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PRIME FILTERS OF COMMUTATIVE BE-ALGEBRAS

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ABSTRACT. Properties of prime filters are studied in BE-algebras as well as in commutative BE-algebras. An equivalent condition is derived for a BE-algebra to become a totally ordered set. A condition **L** is introduced in a commutative BE-algebra in ordered to study some more properties of prime filters in commutative BE-algebras. A set of equivalent conditions is derived for a commutative BE-algebra to become a chain. Some topological properties of the space of all prime filters of BE-algebras are studied.

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1. Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [5]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [4]. Some properties of filters of BE-algebras were studied by S.S. Ahn and K.S. So in [1]. In [6, 7], the notion of normal filters is introduced in BE-algebras. In [2, 3], S.S. Ahn and K.S. So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. Also they generalized the notion of upper sets in BE-algebras, and discussed some properties of the characterizations of generalized upper sets related to the structure of ideals in transitive and self-distributive BE-algebras. Recently in 2012, S.S. Ahn, Y.H. Kim and J.M. Ko [1] introduced the notion of a terminal section of BE-algebras and derived some characterizations of commutative BE-algebras in terms of lattice ordered relations and terminal sections.

In this paper, the notion of prime filters is introduced in BE-algebras. Some properties of prime filters and maximal filters are then studied. An equivalent condition is derived, in terms of prime filters, for the class of all filters of a BE-algebra to become a totally ordered set. Properties of prime filters are also

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studied in commutative BE-algebras. A condition **L** is introduced to study some properties of prime filters of BE-algebras. Prime filters of a commutative BE-algebra are characterized. A set of equivalent conditions is derived for a commutative BE-algebra to become a chain. Some topological properties of the space of all prime filters of a BE-algebra are studied. An equivalent condition is derived for every prime filter of a BE-algebra to become a maximal filter.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [5] and [7] for the ready reference of the reader.

Definition 2.1 ([5])**.** An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:
(1) $x * x = 1$ (2) $x * 1 = 1$
(3) $1 * x = x$ (4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$

Theorem 2.2 ([5])**.** *Let $(X, *, 1)$ be a BE-algebra. Then we have the following:*

$$(1)\ x * (y * x) = 1 \qquad (2)\ x * ((x * y) * y)) = 1$$

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$.

Definition 2.3 ([1])**.** A BE-algebra $(X, *, 1)$ is said to transitive if for all $x, y, z \in X$, it satisfies $y * z \leq (x * y) * (x * z)$.

Definition 2.4 ([1])**.** Let $(X, *, 1)$ be a BE-algebra. A non-empty subset F of X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:
(a) $1 \in F$
(b) $x \in F$ and $x * y \in F$ imply that $y \in F$

Definition 2.5 ([5])**.** Let $(X_1, *, 1)$ and $(X_2, \circ, 1')$ be two BE-algebras. Then a mapping $f : X_1 \longrightarrow X_2$ is called a homomorphism if $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X_1$.

It it clear that if $f : X_1 \longrightarrow X_2$ is a homomorphism, then $f(1) = 1'$. For any $x, y \in X$, A. Walendzaik [8] defined the operation $\vee$ as $x \vee y = (y * x) * x$. However, in a commutative BE-algebra X, we can obtain for any $x, y \in X$, that $x \vee y = (y * x) * x = (x * y) * y = y \vee x$. For any non-empty subset A of a BE-algebra X, $\langle A \rangle$ is the smallest filter containing A.

Theorem 2.6 ([1])**.** *If A is a non-empty subset of a self-distributive BE-algebra X, then*

$$\langle A \rangle = \{x \in X \mid a_n * (... * (a_1 * x)...) = 1 \text{ for some } a_1, a_2, ..., a_n \in A\}.$$

Let F be a filter of a BE-algebra X. Then $\langle F \cup \{a\} \rangle = \{x \in X \mid a^n * x \in F$ for some $n \in \mathbb{N}\}$. For $A = \{a\}$, we will denote $\langle \{a\} \rangle$, briefly by $\langle a \rangle$, we call it a principal filter of X. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$.

3. Prime filters of BE-algebras

In this section, some properties of prime filters and maximal filters of BE-algebra are studied. A necessary and sufficient condition is derived for a proper filter of a BE-algebra to become a prime filter. Throughout this section, X stands for a BE-algebra unless otherwise mentioned.

Definition 3.1. A proper filter P of a BE-algebra X is called prime if $F\cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$ for any two filters F and G of X.

