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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR A SYSTEM OF EVEN ORDER DYNAMIC EQUATION ON TIME SCALES

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Abstract. We determine interval of two eigenvalues for which there existence and nonexistence of positive solution for a system of even-order dynamic equation on time scales subject to Sturm-Liouville boundary conditions.

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1. Introduction

The theory of time scales was introduced and developed by Hilger [9] to unify both continuous and discrete analysis. Time scales theory presents us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamic systems and allows us to connect them. The theory is widely applied to various situations like epidemic models, the stock market and mathematical modeling of physical and biological systems. Certain economically important phenomena contain processes that feature elements of both the continuous and discrete. The book on the subject of time scales by Bohner and Peterson [4, 5], summarizes and organizes much of the time scale calculus.

In recent years, the existence and nonexistence of positive solutions of the higher order boundary value problems (BVPs) on time scales have been studied extensively due to their striking applications to almost all area of science, engineering and technology, Anderson [2, 3], Chyan and Henderson [6], Erbe and Peterson [7], Kameswararao and Nageswararao [14], Sun [16].

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We are concerned with determining values of λ and μ for which there exist and nonexist of positive solutions for the system of dynamic equations,

$$\begin{aligned}(-1)^n u^{(\Delta\nabla)^n}(t) + \lambda p(t) f(u(t), v(t)) = 0,\ t \in [a,b],\\ (-1)^n v^{(\Delta\nabla)^n}(t) + \mu q(t) g(u(t), v(t)) = 0,\ t \in [a,b],\end{aligned} \tag{1.1}$$

with the Sturm-Liouville boundary conditions,

$$\begin{aligned}\alpha_{i+1} u^{(\Delta\nabla)^i}(a) - \beta_{i+1} u^{(\Delta\nabla)^i\Delta}(a) = 0,\ \gamma_{i+1} u^{(\Delta\nabla)^i}(b) + \delta_{i+1} u^{(\Delta\nabla)^i\Delta}(b) = 0,\\ \alpha_{i+1} v^{(\Delta\nabla)^i}(a) - \beta_{i+1} v^{(\Delta\nabla)^i\Delta}(a) = 0,\ \gamma_{i+1} v^{(\Delta\nabla)^i}(b) + \delta_{i+1} v^{(\Delta\nabla)^i\Delta}(b) = 0,\end{aligned} \tag{1.2}$$

for $0 \le i \le n-1,\ n \ge 1$ with $a \in \mathbb{T}_{k^n},\ b \in \mathbb{T}^{k^n}$ for a time scale $\mathbb{T}$ and $\sigma^n(a) < \rho^n(b)$. Our interest in this paper is to investigate the existence and nonexistence of eigenvalues λ and μ that yields positive and no positive solutions to the associated boundary value problems, (1.1)-(1.2).
We assume that:

- (A1) $\alpha_j,\ \beta_j,\ \gamma_j, \delta_j \ge 0$ and $d_j = \gamma_j\beta_j + \alpha_j\delta_j + \alpha_j\gamma_j(b-a) > 0$;
- (A2) $f, g \in C([0,\infty) \times [0,\infty), [0,\infty))$;
- (A3) $p, q \in C([a,b], [0,\infty))$, and each does not vanish identically on any subinterval;
- (A4) All of
  $f_0 := \lim_{u+v\to 0^+} \frac{f(u,v)}{u+v},\ g_0 := \lim_{u+v\to 0^+} \frac{g(u,v)}{u+v}$,
  $f_\infty := \lim_{u+v\to\infty} \frac{f(u,v)}{u+v},\ g_\infty := \lim_{u+v\to\infty} \frac{g(u,v)}{u+v}$
  exist as positive real numbers.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of a positive solution of the system (1.1)-(1.2). The intervals in which the parameters λ and μ can guarantee the existence of a solution are obtained. In Section 4, we will consider the conditions of the nonexistence of a positive solution.

2. Preliminary results

In this section, we state some lemmas that will be used to prove our results. Shortly we will be concerned with a completely continuous operator whose kernel is the Green's function for the related homogeneous problem $(-1)^n u^{(\Delta\nabla)^n}(t) = 0,\ t \in [a,b]$ satisfying boundary conditions (1.2). For $1 \le j \le n$, let $G_j(t,s)$ be the Green's function for the boundary value problems,

$$-u^{\Delta\nabla}(t) = 0,\ t \in [a,b] \tag{2.1}$$

$$\alpha_j u(a) - \beta_j u^{\Delta}(a) = 0,\ \gamma_j u(b) + \delta_j u^{\Delta}(b) = 0. \tag{2.2}$$

First, we need few results on the related second order homogeneous boundary value problem (2.1)-(2.2).

Lemma 2.1. *For* $1 \leq j \leq n$, *let* $d_j = \gamma_j\beta_j + \alpha_j\delta_j + \alpha_j\gamma_j(b-a)$. *The homogeneous boundary value problem* (2.1)-(2.2) *has only the trivial solution if and only if* $d_j > 0$.

