X)) = 0$$

$$x_1 + x_2 + ... + x_n - n(n-1) = 0 \qquad (11)$$

$$\mu_2 \geq 0,\ \mu_2 h(X) = 0,\ \mu_3 \geq 0,\ \mu_3 k(X) = 0,\ \mu_4 \geq 0,\ \mu_4 l(X) = 0 \qquad (12)$$

$$\rho_i \geq 0,\ \rho_i m_i(X) = 0,\ i = 1, 2, ..., n \qquad (13)$$

$$\gamma_i \geq 0,\ \gamma_i n_i(X) = 0,\ i = 1, 2, ..., n \qquad (14)$$

By the choice of δ, we have $n_i(X) < 0$ for $i = 1, 2, ..., n$ and hence by (3.14), $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$. We assume that $D = \prod_{i=1}^{n} x_i$, then (3.10) can be written as

$$px_i^{p-1} + \mu_1 - 2\mu_2 x_i + 2\mu_3 x_i + \frac{\mu_4 D}{x_i} - \rho_i = 0 \ for\ i = 1, 2, ..., n \qquad (15)$$

We consider the following cases:

Case 1:

Let $x_1 = (n-1)^2$. Then by (3.11) and since X satisfies (3.7), we have

$$1 = \frac{x_2 + ... + x_n}{n-1} \geq (x_2 x_3 ... x_n)^{\frac{1}{n-1}} \geq 1. \qquad (16)$$

It turns out that $x_2 = x_3 = ... = x_n = 1$ and we have $f(X) = (n-1)^{2p} + (n-1)$.

Case 2:

Let $x_1 < (n-1)^2$. So, by (3.13), $\rho_1 = \rho_2 = ... = \rho_n = 0$. It turns out that $x_1, x_2, ..., x_n$ must satisfy the following equation:

$$px^p = -\mu_4 D - \mu_1 x + 2(\mu_2 - \mu_3)x^2 \qquad (17)$$

Assume that $\mu = \mu_2 - \mu_3$, then we have:

$$px^p = -\mu_4 D - \mu_1 x + 2\mu x^2 \qquad (18)$$

The curves of $y = px^p$ and Parabolic curve $y = -\mu_4 D - \mu_1 x - 2\mu x^2$ intersect in at most two points, i.e., the formula (3.17) at most two distinct positive roots. Now, we have two subcases:

Subcase(i): We have one positive root. Then by (3.11), $x_1 = x_2 = ... = x_n = n-1$. Hence $f(X) = n(n-1)^p$ which is smaller than $(n-1)^{2p} + (n-1)$, for $n > 3,\ p > 1$.

Subcase(ii): If (3.18) has two positive roots, say a and b, then by Lemma 3.8, we assume that $c = (n-1)^2$ and $d = 1$, we have $f(X) \leq (n-1)^{2p} + (n-1)$, which is the desired conclusion and the proof is completed. □

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