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SOME SMALL DEVIATION THEOREMS FOR ARBITRARY RANDOM FIELDS WITH RESPECT TO BINOMIAL DISTRIBUTIONS INDEXED BY AN INFINITE TREE ON GENERALIZED RANDOM SELECTION SYSTEMS†

FANG LI* AND KANGKANG WANG

Abstract. In this paper, we establish a class of strong limit theorems, represented by inequalities, for the arbitrary random field with respect to the product binomial distributions indexed by the infinite tree on the generalized random selection system by constructing the consistent distribution and a nonnegative martingale with pure analytical methods. As corollaries, some limit properties for the Markov chain field with respect to the binomial distributions indexed by the infinite tree on the generalized random selection system are studied.

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1. Introduction

A tree is a graph $S = \{T, E\}$ which is connected and contains no circuits. Given any two vertices $\sigma, t(\ \sigma \neq t \in T)$, let $\overline{\sigma t}$ be the unique path connecting σ and t. Define the graph distance $d(\sigma, t)$ to be the number of edges contained in the path $\overline{\sigma t}$.

Let T be an arbitrary infinite tree that is partially finite (i.e. it has infinite vertices, and each vertex connects with finite vertices) and has a root o. For a better explanation of the infinite root tree T, we take Cayley tree $T_{C,N}$ for example. It's a special case of the tree T, the root o of Cayley tree has N neighbors and all the other vertices of it have $N + 1$ neighbors each (see Fig.1).

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Let σ, t be vertices of the infinite tree T. Write $t \le \sigma$ $(\sigma, t \neq -1)$ if t is on the unique path connecting o to σ, and $|\sigma|$ for the number of edges on this path. For any two vertices σ, t of the tree T, denote by $\sigma \wedge t$ the vertex farthest from o satisfying $\sigma \wedge t \le \sigma$ and $\sigma \wedge t \le t$.

The set of all vertices with distance n from root o is called the n-th generation of T, which is denoted by L_n. We say that L_n is the set of all vertices on level n. We denote by $T^{(n)}$ the subtree of the tree T containing the vertices from level 0 (the root o) to level n. Let $t(\neq o)$be a vertex of the tree T. We denote the first predecessor of t by 1_t, the second predecessor of t by 2_t, and denote by n_t the n-th predecessor of t. Let $X^A = \{X_t, t \in A\}$, and let x^A be a realization of X^A and denote by $|A|$ the number of vertices of A .

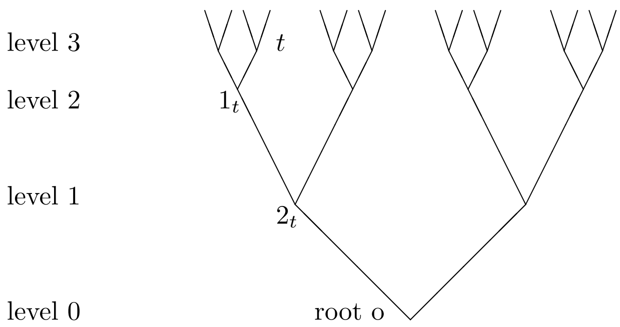

Fig.1 An infinite tree $T_{C,2}$

Suppose that $S = \{0, 1, 2, 3, \cdots, N\}$ is a finite state space. Let $\Omega = S^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on T and taking values in S, and $\mathcal{F}$ be the smallest Borel field containing all cylinder sets in Ω, μ be the probability measure on $(\Omega, \mathcal{F})$. Let $X = \{X_t, t \in T\}$ be the coordinate stochastic process defined on the measurable space $(\Omega, \mathcal{F})$; that is, for any $\omega = \{\omega(t), t \in T\}$, define

$$X_t(\omega) = \omega(t),\ t \in T^{(n)}$$

$$X^{T^{(n)}} \triangleq \{X_t, t \in T^{(n)}\},\ \mu(X^{T^{(n)}} = x^{T^{(n)}}) = \mu(x^{T^{(n)}}). \tag{1}$$

Now we give a definition of Markov chain fields on the tree T by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see[4]).