Lemma 2.2. *For* $1 \leq j \leq n$, *the Green's function* $G_j(t,s)$ *for the homogeneous boundary value problem* (2.1)-(2.2), *is given by*

$$G_j(t,s) = \begin{cases} \dfrac{1}{d_j}\{\alpha_j(t-a)+\beta_j\}\{\gamma_j(b-s)+\delta_j\} \ : \ a \leq t \leq s \leq b \\ \dfrac{1}{d_j}\{\alpha_j(s-a)+\beta_j\}\{\gamma_j(b-t)+\delta_j\} \ : \ a \leq s \leq t \leq b. \end{cases} \tag{2.3}$$

Lemma 2.3. *Assume that condition* $(A1)$ *is satisfied. Then, the Green's function* $G_j(t,s)$ *satisfies the following inequality*

$$g_j(t)G_j(s,s) \leq G_j(t,s) \leq G_j(s,s), \ \ for\ any\ s,t \in [a,b], \tag{2.4}$$

where

$$g_j(t) = \min\left\{\frac{\alpha_j(t-a)+\beta_j}{\alpha_j(b-a)+\beta_j}, \ \frac{\gamma_j(b-t)+\delta_j}{\gamma_j(b-a)+\delta_j}\right\} < 1, \tag{2.5}$$

for $1 \leq j \leq n$.

Proof. It is straightforward to see that

$$\frac{G_j(t,s)}{G_j(s,s)} = \begin{cases} \dfrac{\alpha_j(t-a)+\beta_j}{\alpha_j(s-a)+\beta_j} : \quad \mathrm{a \leq t \leq s \leq b} \\ \dfrac{\gamma_j(b-t)+\delta_j}{\gamma_j(b-s)+\delta_j} : \quad \mathrm{a \leq s \leq t \leq b} \end{cases}$$

this expression yields both inequalities in (2.4) for g_j as in (2.5). □

Lemma 2.4. *Assume that the condition* $(A1)$ *is satisfied, and* $G_j(t,s)$ *as in* (2.3). *Let us define* $H_1(t,s) = G_1(t,s)$, *and recursively define*

$$H_j(t,s) = \int_a^b H_{j-1}(t,r)G_j(r,s)\nabla r, \tag{2.6}$$

for $2 \leq j \leq n$. *Then* $H_n(t,s)$ *is the Green's function for the corresponding homogeneous problem* (1.1)-(1.2).

Let ξ and ω are chosen from $\mathbb{T}$ such that $a < \xi < \omega < b$ and also

$$m_j = \min_{t\in[\xi,\omega]} g_j(t) \tag{2.7}$$

for g_j as in (2.5).
Let $\tau \in [\xi,\omega]$ be defined by

$$\int_\xi^\omega G_j(\tau,s)p(s)\nabla s = \max_{t\in[\xi,\omega]} \int_\xi^\omega G_j(t,s)p(s)\nabla s$$

Lemma 2.5. *Assume that the condition* $(A1)$ *holds. If we define*

$$K = \prod_{j=1}^{n-1} K_j, \ \ L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ *in* Lemma 2.4 *satisfies*

$$0 \le H_n(t,s) \le K G_n(s,s), \ \ (t,s) \in [a,b] \times [a,b] \tag{2.8}$$

and

$$H_n(t,s) \ge m_n L G_n(s,s), \ \ (t,s) \in [\xi,\omega] \times [a,b] \tag{2.9}$$

where m_n *is given in* (2.7),

$$K_j = \int_a^b G_j(s,s)\nabla s > 0 \text{ and } L_j = \int_\xi^\omega G_j(s,s)\nabla s > 0, \ \ 1 \le j \le n.$$

Proof. We using mathematical induction on n it is straightforward. □

By using Green's function, our problem (1.1)-(1.2) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t) = \lambda \displaystyle\int_a^b H_n(t,s)p(s)f(u(s),v(s))\nabla s, \ \ a \le t \le b, \\ v(t) = \mu \displaystyle\int_a^b H_n(t,s)q(s)g(u(s),v(s))\nabla s, \ \ a \le t \le b. \end{cases}$$

We consider the Banach space $\mathcal{B} = C[a,b] \times C[a,b]$ with the norm

$$\|(u,v)\| = \|u\| + \|v\| = \max_{t\in[a,b]} |u(t)| + \max_{t\in[a,b]} |v(t)|.$$

We define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{(u,v) : \mathcal{B} : u(t) \ge 0, \ v(t) \ge 0, \ \forall \ t \in [a,b] \text{ and } \min_{t\in \ [\xi,\omega]} (u(t)+v(t)) \ge \frac{m_n L}{K} \parallel (u,v) \parallel \Big\}.$$

For $\lambda, \mu > 0$, we introduce the operators $Q_\lambda, Q_\mu : C[a,b] \times C[a,b] \to C[a,b]$ by

$$Q_\lambda(u,v)(t) = \lambda \int_a^b H_n(t,s)p(s)f(u(s),v(s))\nabla s, \quad a \le t \le b,$$

$$Q_\mu(u,v)(t) = \mu \int_a^b H_n(t,s)q(s)g(u(s),v(s))\nabla s, \quad a \le t \le b,$$

and an operator $Q : C[a,b] \times C[a,b] \to C[a,b] \times C[a,b]$ as

$$Q(u,v) = \big(Q_\lambda(u,v), Q_\mu(u,v)\big), (u,v) \in C[a,b] \times C[a,b]. \tag{2.10}$$

Then seeking solution to our BVP (1.1)-(1.2) is equivalent to looking for fixed points of the equation $Q(u,v) = (u,v)$ in the Banach space $\mathcal{B}$.