Theorem 3.2. *A proper filter P of a BE-algebra is prime if and only if $\langle x\rangle \cap \langle y\rangle \subseteq P$ implies $x \in P$ or $y \in P$ for all $x, y \in X$.*

Proof. Assume that P is a prime filter of X. Let $x, y \in X$ be such that $\langle x\rangle\cap\langle y\rangle \subseteq P$. Since P is prime, it implies that $x \in \langle x\rangle \subseteq P$ or $y \in \langle y\rangle \subseteq P$. Conversely, assume that the condition holds. Let F and G be two filters of X such that $F \cap G \subseteq P$. Let $x \in F$ and $y \in G$ be the arbitrary elements. Then $\langle x\rangle \subseteq F$ and $\langle y\rangle \subseteq G$. Hence $\langle x\rangle \cap \langle y\rangle \subseteq F \cap G \subseteq P$. Then by the assumed condition, we get $x \in P$ or $y \in P$. Thus $F \subseteq P$ or $G \subseteq P$. Therefore P is prime. □

Theorem 3.3. *Let X be a BE-algebra and F a filter of X. Then for any $a, b \in X$,*

$$\langle a\rangle \cap \langle b\rangle \subseteq F \text{ if and only if } \langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle = F.$$

Proof. Assume that $\langle F\cup\{a\}\rangle\cap\langle F\cup\{b\}\rangle = F$ for any $a, b \in X$. Since $a \in \langle F\cup\{a\}\rangle$ and $b \in \langle F \cup \{b\}\rangle$, we get $\langle a\rangle \subseteq \langle F \cup \{a\}\rangle$ and $\langle b\rangle \subseteq \langle F \cup \{b\}\rangle$. Hence, it yields $\langle a\rangle\cap\langle b\rangle \subseteq \langle F\cup\{a\}\rangle\cap\langle F\cup\{b\}\rangle = F$. Therefore, it concludes that $\langle a\rangle\cap\langle b\rangle \subseteq F$.

Conversely, assume that $\langle a\rangle \cap \langle b\rangle \subseteq F$. Clearly $F \subseteq \langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle$. Let $x \in \langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle$. Since F is a filter, there exists $m, n \in \mathbb{N}$ such that $a^m * x \in F$ and $b^n * x \in F$. Hence, there exists $m_1, m_2 \in F$ such that $a^m * x = m_1$ and $b^n * x = m_2$. Hence

$$a^m * (m_1 * x) = m_1 * (a^m * x) = m_1 * m_1 = 1$$

Hence $m_1 * x \in \langle a\rangle$. Similarly, we get $m_2 * x \in \langle b\rangle$. Since

$$m_1 * x \le m_2 * (m_1 * x) = m_1 * (m_2 * x) \text{ and } m_2 * x \le m_1 * (m_2 * x)$$

we get that $m_1 * (m_2 * x) \in \langle a\rangle$ and $m_1 * (m_2 * x) \in \langle b\rangle$. Hence

$$m_1 * (m_2 * x) \in \langle a\rangle \cap \langle b\rangle \subseteq F$$

Since $m_1, m_2 \in F$ and F is a filter, we get $x \in F$. Hence $\langle F\cup\{a\}\rangle\cap\langle F\cup\{b\}\rangle \subseteq F$. Therefore, it concludes that $\langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle = F$. □

Definition 3.4. A filter F of a BE-algebra X is called proper if $F \neq X$.

Definition 3.5. A proper filter M of a BE-algebra X is called a maximal filter if $\langle M \cup \{x\}\rangle = X$ for any $x \in X - M$.

Theorem 3.6. *Every maximal filter of a BE-algebra is a prime filter.*

Proof. Let M be a maximal filter of a BE-algebra X. Let $\langle x\rangle\cap\langle y\rangle \subseteq M$ for some $x, y \in X$. Suppose $x \notin M$ and $y \notin M$. Then $\langle M\cup\{x\}\rangle = X$ and $\langle M\cup\{y\}\rangle = X$. Hence

$$\langle M \cup \{x\}\rangle \cap \langle M \cup \{y\}\rangle = X$$

Hence, by the Theorem 3.3, it yields that $\langle x\rangle\cap\langle y\rangle \nsubseteq M$, which is a contradiction. Hence $x \in M$ or $y \in M$. Therefore M is a prime filter of X. □

Theorem 3.7. *Let X and Y be two BE-algebras and $f : X \to Y$ a homomorphism such that $f(X)$ is a filter in Y. If F is a prime filter of Y and $f^{-1}(F) \neq X$, then $f^{-1}(F)$ is a prime filter of X.*

Proof. Since $f(1) = 1 \in F$, we get $1 \in f^{-1}(F)$. Let $x, x * y \in f^{-1}(F)$. Then $f(x) \in F$ and $f(x) * f(y) = f(x * y) \in F$. Since F is a filter in Y, it yields that $f(y) \in F$. Hence $y \in f^{-1}(F)$. Therefore $f^{-1}(F)$ is a filter of X. Let $x, y \in X$ be such that $\langle x\rangle \cap \langle y\rangle \subseteq f^{-1}(F)$. Let $u \in \langle f(x)\rangle \cap \langle f(y)\rangle$. Then there exists $m, n \in \mathbb{N}$ such that $f(x)^n * u = 1 \in F$ and $f(y)^m * u = 1 \in F$. Since $f(x) \in f(X)$ and $f(X)$ is a filter, it implies that $u \in f(X)$. Hence $u = f(a)$ for some $a \in X$. Moreover, $f(x^n * a) = f(y^m * a) = 1 \in F$ because of f is a homomorphism. Hence

$$x^n * a \in f^{-1}(F) \text{ and } y^m * a \in f^{-1}(F).$$

Hence

$$a \in \langle f^{-1}(F) \cup \{x\}\rangle \cap \langle f^{-1}(F) \cup \{y\}\rangle.$$