Lemma 2.6. $Q : \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof. By using standard arguments, we can easily show that, under assumptions $(A1) - (A2)$, the operator Q is completely continuous, we need only to prove $Q(\mathcal{P}) \subseteq \mathcal{P}$. Choose some $(u, v) \in \mathcal{P}$. Then by Lemma 2.5 we have

$$\min_{t\in[\xi,\omega]} Q_\lambda(u,v)(t) \geq \frac{m_n L}{K} \parallel Q_\lambda(u,v) \parallel, \quad \min_{t\in[\xi,\omega]} Q_\mu(u,v)(t) \geq \frac{m_n L}{K} \parallel Q_\mu(u,v) \parallel$$

and thus

$$\begin{aligned}\min_{t\in[\xi,\omega]} \left[\parallel Q_\lambda(u,v) \parallel + \parallel Q_\mu(u,v) \parallel \right] &\geq \min_{t\in[\xi,\omega]} Q_\lambda(u,v)(t) + \min_{t\in[\xi,\omega]} Q_\mu(u,v)(t) \\ &\geq \frac{m_n L}{K} \parallel Q_\lambda(u,v) \parallel + \frac{m_n L}{K} \parallel Q_\mu(u,v) \parallel \\ &= \frac{m_n L}{K} \left[\parallel Q_\lambda(u,v) \parallel + \parallel Q_\mu(u,v) \parallel \right]\end{aligned}$$

which implies that $Q(\mathcal{P}) \subseteq \mathcal{P}$ for every $(u, v) \in \mathcal{P}$.

As Q_λ and Q_μ are integral operators, it is not difficult to see that using standard arguments we may conclude that both Q_λ and Q_μ are completely continuous, hence Q is completely continuous operator. $\square$

3. Existence results

In this section, we apply Krasnosel'skii fixed point theorem [13] to obtain the solutions in a cone (that is, positive solution) of (1.1)-(1.2).

Theorem 3.1 (**Krasnosel'skii**). *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that either*

(i) $\parallel Tu \parallel \leq \parallel u \parallel,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\parallel Tu \parallel \geq \parallel u \parallel,\ u \in \mathcal{P} \cap \partial\Omega_2,$ *or*
(ii) $\parallel Tu \parallel \geq \parallel u \parallel,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\parallel Tu \parallel \leq \parallel u \parallel,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

For our first result, define positive numbers M_1 and M_2 by

$$\begin{aligned}M_1 :&= \max\left\{ \frac{1}{2}\left[\frac{m_n^2 L^2}{K}\int_\xi^\omega G_n(\tau,s)p(s)\nabla s f_\infty\right]^{-1},\ \frac{1}{2}\left[\frac{m_n^2 L^2}{K}\int_\xi^\omega G_n(\tau,s)q(s)\nabla s g_\infty\right]^{-1}\right\},\\ M_2 :&= \min\left\{ \frac{1}{2}\left[K\int_a^b G_n(s,s)p(s)\nabla s f_0\right]^{-1},\ \frac{1}{2}\left[K\int_a^b G_n(s,s)q(s)\nabla s g_0\right]^{-1}\right\}.\end{aligned}$$

Theorem 3.2. *Assume that conditions $(A1) - (A4)$ are satisfied. Then, for each λ, μ satisfying*

$$M_1 < \lambda,\ \mu < M_2, \tag{3.1}$$

there exists a pair (u, v) satisfying (1.1)-(1.2) such that $u(t) > 0$ and $v(t) > 0$ on (a, b).

Proof. Let $\lambda,\ \mu$ be as in (3.1). Let $\epsilon > 0$ be chosen such that

$$\max\left\{\frac{1}{2}\left[\frac{m_n^2L^2}{K}\int_\xi^\omega G_n(\tau,s)p(s)\nabla s(f_\infty-\epsilon)\right]^{-1}, \frac{1}{2}\left[\frac{m_n^2L^2}{K}\int_\xi^\omega G_n(\tau,s)q(s)\nabla s(g_\infty-\epsilon)\right]^{-1}\right\} \le \lambda,\mu$$

$$\lambda,\mu \le \min\left\{\frac{1}{2}\left[K\int_a^b G_n(s,s)p(s)\nabla s(f_0+\epsilon)\right]^{-1},\ \frac{1}{2}\left[K\int_a^b G_n(s,s)q(s)\nabla s(g_0+\epsilon)\right]^{-1}\right\}.$$

Let Q be defined as in (2.10), then Q is a cone preserving, completely continuous operator. By the definitions of f_0 and g_0, there exists $H_1 > 0$ such that $f(u,v) \le (f_0+\epsilon)(u+v)$ for $(u,v) \in \mathcal{P}$ with $0 < (u,v) \le H_1$, and $g(u,v) \le (g_0+\epsilon)(u+v)$ for $(u,v) \in \mathcal{P}$ with $0 < (u,v) \le H_1$. Set $\Omega_1 = \{(u,v) \in \mathcal{B} : \parallel (u,v) \parallel < H_1\}$. Now let $(u,v) \in \mathcal{P} \cap \partial\Omega_1$, i.e., let $(u,v) \in \mathcal{P}$ with $\parallel (u,v) \parallel = H_1$. Then, in view of the inequality (2.8) and choice of ϵ, for $a \le s \le b$, we have

$$\begin{aligned}
Q_\lambda(u,v)(t) &= \lambda\int_a^b H_n(t,s)p(s)f(u(s),v(s))\nabla s\\
&\le \lambda K\int_a^b G_n(s,s)p(s)f(u(s),v(s))\nabla s\\
&\le \lambda K\int_a^b G_n(s,s)p(s)(f_0+\epsilon)(u(s)+v(s))\nabla s\\
&\le \lambda K\int_a^b G_n(s,s)p(s)\nabla s(f_0+\epsilon)[\parallel u \parallel + \parallel v \parallel]\\
&\le \frac{1}{2}[\parallel u \parallel + \parallel v \parallel] = \frac{1}{2}\parallel (u,v) \parallel
\end{aligned}$$

and so,

$$\parallel Q_\lambda(u,v) \parallel \le \frac{1}{2}\parallel (u,v) \parallel.$$

Similarly, we prove that $\parallel Q_\mu(u,v) \parallel \le \frac{1}{2} \parallel (u,v) \parallel$. Thus, for $(u,v) \in \mathcal{P} \cap \partial\Omega_1$ it follows that

$$\begin{aligned}
\parallel Q(u,v) \parallel &= \parallel (Q_\lambda(u,v), Q_\mu(u,v)) \parallel\\
&= \parallel Q_\lambda(u,v) \parallel + \parallel Q_\mu(u,v) \parallel\\
&\le \frac{1}{2}\parallel (u,v) \parallel + \frac{1}{2}\parallel (u,v) \parallel\\
&= \parallel (u,v) \parallel,
\end{aligned}$$

that is,

$$\parallel Q(u,v) \parallel \le \parallel (u,v) \parallel \ \text{ for all } (u,v) \in \mathcal{P} \cap \partial\Omega_1. \tag{3.2}$$

Due to the definition of f_∞ and g_∞, there exists an $\overline{H}_2 > 0$ such that $f(u,v) \ge (f_\infty-\epsilon)(u+v)$ for all $u,v \ge \overline{H}_2$ and $g(u,v) \ge (g_\infty-\epsilon)(u+v)$ for all $u,v \ge \overline{H}_2$. Set $H_2 = \max\{2H_1, \frac{K}{m_nL}\overline{H}_2\}$ and define $\Omega_2 = \{(u,v) \in \mathcal{P} : \parallel (u,v) \parallel < H_2\}$. If $(u,v) \in \mathcal{P}$ with $\parallel (u,v) \parallel = H_2$ then, $\min_{t\in[\xi,\omega]}(u+v)(t) \ge \frac{m_nL}{K} \parallel (u,v) \parallel \ge \overline{H}_2$, by consequently, from (2.9) and choice of ϵ, for $a \leq s \leq b$, we have that

$$\begin{aligned} Q_\lambda(u,v)(\tau) &= \lambda \int_a^b H_n(\tau,s)p(s)f(u(s),v(s))\nabla s \\ &\geq \lambda m_n L \int_\xi^\omega G_n(\tau,s)p(s)(f_\infty - \epsilon)(u(s)+v(s))\nabla s \\ &\geq \lambda \frac{m_n^2 L^2}{K} \int_\xi^\omega G_n(\tau,s)p(s)\nabla s(f_\infty - \epsilon)[\| u \| + \| v \|] \\ &\geq \frac{1}{2} \| (u,v) \| . \end{aligned}$$

that is, $Q_\lambda(u,v)(t) \geq \frac{1}{2} \| (u,v) \|$ for all $t \geq \tau$ and so, $Q_\lambda(u,v)(t) \geq \frac{1}{2} \| (u,v) \|$. Similarly, we find that $Q_\mu(u,v) \geq \frac{1}{2} \| (u,v) \|$. Thus, for $(u,v) \in \mathcal{P} \cap \partial\Omega_2$ it follows that

$$\begin{aligned} \| Q(u,v) \| &= \| (Q_\lambda(u,v), Q_\mu(u,v)) \| \\ &= \| Q_\lambda(u,v) \| + \| Q_\mu(u,v) \| \\ &\geq \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| \\ &= \| (u,v) \|, \end{aligned}$$

that is,

$$\| Q(u,v) \| \geq \| (u,v) \|, \text{ for all } (u,v) \in \mathcal{P} \cap \partial\Omega_2. \tag{3.3}$$

Applying Theorem 3.1 to (3.2) and (3.3), we obtain that Q has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $H_1 \leq \| (u,v) \| \leq H_2$, and so (1.1)-(1.2) has a positive solution. The proof is complete. □

For our next result we define the positive numbers

$$M_3 := \max\Big\{\frac{1}{2}\Big[\frac{m_n^2L^2}{K}\int_\xi^\omega G_n(\tau,s)p(s)\nabla s f_0\Big]^{-1}, \frac{1}{2}\Big[\frac{m_n^2L^2}{K}\int_\xi^\omega G_n(\tau,s)q(s)\nabla s g_0\Big]^{-1}\Big\},$$

$$M_4 := \min\Big\{\frac{1}{2}\Big[K\int_a^b G_n(s,s)p(s)\nabla s f_\infty\Big]^{-1}, \frac{1}{2}\Big[K\int_a^b G_n(s,s)q(s)\nabla s g_\infty\Big]^{-1}\Big\}.$$

We are now ready to state and prove our main result.