Since $\langle x\rangle \cap \langle y\rangle \subseteq f^{-1}(F)$, then by Theorem 3.3, we get $a \in f^{-1}(F)$. Hence $u = f(a) \in F$. It concludes that $\langle f(x)\rangle \cap \langle f(y)\rangle \subseteq F$. Since F is a prime filter of Y, we get that $\langle f(x)\rangle \subseteq F$ or $\langle f(y)\rangle \subseteq F$. Thus it yields that $f(x) \in F$ or $f(y) \in F$. Therefore $x \in f^{-1}(F)$ or $y \in f^{-1}(F)$, which concludes that $f^{-1}(F)$ is a prime filter of X. □

Let us now denote that the class of all filters of a BE-algebra X by $\mathcal{F}(X)$. Then in the following theorem, a necessary and sufficient condition is derived, in terms of primeness of filters, for the class $\mathcal{F}(X)$ to become a chain.

Theorem 3.8. *Let X be a BE-algebra. Then $\mathcal{F}(X)$ is a totally ordered set or a chain if and only if every proper filter of X is prime.*

Proof. Assume that $\mathcal{F}(X)$ is a totally ordered set. Let F be a proper filter of X. Let $a, b \in X$ be such that $\langle a\rangle \cap \langle b\rangle \subseteq F$. Since $\langle a\rangle$ and $\langle b\rangle$ are filters of X, we get that either $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$. Hence, it concludes that $a \in F$ or $b \in F$. Therefore F is prime.

Conversely assume that every proper filter of X is prime. Let F and G be two proper filters of X. Since $F \cap G$ is a proper filter of X, we get that

$$F \subseteq F \cap G \text{ or } G \subseteq F \cap G$$

Hence $F \subseteq G$ or $G \subseteq F$. Therefore $\mathcal{F}(X)$ is a totally ordered set. □

4. Prime filters of commutative BE-algebras

In this section, a condition **L** is introduced to study the properties of prime filters of commutative BE-algebras. A set of equivalent conditions is derived for a commutative BE-algebra to become a totally ordered set.

Proposition 4.1. *Let $(X, *, 1)$ be a commutative BE-algebra and $x, y, z \in X$. Then the following conditions hold.*
(1) $x * (y \vee z) = (z * y) * (x * y)$;
(2) $x \leq y$ *implies* $x \vee y = y$;
(3) $z \leq x$ *and* $x * z \leq y * z$ *imply* $y \leq x$.

Proof. (1). Let $x, y, z \in X$. Then $x * (y \vee z) = x * ((z * y) * y) = (z * y) * (x * y)$.
(2). Let $x \leq y$. Then $x * y = 1$. Hence $y = 1 * y = (x * y) * y = (y * x) * x = x \vee y$.
(3). Let $z \leq x$ and $x * z \leq y * z$. Then $z * x = 1$ and $(x * z) * (y * z) = 1$. Hence

$$\begin{aligned} y * x &= y * (1 * x) \\ &= y * ((z * x) * x) \\ &= y * ((x * z) * z) \\ &= (x * z) * (y * z) \end{aligned}$$

Therefore, it concludes that $y \leq x$. □

Definition 4.2. A BE-algebra X is said to satisfy the condition **L** if for all $x, y \in X$, there exists $u \in X$ such that $u \leq x$ and $u \leq y$.

Theorem 4.3. *Let X be a commutative BE-algebra. Then X satisfies the condition **L** if and only if for all $x, y \in X$, the greatest lower bound* $\inf\{x, y\} = x \wedge y$ *for brevity, is* $x \wedge y = [(x * u) \vee (y * u)] * u$ *where* $u \leq x, y$.

Proof. Assume that X satisfies the condition **L**. Let $u \leq x, y$. Clearly $u \leq x \wedge y$. Since $x * u \leq (x * u) \vee (y * u)$, we get

$$\begin{aligned} [(x * u) \vee (y * u)] * u &\leq (x * u) * u \\ &= u \vee x \\ &= x \end{aligned}$$

Hence $x \wedge y \leq x$. Similarly, we can obtain that $x \wedge y \leq y$. Hence $x \wedge y$ is a lower bound of x and y. Suppose v is another lower bound for x and y, i.e. $v \leq x, y$. Hence $x * u \leq v * u$ and $y * u \leq v * u$. Hence $(x * u) \vee (y * u) \leq v * u$. Therefore we get

$$\begin{aligned} v &\leq v \vee u \\ &= (u * v) * v \\ &= (v * u) * u \\ &\leq [(x * u) \vee (y * u)] * u \\ &= x \wedge y \end{aligned}$$

Hence $x \wedge y$ is the greatest lower bound of x and y. Converse is clear. □

In the following proposition, some properties of a commutative BE-algebra with condition **L** are derived. Throughout this section, X stands for a commutative BE-algebra which satisfies the condition **L**, unless otherwise mentioned.