Theorem 3.3. *Assume that conditions* $(A1)-(A4)$ *are satisfied. Then, for each* λ, μ *satisfying*

$$M_3 < \lambda, \mu < M_4, \tag{3.4}$$

there exists a pair (u,v) *satisfying* (1.1)-(1.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* (a,b).

Proof. Let λ, μ be as in (3.4) and choose a sufficiently small $\epsilon > 0$ such that

$$\max\Big\{\frac{1}{2}\Big[\frac{m_n^2L^2}{K}\int_\xi^\omega G_n(\tau,s)p(s)\nabla s(f_0-\epsilon)\Big]^{-1}, \frac{1}{2}\Big[\frac{m_n^2L^2}{K}\int_\xi^\omega G_n(\tau,s)q(s)\nabla s(g_0-\epsilon)\Big]^{-1}\Big\} \leq \lambda, \mu$$

$$\lambda, \mu \leq \min\Big\{\frac{1}{2}\Big[K\int_a^b G_n(s,s)p(s)\nabla s(f_\infty+\epsilon)\Big]^{-1}, \frac{1}{2}\Big[K\int_a^b G_n(s,s)q(s)\nabla s(g_\infty+\epsilon)\Big]^{-1}\Big\}.$$

By the definition of f_0 and g_0, there exists an $H_3 > 0$ such that$f(u,v) \geq (f_0 - \epsilon)(u+v)$, for all (u,v) with $0 < (u,v) \leq H_3$ and $g(u,v) \geq (g_0 - \epsilon)(u+v)$, for all (u,v) with $0 < (u,v) \leq H_3$. Set $\Omega_3 = \big\{(u,v) \in \mathcal{P} : \| (u,v) \| < H_3\big\}$ and let $(u,v) \in \mathcal{P} \cap \partial\Omega_3$. Thus we have, from (2.9) and choice of ϵ, for $a \leq s \leq b$,

$$\begin{aligned}
Q_\lambda(u,v)(\tau) &= \lambda \int_a^b H_n(\tau,s)p(s)f(u(s),v(s))\nabla s \\
&\geq \lambda m_n L \int_\xi^\omega G_n(\tau,s)p(s)f(u(s),v(s))\nabla s \\
&\geq \lambda m_n L \int_\xi^\omega G_n(\tau,s)p(s)(f_0-\epsilon)(u(s)+v(s))\nabla s \\
&\geq \lambda \frac{m_n^2 L^2}{K} \int_\xi^\omega G_n(\tau,s)p(s)(f_0-\epsilon)[\| u \| + \| v \|]\nabla s \\
&\geq \frac{1}{2} \| (u,v) \|,
\end{aligned}$$

that is, $\| Q_\lambda(u,v) \| \geq \frac{1}{2} \| (u,v) \|$. In a similar manner, $\| Q_\mu(u,v) \| \geq \frac{1}{2} \| (u,v) \|$. Thus, for an arbitrary $(u,v) \in \mathcal{P} \cap \partial\Omega_3$ it follows that

$$\begin{aligned}
\| Q(u,v) \| &= \| (Q_\lambda(u,v), Q_\mu(u,v)) \| \\
&= \| Q_\lambda(u,v) \| + \| Q_\mu(u,v) \| \\
&\geq \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| \\
&= \| (u,v) \|,
\end{aligned}$$

and so,

$$\| Q(u,v) \| \geq \| (u,v) \| \text{ for all } (u,v) \in \mathcal{P} \cap \partial\Omega_3. \tag{3.5}$$

Now let us define two functions $f^*,\ g^* :\ [0,\infty) \to [0,\infty)$ by

$$f^*(t) = \max_{0 \leq u+v \leq t} f(u,v) \text{ and } g^*(t) = \max_{0 \leq u+v \leq t} g(u,v).$$

It follows that $f(u,v) \leq f^*(t)$ and $g(u,v) \leq g^*(t)$ for all (u,v) with $0 \leq u+v \leq t$. It is clear that the function f^* and $g*$ are nondecreasing. Also, there is no difficulty to see that

$$\lim_{t\to\infty} \frac{f^*(t)}{t} = f_\infty \text{ and } \lim_{t\to\infty} \frac{g^*(t)}{t} = g_\infty.$$

In view of the definitions of f_∞ and g_∞, there exists an $\overline{H}_4$ such that

$$f^*(t) < (f_\infty + \epsilon)t \text{ for all } t \geq \overline{H}_4,\ g^*(t) < (g_\infty + \epsilon)t \text{ for all } t \geq \overline{H}_4.$$