Proposition 4.4. *Let $(X, *, 1)$ be a commutative BE-algebra and $x, y, z \in X$. Then the following conditions hold.*
(1) $(x \vee y) * z = (x * z) \wedge (y * z)$
(2) $x * (y \wedge z) = (x * y) \wedge (x * z)$
(3) $x * (x \wedge y) = x * y$
(4) $(x * y) \vee (y * x) = 1$
(5) $(x \wedge y) * z = (x * z) \vee (y * z)$

Proof. (1). Since $x, y \leq x \vee y$, we get that $(x \vee y) * z \leq x * z$ and $(x \vee y) * z \leq y * z$. Hence $(x \vee y) * z$ is a lower bound for $x * z$ and $y * z$. Let u be a lower bound for $x * z$ and $y * z$. Hence $u \leq x * z$ and $u \leq y * z$ and so $x \leq u * z$ and $y \leq u * z$. Therefore $x \vee y \leq u * z$ and thus $u \leq (x \vee y) * z$. Therefore $(x \vee y) * z$ is the greatest lower bound for $x * z$ and $y * z$. Hence $(x \vee y) * z = (x * z) \wedge (y * z)$.
(2). Let $x, y, z \in X$. By the Theorem 4.3, we know that $y \wedge z = ((y * u) \vee (z * u)) * u$ where $u \leq y, z$. Since $u \leq y$, we get that $(y * u) * u = (u * y) * y = 1 * y = y$. Similarly, we get that $(z * u) * u = z$. Hence we get that

$$\begin{aligned} x * (y \wedge z) &= x * [((y * u) \vee (z * u)) * u] && \text{where } u \leq y, z \\ &= ((y * u) \vee (z * u)) * (x * u) \\ &= ((y * u) * (x * u)) \wedge ((z * u) * (x * u)) && \text{by (1)} \\ &= (x * ((y * u) * u)) \wedge (x * ((z * u) * u)) \\ &= (x * y) \wedge (x * z) \end{aligned}$$

(3). By replacing y by x and z by y in (2), we get

$$x * (x \wedge y) = (x * x) \wedge (x * y) = 1 \wedge (x * y) = x * y.$$

(4). Let $x, y, z \in X$. Then

$$\begin{aligned} (x * y) \vee (y * x) &= ((y * x) * (x * y)) * (x * y) \\ &= ((y * (y \wedge x)) * (x * (x \wedge y))) * (x * y) \\ &= ((y * (y \wedge x)) * (x * (y \wedge x))) * (x * y) \\ &= (x * ((y * (y \wedge x)) * (y \wedge x))) * (x * y) \\ &= (x * (((y \wedge x) * y) * y)) * (x * y) \\ &= (x * (1 * y)) * (x * y) && \text{since } y \wedge x \leq y \\ &= (x * y) * (x * y) \\ &= 1 \end{aligned}$$

(5). By using the dual argument, it can be followed by (1). □

Definition 4.5. A filter P of a commutative BE-algebra is called prime if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in F$.

Lemma 4.6. *Let X be a self-distributive and commutative BE-algebra. Then for any $a, b \in X$, the following conditions hold:*
(1) *$a \leq b$ implies $\langle b \rangle \subseteq \langle a \rangle$*
(2) *$\langle a \vee b \rangle = \langle a \rangle \cap \langle b \rangle$.*

Proof. (1). Suppose $a \leq b$. Let $x \in \langle b \rangle$. Then $b * x = 1$. Hence $1 = b * x \leq a * x$. Thus it yields that $x \in \langle a \rangle$. Therefore $\langle b \rangle \subseteq \langle a \rangle$.

(2). Since $a, b \leq a \vee b$, we get that $\langle a \vee b \rangle \subseteq \langle a \rangle$ and $\langle a \vee b \rangle \subseteq \langle b \rangle$. Hence $\langle a \vee b \rangle \subseteq \langle a \rangle \cap \langle b \rangle$. Conversely, let $x \in \langle a \rangle \cap \langle b \rangle$. Then $a * x = b * x = 1$. Since X is commutative, by proposition 4.4(1), we get $(a \vee b) * x = (a * x) \wedge (b * x) = 1 \wedge 1 = 1$. Hence $x \in \langle a \vee b \rangle$. Thus $\langle a \rangle \cap \langle b \rangle \subseteq \langle a \vee b \rangle$. Therefore $\langle a \vee b \rangle = \langle a \rangle \cap \langle b \rangle$. □

In the following theorem, the class of all prime filters of a commutative BE-algebra is characterized in terms of principal filters.

Theorem 4.7. *Let X be a self-distributive and commutative BE-algebra and P a proper filter of X. Then the following conditions are equivalent.*
(1) *P is prime;*
(2) *For any two filters F and G of X, $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$;*
(3) *For any $x, y \in X$, $\langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$.*

Proof. The equivalency between (2) and (3) is proved in Theorem 3.2.