Set $H_4 = \max\Big\{2H_3, \frac{K}{m_n L}\overline{H}_4\Big\}$, and $\Omega_4 = \{(u,v) : (u,v) \in \mathcal{P}$ and $\| (u,v) \| < H_4\}$. Let $(u,v) \in \mathcal{P} \cap \partial\Omega_4$ and observe that, by the definition of f^*, it follows that for any $s \in [a,b]$, we have

$$f(u(s),v(s)) \leq f^*(\| u \| + \| v \|) = f^*(\| (u,v) \|).$$

In view of the observation and by the use of inequality (2.8),

$$\begin{aligned} Q_\lambda(u,v)(t) &= \lambda \int_a^b H_n(t,s)p(s)f(u(s),v(s))\nabla s \\ &\leq \lambda K \int_a^b G_n(s,s)p(s)f^*(u(s)+v(s))\nabla s \\ &\leq \lambda K \int_a^b G_n(s,s)p(s)(f_\infty+\epsilon)(\| u \| + \| v \|)\nabla s \\ &\leq \frac{1}{2} \| (u,v) \|, \end{aligned}$$

which implies $\| Q_\lambda(u,v) \| \leq \frac{1}{2} \| (u,v) \|$. In a similar manner, we can prove that $\| Q_\mu(u,v) \| \leq \frac{1}{2} \| (u,v) \|$. Thus, for $(u,v) \in \mathcal{P} \cap \partial\Omega_4$, it follows that

$$\begin{aligned} \| Q(u,v) \| &= \| (Q_\lambda(u,v), Q_\mu(u,v)) \| \\ &= \| Q_\lambda(u,v) \| + \| Q_\mu(u,v) \| \\ &\leq \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| \\ &= \| (u,v) \|, \end{aligned}$$

and so,

$$\| Q(u,v) \| \leq \| (u,v) \|, \text{ for } (u,v) \in \mathcal{P} \cap \partial\Omega_4. \tag{3.6}$$

Applying Theorem 3.1 to (3.5) and (3.6), we obtain that Q has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $H_3 \leq \| (u,v) \| \leq H_4$, and so (1.1)-(1.2) has a positive solution. The proof is complete. □

4. Nonexistence results

In this section, we give some sufficient conditions for the nonexistence of positive solutions to the BVP (1.1)-(1.2).

Theorem 4.1. *Assume that* $(A1)-(A4)$ *hold. If* $f_0, f_\infty, g_0, g_\infty < \infty$, *then there exist positive constants* λ_0, μ_0 *such that for every* $\lambda \in (0,\lambda_0)$ *and* $\mu \in (0,\mu_0)$*, the boundary value problem* (1.1)-(1.2) *has no positive solution.*

Proof. Since $f_0, f_\infty < \infty$, we deduce that there exist $M_1^{'}, M_1^{''}, r_1, r_1^{'} > 0, r_1 < r_1^{'}$ such that

$$\begin{aligned} f(u,v) &\leq M_1^{'}(u+v),\ \forall\ u,v \geq 0,\ u+v \in [0,r_1], \\ f(u,v) &\leq M_1^{''}(u+v), \forall\ u,v \geq 0,\ u+v \in [r_1^{'},\infty). \end{aligned}$$

We consider $M_1 = \max\left\{M_1^{'}, M_1^{''}, \max_{r_1 \leq u+v \leq r_1^{'}} \frac{f(u,v)}{u+v}\right\} > 0$. Then, we obtain $f(u,v) \leq M_1(u+v),\ \forall\ u,v \geq 0$. Since $g_0, g_\infty < \infty$, we deduce that there exist $M_2^{'}, M_2^{''}, r_2, r_2^{'} > 0, r_2 < r_2^{'}$ such that

$$\begin{aligned} g(u,v) &\leq M_2^{'}(u+v),\ \forall\ u,v \geq 0,\ u+v \in [0,r_2], \\ g(u,v) &\leq M_2^{''}(u+v), \forall\ u,v \geq 0,\ u+v \in [r_2^{'},\infty). \end{aligned}$$

We consider $M_2 = \max\left\{M_2^{'}, M_2^{''}, \max_{r_2 \le u+v \le r_2^{'}} \frac{g(u,v)}{u+v}\right\} > 0$. Then, we obtain $g(u,v) \le M_2(u+v),\ \forall\ u,v \ge 0$. We define $\lambda_0 = \frac{1}{2M_1B}$ and $\mu_0 = \frac{1}{2M_2D}$, where $B = K\int_a^b G_n(s,s)p(s)\nabla s$ and $D = K\int_a^b G_n(s,s)q(s)\nabla s$. We shall show that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, the problem (1.1)-(1.2) has no positive solution.