(1) $\Rightarrow$ (2): Assume that P is a prime filter of X. Let F and G be two filters of X such that $F \cap G \subseteq P$. Without loss of generality, assume that $F \nsubseteq P$. Then there exists $a \in X$ such that $a \in F$ and $a \notin P$. Let $b \in G$ be an arbitrary element. Clearly $\langle a \rangle \cap \langle b \rangle = F \cap G \subseteq P$. Hence $\langle a \vee b \rangle \subseteq F \cap G \subseteq P$. Thus $a \vee b \in P$. Since P is prime and $a \notin P$, we get that $b \in P$. Therefore $G \subseteq P$.

(2) $\Rightarrow$ (1): Assume that the condition (2) holds. Let $x, y \in X$ be such that $x \vee y \in P$. Then we get that $\langle x \rangle \cap \langle y \rangle \subseteq P$. Hence, by condition (2), either $\langle x \rangle \subseteq P$ or $\langle y \rangle \subseteq P$. Therefore $x \in P$ or $y \in P$. □

The following theorem provides another characterization of prime filters in commutative BE-algebras with condition **L**.

Theorem 4.8. *Let X be a commutative BE-algebra with condition **L** and F a filter of X. Then F is prime if and only if $x * y \in F$ or $y * x \in F$ for all $x, y \in X$.*

Proof. Assume that F is a a prime filter in X. Since $(x * y) \vee (y * x) = 1 \in F$, we get either $x * y \in F$ or $y * x \in F$. Conversely, assume that $x * y \in F$ or $y * x \in F$ for all $x, y \in X$. Let $x \vee y \in F$. Suppose $x * y \in F$. Then $(x * y) * y = y \vee x \in F$. Since F is a filter and $x * y \in F$, we get that $y \in F$. Suppose $y * x \in F$. Then $(y * x) * x = x \vee y \in F$. Since F is a filter and $y * x \in F$, we get that $x \in F$. □

The following extension property of prime filters is a direct consequence of the above theorem.

Corollary 4.9. *Let X be a commutative BE-algebra with condition $\boldsymbol{L}$ and F a prime filter of X. If G is a filter of X such that $F \subseteq G$, then G is also prime.*

Theorem 4.10. *Let X be a commutative BE-algebra with condition $\boldsymbol{L}$. Then the following conditions are equivalent.*
(1) *Every proper filter is a prime filter;*
(2) *The filter $\{1\}$ is a prime filter;*
(3) *X is a totally ordered set with respect to BE-ordering.*

Proof. (1) $\Rightarrow$ (2): It is obvious.
(2) $\Rightarrow$ (3): Assume that $\{1\}$ is a prime filter. Let $x, y \in X$. Since $\{1\}$ is prime, we get that either $x * y \in \{1\}$ or $y * x \in \{1\}$. Hence $x \leq y$ or $y \leq x$. Therefore X is totally ordered.
(3) $\Rightarrow$ (1): Assume that X is a totally ordered set with respect to BE-ordering $\leq$. Let F be a proper filter of X. Let $x, y \in X$. Hence $x \leq y$ or $y \leq x$ and thus $x * y = 1 \in F$ or $y * x = 1 \in F$. Therefore F is prime. $\square$

Theorem 4.11. *Let F be a filter of a commutative BE-algebra with condition $\boldsymbol{L}$. For any $x, y \in X$, define a relation θ on X by*

$$(x, y) \in \theta \text{ if and only if } x * y \in F \text{ and} y * x \in F$$

Then θ is a congruence on X.

Proof. Clearly θ is reflexive and symmetric. Let $x, y, z \in X$ be such that $(x, y) \in \theta_F$ and $(y, z) \in \theta$. Then $x * y \in F, y * x \in F, y * z \in F$ and $z * y \in F$. Since $y * z \in F$, we get $x * (y * z) \in F$. By a known property of filters, we get $\{[x * (y * z)] * [(x * y) * (x * z)]\} = 1 \in F$. Since $x * (y * z) \in F$ and $x * y \in F$, we get $x * z \in F$. Similarly, we get $z * x \in F$. Thus $(x, z) \in \theta$. Therefore θ is an equivalence relation on X. Let $(x, y) \in \theta$ and $(u, v) \in \theta$. Then $x * y \in F, y * x \in F, u * v \in F$ and $v * u \in F$. Since $x * y \in F$, we get $(u*x)*(u*y) = u*(x*y) \in F$. Since $y*x \in F$, we get $(u*y)*(u*x) = u*(y*x) \in F$. Hence $(u * x, u * y) \in \theta$. Again,

$$\begin{aligned}(v * y) * (u * y) &= u * ((v * y) * y) \\ &= (u * (v * y)) * (u * y) \\ &= ((u * v) * (u * y)) * (u * y)\end{aligned}$$