Let $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$. We suppose that (1.1)-(1.2) has a positive solution $(u(t),v(t)), t \in [a,b]$. Then, we have

$$\begin{aligned}
u(t) = Q_\lambda(u,v)(t) &= \lambda\int_a^b H_n(t,s)p(s)f(u(s),v(s))\nabla s\\
&\le \lambda K\int_a^b G_n(s,s)p(s)f(u(s),v(s))\nabla s\\
&\le \lambda K\int_a^b G_n(s,s)p(s)M_1(u(s)+v(s))\nabla s\\
&\le \lambda M_1K\int_a^b G_n(s,s)p(s)(\| u \| + \| v \|)\nabla s\\
&= \lambda M_1B \| (u,v) \|,\ \forall\ t \in [a,b].
\end{aligned}$$

Therefore, we conclude

$$\| u \| \le \lambda M_1B \| (u,v) \| < \lambda_0 M_1 B \| (u,v) \| = \frac{1}{2} \| (u,v) \| .$$

In a similar manner,

$$\begin{aligned}
v(t) = Q_\mu(u,v)(t) &= \mu\int_a^b H_n(t,s)q(s)g(u(s),v(s))\nabla s\\
&\le \mu K\int_a^b G_n(s,s)q(s)g(u(s),v(s))\nabla s\\
&\le \mu K\int_a^b G_n(s,s)q(s)M_2(u(s)+v(s))\nabla s\\
&\le \mu M_2K\int_a^b G_n(s,s)q(s)(\| u \| + \| v \|)\nabla s\\
&= \mu M_2D \| (u,v) \|,\ \forall\ t \in [a,b].
\end{aligned}$$

Therefore, we conclude

$$\| v \| \le \mu M_2D \| (u,v) \| < \mu_0 M_2 D \| (u,v) \| = \frac{1}{2} \| (u,v) \| .$$

Hence, $\| (u,v) \| = \| u \| + \| v \| < \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| = \| (u,v) \|$, which is a contradiction. So, the boundary value problem (1.1)-(1.2) has no positive solution. □

Theorem 4.2. *Assume that* $(A1)-(A4)$ *hold.*
(i) If $f_0, f_\infty > 0$, *then there exists a positive constant* $\tilde{\lambda_0}$ *such that for every* $\lambda >$

$\tilde{\lambda_0}$ *and* $\mu > 0$, *the boundary value problem* (1.1)-(1.2) *has no positive solution.*
(ii) *If* $g_0, g_\infty > 0$, *then there exists a positive constant* $\tilde{\mu_0}$ *such that for every* $\mu > \tilde{\mu_0}$ *and* $\lambda > 0$, *the boundary value problem* (1.1)-(1.2) *has no positive solution.*
(iii)*If* $f_0, f_\infty, g_0, g_\infty > 0$, *then there exist positive constants* $\tilde{\tilde{\lambda_0}}$ *and* $\tilde{\tilde{\mu_0}}$ *such that for every* $\lambda > \tilde{\tilde{\lambda_0}}$ *and* $\mu > \tilde{\tilde{\mu_0}}$, *the boundary value problem* (1.1)-(1.2) *has no positive solution.*

Proof. (i) Since $f_0, f_\infty > 0$, we deduce that there exist $m_1^{'}, m_1^{''}, r_3, r_3^{'} > 0, r_3 < r_3^{'}$ such that

$$f(u,v) \geq m_1^{'}(u+v),\ \forall\ u,v \geq 0,\ \ u+v \in [0, r_3],$$
$$f(u,v) \geq m_1^{''}(u+v), \forall\ u,v \geq 0,\ \ u+v \in [r_3^{'}, \infty).$$

We introduce $m_1 = \min\left\{m_1^{'}, m_1^{''}, \min_{r_3 \leq u+v \leq r_3^{'}} \frac{f(u,v)}{u+v}\right\} > 0$. Then, we obtain $f(u,v) \geq m_1(u+v),\ \forall\ u,v \geq 0$. We define $\tilde{\lambda_0} = \frac{K}{m_n^2 L^2 m_1 A} > 0$, where $A = \int_\xi^\omega G_n(s,s)p(s)\nabla s$. We shall show that for every $\lambda > \tilde{\lambda_0}$ and $\mu > 0$ the problem (1.1)-(1.2) has no positive solution.

Let $\lambda > \tilde{\lambda_0}$ and $\mu > 0$. We suppose that (1.1)-(1.2) has a positive solution $(u(t), v(t)), t \in [a,b]$. Then, we obtain

$$\begin{aligned} u(t) = Q_\lambda(u,v)(t) &= \lambda \int_a^b H_n(t,s)p(s)f(u(s),v(s))\nabla s \\ &\geq \lambda m_n L \int_\xi^\omega G_n(s,s)p(s)f(u(s),v(s))\nabla s \\ &\geq \lambda m_n L \int_\xi^\omega G_n(s,s)p(s)m_1(u(s)+v(s))\nabla s \\ &\geq \lambda \frac{m_n^2 L^2}{K} m_1 \int_\xi^\omega G_n(s,s)p(s) \parallel (u,v) \parallel \nabla s \\ &= \lambda \frac{m_n^2 L^2}{K} m_1 A \parallel (u,v) \parallel . \end{aligned}$$

Therefore, we deduce

$$\parallel u \parallel \geq u(t) \geq \lambda \frac{m_n^2 L^2}{K} m_1 A \parallel (u,v) \parallel > \tilde{\lambda_0} \frac{m_n^2 L^2}{K} m_1 A \parallel (u,v) \parallel = \parallel (u,v) \parallel .$$

and so, $\parallel (u,v) \parallel = \parallel u \parallel + \parallel v \parallel \geq \parallel u \parallel > \parallel (u,v) \parallel$, which is a contradiction. Therefore, the boundary value problem (1.1)-(1.2) has no positive solution.
(ii) Since $g_0, g_\infty > 0$, we deduce that there exist $m_2^{'}, m_2^{''}, r_4, r_4^{'} > 0, r_4 < r_4^{'}$ such that

$$g(u,v) \geq m_2^{'}(u+v),\ \forall\ u,v \geq 0,\ \ u+v \in [0, r_4],$$
$$g(u,v) \geq m_2^{''}(u+v),\ \forall\ u,v \geq 0,\ \ u+v \in [r_4^{'}, \infty).$$

We introduce $m_2 = \min\left\{m_2^{'}, m_2^{''}, \min_{r_4 \leq u+v \leq r_4^{'}} \frac{g(u,v)}{u+v}\right\} > 0$. Then, we obtain $g(u,v) \geq m_2(u+v),\ \forall\ u,v \geq 0$. We define $\tilde{\mu_0} = \frac{K}{m_n^2 L^2 m_2 C} > 0$, where $C =$

$\int_\xi^\omega G_n(s,s)q(s)\nabla s$. We shall show that for every $\mu > \tilde{\mu_0}$ and $\lambda > 0$ the problem (1.1)-(1.2) has no positive solution.

Let $\mu > \tilde{\mu_0}$ and $\lambda > 0$. We suppose that (1.1)-(1.2) has a positive solution $(u(t), v(t)), t \in [a,b]$. Then, we obtain

$$\begin{aligned} v(t) = Q_\mu(u,v)(t) &= \mu \int_a^b H_n(t,s)q(s)g(u(s),v(s))\nabla s \\ &\geq \mu\ m_n L \int_\xi^\omega G_n(s,s)q(s)g(u(s),v(s))\nabla s \\ &\geq \mu\ m_n L \int_\xi^\omega G_n(s,s)q(s)m_2(u(s)+v(s))\nabla s \\ &\geq \mu\ \frac{m_n^2 L^2}{K} m_2 \int_\xi^\omega G_n(s,s)q(s)[\| u \| + \| v \|]\nabla s \\ &= \mu \frac{m_n^2 L^2}{K} m_2 C \| (u,v) \| . \end{aligned}$$

Therefore, we deduce

$$\| v \| \geq v(t) \geq \mu \frac{m_n^2 L^2}{K} m_2 C \| (u,v) \| > \tilde{\mu_0} \frac{m_n^2 L^2}{K} m_2 C \| (u,v) \| = \| (u,v) \| .$$

and so, $\| (u,v) \| = \| u \| + \| v \| \geq \| v \| > \| (u,v) \|$, which is a contradiction. Therefore, the boundary value problem (1.1)-(1.2) has no positive solution.

(iii) Because $f_0, f_\infty, g_0, g_\infty > 0$, we deduce as above, that there exist $m_1, m_2 > 0$ such that $f(u,v) \geq m_1(u+v),\ g(u,v) \geq m_2(u+v), \forall\ u,v \geq 0$. We define $\tilde{\tilde{\lambda}}_0 = \frac{k}{2m_n^2 L^2 m_1 A}\left(= \frac{\tilde{\lambda_0}}{2}\right)$ and $\tilde{\tilde{\mu}}_0 = \frac{k}{2m_n^2 L^2 m_2 C}\left(= \frac{\tilde{\mu_0}}{2}\right)$. Then for every $\lambda > \tilde{\tilde{\lambda}}_0$ and $\mu > \tilde{\tilde{\mu}}_0$, the problem (1.1)-(1.2) has no positive solution.

Indeed, let $\lambda > \tilde{\tilde{\lambda}}_0$ and $\mu > \tilde{\tilde{\mu}}_0$. We suppose that (1.1)-(1.2) has a positive solution $(u(t), v(t)), t \in [a,b]$. Then in a similar manner as above, we deduce

$$\| u \| \geq \lambda \frac{m_n^2 L^2}{K} m_1 A \| (u,v) \|,\ \| v \| \geq \mu \frac{m_n^2 L^2}{K} m_2 C \| (u,v) \|,$$

and so,

$$\begin{aligned} \| (u,v) \| &= \| u \| + \| v \| \\ &\geq \lambda \frac{m_n^2 L^2}{K} m_1 A \| (u,v) \| + \mu \frac{m_n^2 L^2}{K} m_2 C \| (u,v) \| \\ &> \tilde{\tilde{\lambda}}_0 \frac{m_n^2 L^2}{K} m_1 A \| (u,v) \| + \tilde{\tilde{\mu}}_0 \frac{m_n^2 L^2}{K} m_2 C \| (u,v) \| \\ &= \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| = \| (u,v) \| \end{aligned}$$

which is a contradiction. Therefore, the boundary value problem (1.1)-(1.2) has no positive solution. $\square$

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