Hence

$$\begin{aligned}(u * v) * ((v * y) * (u * y)) &= (u * v) * (((u * v) * (u * y)) * (u * y)) \\ &= ((u * v) * (u * y)) * ((u * v) * (u * y)) \\ &= 1 \in F\end{aligned}$$

Since $u * v \in F$, we get $(v * y) * (u * y) \in F$. Similarly $(u * y) * (v * y) \in F$. Hence $(u * y, v * y) \in \theta$. Thus $(u * x, v * y) \in \theta$. Hence θ is a congruence on X. $\square$

For any commutative BE-algebra X, let C_x be the congruence class generated by $x \in X$, i.e. $C_x = \{y \in X \mid x \text{ is congruent to } y\}$. Define $X_{/F} = \{C_x \mid x \in F\}$.

Then clearly $X_{/F}$ is a commutative BE-algebra with respect to the operation $*$ defined on $X_{/F}$ as follows:

$$C_x * C_y = C_{x*y} \qquad \text{for all } x, y \in X$$

It can also be observed that, for any $x, y \in X$, $C_x \leq C_y$ if and only if $C_x * C_y = C_1$ is a BE-ordering on $X_{/F}$.

Theorem 4.12. *Let X be a commutative BE-algebra with condition $\boldsymbol{L}$ and F a proper filter of X. Then F is prime if and only if $X_{/F}$ is a totally ordered set(chain).*

Proof. Assume that F is a prime filter in X. Then $x * y \in F$ or $y * x \in F$ for all $x, y \in X$. If $x * y \in F$, then $C_x * C_y = C_{x*y} = C_1$. Hence $C_x \leq C_y$. If $y * x \in F$, then similar argument yields $C_y \leq C_x$. Therefore X/F is a totally ordered set. Conversely, assume that $X_{/F}$ is a totally ordered set. Let $x, y \in X$. then clearly $C_x \leq C_y$ or $C_y \leq C_x$. Hence $C_{x*y} = C_x * C_y = C_1$ or $C_{y*x} = C_y * C_x = C_1$. Thus, it yields $x * y \in F$ or $y * x \in F$. Therefore F is a prime filter in X. □

5. The space of prime filters of BE-algebras

In this section, some topological properties of the space of all prime filters of BE-algebras are studied. A necessary and sufficient condition is derived for a prime filter of a BE-algebra to become maximal.

Theorem 5.1. *Let X ba a BE-algebra and $a \in X$. If F is a filter in X such that $a \notin F$, then there exists a prime filter P such that $a \notin P$ and $F \subseteq P$.*

Proof. Let F be a filter of X such that $a \notin F$. Consider $\Im = \{G \in \mathcal{F}(X) \mid a \notin G \text{ and } F \subseteq G\}$. Clearly $F \in \Im$. Then by the Zorn's Lemma, $\Im$ has a maximal element, say M. Clearly $a \notin M$. We now prove that M is prime. Let $x, y \in X$ be such that $\langle x \rangle \vee \langle y \rangle \subseteq M$. Then by Theorem 3.3, we get

$$\langle M \cup \{x\} \rangle \cap \langle M \cup \{y\} \rangle = M$$

Since $a \notin M$, we can obtain that $a \notin \langle M \cup \{x\} \rangle$ or $a \notin \langle M \cup \{y\} \rangle$. By the maximality of M, we get that $\langle M \cup \{x\} \rangle = M$ or $\langle M \cup \{y\} \rangle = M$. Hence $x \in M$ or $y \in M$. Therefore M is prime. □

Corollary 5.2. *Let X be a commutative BE-algebra and $1 \neq a \in X$. Then there exists a prime filter P such that $a \notin P$.*

Let X be a commutative BE-algebra and $Spec_F(X)$ denote the set of all prime filters of X. For any $A \subseteq X$, let $K(A) = \{P \in Spec_F(X) \mid A \nsubseteq P\}$ and for any $x \in L, K(x) = K(\{x\})$. Then we have the following observations:

Lemma 5.3. *Let X be a commutative BE-algebra with condition $\boldsymbol{L}$. For any $x, y \in L$, the following holds:*
(1) $K(x) \cap K(y) = K(x \vee y)$
(2) $K(x) \cup K(y) = K(x \wedge y)$
(3) $K(x) = \emptyset \Leftrightarrow x = 1$

Proof. (1). Let $P \in Spec_F(X)$ be such that $P \in K(x) \cap K(y)$. Then $x \notin P$ and $y \notin P$. Since P is prime, we get $x \vee y \notin P$. Hence $P \in K(x \vee y)$. Therefore $K(x) \cap K(y) \subseteq K(x \vee y)$. Conversely, assume that $P \in Spec_F(X)$. Suppose $P \in K(x \vee y)$. Hence $x \vee y \notin P$. If $x \in P$, then $x \vee y \in P$ because of $x \leq x \vee y$. Thus it yields that $x \notin P$. Therefore $P \in K(x)$. Similarly, we get $P \in K(y)$. Hence $P \in K(x) \cap K(y)$. Therefore $K(x \vee y) \subseteq K(x) \cap K(y)$.
(2). Let $P \in Spec_F(X)$ be such that $P \in K(x) \cup K(y)$. Then $P \in K(x)$ or $P \in K(y)$. Hence $x \notin P$ or $y \notin P$. If $x \wedge y \in P$, then we get that both x and y must be in P. Hence $x \wedge y \notin P$. Thus $P \in K(x \wedge y)$. Therefore $K(x) \cup K(y) \subseteq K(x \wedge y)$. Conversely, let $P \in Spec_F(X)$ be such that $P \in K(x \wedge y)$. Then $x \wedge y \notin P$. Since $x \wedge y$ is the g.l.b of x and y, it concludes that $x \notin P$ and $y \notin P$. Hence $P \in K(x) \cup K(y)$. Therefore $K(x \wedge y) \subseteq K(x) \cup K(y)$.
(3). Since $\{1\} \subseteq P$ for all $P \in Spec_F(X)$, it is obvious. □

Proposition 5.4. *For any commutative BE-algebra X,* $\bigcup_{x \in X} K(x) = Spec_F(X)$.

Proof. Let $P \in Spec_F(X)$. Since P is a proper filter, there exists $a \in X$ such that $a \notin P$. Hence $P \in K(a) \subseteq \bigcup_{x \in X} K(x)$. Therefore $Spec_F(X) \subseteq \bigcup_{x \in X} K(x)$. Clearly $\bigcup_{x \in X} K(x) \subseteq Spec_F(X)$. Therefore $\bigcup_{x \in X} K(x) = Spec_F(X)$. □

Form the above proposition, it can be seen that $\{K(x) \mid x \in X\}$ forms a covering of $Spec_F(X)$. Hence $\{K(x) \mid x \in X\}$ is an open base for a topology on $Spec_F(X)$ which is called a hull-kernel technology . In the following, we will discuss the properties of this topology.

Lemma 5.5. *Let X be a commutative BE-algebra. Then the following hold.*
(1) *For any* $x \in X, K(\langle x \rangle) = K(x)$*;*
(2) *For any two filters* F, G *of* X, $K(F) \cap K(G) = K(F \cap G)$.

Proof. (1) Let $P \in Spec_F(X)$ be such that $P \in K(\langle x \rangle)$. Then $\langle x \rangle \nsubseteq P$. Hence $x \notin P$. Therefore $P \in K(x)$. Thus $K(\langle x \rangle) \subseteq K(x)$. Conversely, let $P \in K(x)$. Then $x \notin P$. Hence $\langle x \rangle \nsubseteq P$. Therefore $P \in K(\langle x \rangle)$. Hence $K(x) \subseteq K(\langle x \rangle)$. Therefore $K(\langle x \rangle) = K(x)$.
(2). Let $P \in Spec_F(X)$ be an arbitrary prime filter. Let $P \in K(F) \cap K(G)$. Then $F \nsubseteq P$ and $G \nsubseteq P$. Then there exists $x \in F$ and $y \in G$ such that $x \notin P$ and $y \notin P$. Since P is prime, we get $x \vee y \notin P$. Since F and G are filters, we get that $x \vee y \in F \cap G$. Hence $F \cap G \nsubseteq P$. Then $P \in K(F \cap G)$. Therefore $K(F) \cap K(G) \subseteq K(F \cap G)$. The opposite inclusion is obvious. Therefore $K(F) \cap K(G) = K(F \cap G)$. □

Lemma 5.6. *Let F be a filter of a commutative BE-algebra X and* $x \in X$*. Then* $x \in F$ *if and only if* $K(x) \subseteq K(F)$.

Proof. Let F be a filter of a commutative BE-algebra X and $x \in X$. Assume that $x \in F$. Let $P \in Spec_F(X)$ be such that $P \in K(x)$. Then we get that

$x \notin P$. Hence $F \nsubseteq P$. Therefore $P \in K(F)$.

Conversely, assume that $K(x) \subseteq K(F)$. Suppose $x \notin F$. Then by Theorem 5.1, there exists $P \in Spec_F(X)$ such that $x \notin P$ and $F \subseteq P$. Hence, we get that $P \in K(x)$ and $P \notin K(F)$. Therefore $K(x) \subsetneq K(F)$, which is a contradiction. Hence, it concludes that $x \in F$. $\square$

Theorem 5.7. *Let X be a commutative BE-algebra. Then for any $x \in L$, $K(x)$ is compact in $Spec_F(X)$.*

Proof. Let $x \in X$. Let $A \subseteq X$ be such that $K(x) \subseteq \bigcup_{y \in A} K(y)$. Let F be the filter generated by A. Suppose $x \notin F$. Then there exists a prime filter P of X such that $F \subseteq P$ and $x \notin F$. Hence $P \in K(x) \subseteq \bigcup_{y \in A} K(y)$. Therefore $y \notin P$ for some $y \in A$, which is a contradiction (because of $y \in A \subseteq F \subseteq P$). Hence $x \in F$. Then there exist $a_1, a_2, ..., a_n \in A$ such that

$$a_n * (...(a_1 * x)...) = 1$$

Let $P \in K(x)$. Then $x \notin P$. Suppose $a_i \in P$ for all $i = 1, 2, ..., n$. Since $a_n * (...(a_1 * x)...) = 1 \in P$ and P is a filter, we get that $x \in P$, which is a contradiction. Hence $a_i \notin P$ for some $i = 1, 2, ..., n$. Hence $P \in K(a_i)$ for some a_i. Therefore $P \in \bigcup_{i=1}^{n} K(a_i)$. Hence $K(x) \subseteq \bigcup_{i=1}^{n} K(a_i)$, which is a finite subcover of $K(x)$. Hence $K(x)$ is compact in $Spec_F(X)$. Therefore for each $x \in X$, $K(x)$ is a compact open subset of $Spec_F(X)$. $\square$

Theorem 5.8. *Let X be a commutative BE-algebra with condition **L** and C a compact open subset of $Spec_F(X)$. Then $C = K(x)$ for some $x \in X$.*

Proof. Let C be a compact open subset of $Spec_F(X)$. Since C is open, we get $C = \bigcup_{a \in A} K(a)$ for some $A \subseteq X$. Since C is compact, there exists $a_1, a_2, ..., a_n \in A$ such that

$$C = \bigcup_{i=1}^{n} K(a_i) = K(\bigwedge_{i=1}^{n} a_i)$$

Therefore $C = K(x)$ for some $x \in L$. $\square$

Corollary 5.9. *For any commutative BE-algebra X with condition **L**, the set $\{K(x) \mid x \in X\}$ is an open base for the prime space $Spec_F(X)$.*

Theorem 5.10. *Let X be a commutative BE-algebra with condition **L**. Then $Spec_F(X)$ is a T_0-space.*

Proof. Let P and Q be two distinct prime filters of X. Without loss of generality assume that $P \nsubseteq Q$. Choose $x \in L$ such that $x \in P$ and $x \notin Q$. Hence $P \notin K(x)$ and $Q \in K(x)$. Therefore $Spec_F(X)$ is a T_0-space. $\square$

The following corollary is a direct consequence of the above results.

Corollary 5.11. *The map $x \mapsto K_0(x)$ is an anti-homomorphism from X onto the lattice of all compact open subsets of $Spec_F(X)$.*

For any $A \subseteq X$, denote $H(A) = \{P \in Spec_F(X) \mid A \subseteq P\}$. Then clearly $H(A) = Spec_F(X) - K(A)$. Therefore $H(A)$ is a closed set in $Spec_F(L)$. Also every closed set in $Spec_F(L)$ is of the form $H(A)$ for some $A \subseteq X$. Then we have the following:

Theorem 5.12. *The closure of any $Y \subseteq Spec_F(X)$ is given by $\overline{Y} = H(\bigcap_{P\in Y}(P))$.*

Proof. Let $Y \subseteq Spec_F(X)$. Let $Q \in Y$. Then $\bigcap_{P\in Y} P \subseteq Q$. Thus $Q \in H(\bigcap_{P\in Y} P)$. Therefore $H(\bigcap_{P\in Y} P)$ is a closed set containing Y. Let C be any closed set in $Spec_F(X)$. Then $C = H(A)$ for some $A \subseteq X$. Since $Y \subseteq C = H(A)$, we get that $A \subseteq P$ for all $P \in Y$. Hence $A \subseteq \bigcap_{P\in Y} P$. Therefore $H(\bigcap_{P\in Y} P) \subseteq H(A) = C$. Hence $H(\bigcap_{P\in Y} P)$ is the smallest closed set containing Y. Therefore $\overline{Y} = H(\bigcap_{P\in Y} P)$. ☐

Theorem 5.13. *For any commutative BE-algebra X with condition **L**, $Spec_F(X)$ is a T_1-space if and only if every prime filter is maximal.*

Proof. Assume that $Spec_F(X)$ is a T_1-space. Let P be a prime filter of X. Suppose there exists a proper filter Q of X such that $P \subseteq Q$. Since $Spec_F(X)$ is a T_1-space, there exists two basic open sets $K(x)$ and $K(y)$ such that $P \in K(x) - K(y)$ and $Q \in K(y) - K(x)$. Since $P \notin K(y)$, we get $y \in P \subset Q$, which is a contradiction to that $Q \in K(y)$. Hence P is a maximal filter.

Conversely, assume that every prime filter is a maximal filter. Let P_1 and P_2 be two distinct elements of $Spec_F(X)$. Hence by the assumption, both P_1 and P_2 are maximal filters in X. Hence $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Then there exists $a, b \in X$ be such that $a \in P_1 - P_2$ and $b \in P_2 - P_1$. Hence $P_1 \in K(b) - K(a)$ and $P_2 \in K(a) - K(b)$. Therefore $Spec_F(X)$ is a T_1-space. ☐

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