Definition 1. Let $\{p_t, t \in T^{(n)}\}$ be a sequence of positive real numbers, denote $p_t \in (0, 1),\ t \in T^{(n)}$. If

$$\mu_P(x^{T^{(n)}}) = \prod_{k=0}^{n} \prod_{t \in L_k} C_N^{X_t} p_t^{X_t} (1 - p_t)^{N - X_t},\ n \ge 0. \tag{2}$$

Then μ_P will be called a random field which obeys the product binomial distributions (2) indexed by the homogeneous tree T.

Definition 2. Let $\{f_n(x_1, \cdots, x_n), n \geq 1\}$ be a sequence of real-valued functions defined on $S^n (n = 1, 2, \cdots)$, which will be called the generalized selection functions if $\{f_n, n \geq 1\}$ take values in a nonnegative interval of $[0, b]$. We let

$$\begin{aligned} &Y_0 = y \ (y \textit{ is an arbitrary real number}), \\ &Y_t = f_{|t|}(X_{1_t}, X_{2_t}, \cdots, X_0), \quad |t| \geq 1, \end{aligned} \tag{3}$$

where $|t|$ stands for the number of the edges on the path from the root o to t. Then $\{Y_t,\ t \in T^{(n)}\}$ is called the generalized gambling system or the generalized random selection system indexed by an infinite tree with uniformly bounded degree. The traditional random selection system $\{Y_n,\ n \geq 0\}$[10] takes values in the set of $\{0, 1\}$.

We first explain the conception of the traditional random selection, which is the crucial part of the gambling system. We give a set of real-valued functions $f_n(x_1, \cdots, x_n)$ defined on $S^n (n = 1, 2, \cdots)$, which will be called the random selection function if they take values in a two-valued set $\{0, 1\}$. Then let

$$\begin{aligned} Y_0 &= y (y \textit{ is an arbitrary real number}), \\ Y_{n+1} &= f_n(X_1, \cdots, X_n), \quad n \geq 0. \end{aligned}$$

where $\{Y_n, n \geq 1\}$ be called the gambling system (the random selection system).

In order to explain the real meaning of the notion of the random selection, we consider the traditional gambling model. Let $\{X_n, n \geq 0\}$ be an independent sequence of random variables, and $\{g_n(x), n \geq 1\}$ be a real-valued function sequence defined on S. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g_n(X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g_n(X_n)$ is determined by the gambling rules, and $\{Y_n, n \geq 0\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $\{Y_n, n \geq 1\}$ by the results of the previous trials. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on Y_{n-1} as $n \geq 1$, and b_0 is a constant. Thus $\sum_{k=1}^n Y_k g_k(X_k)$ represents the total gain in the first n trials, $\sum_{k=1}^n b_k$ the accumulated entrance fees, and $\sum_{k=1}^n [Y_k g_k(X_k) - b_k]$ the accumulated net gain. Motivated by the classical definition of "fairness" of game of chance (see Kolmogorov[10]), we introduce the following definition:

Definition 3. The game is said to be fair, if for almost all $\omega \in \{\omega : \sum_{k=1}^\infty Y_k = \infty\}$, the accumulated net gain in the first n trial is to be of smaller order of magnitude than the accumulated stake $\sum_{k=1}^n Y_k$ as n tends to infinity, that is

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^n Y_k} \sum_{k=1}^n [Y_k g_k(X_k) - b_k] = 0 \ a.s. \ on \ \{\omega : \sum\nolimits_{k=1}^\infty Y_k = \infty\}.$$

Definition 4. Let $\{X_t, t \in T\}$ be an arbitrary random field defined by (1), and $\{p_t, t \in T\}$ be a sequence of positive real numbers, $p_t \in (0, 1)$. We set

$$R_n(\omega) = \frac{\mu_P(X^{T^{(n)}})}{\mu(X^{T^{(n)}})} = \frac{\prod_{k=0}^{n}\prod_{t\in L_k} C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t}}{\mu(X^{T^{(n)}})}, \tag{4}$$

$$r_n(\omega) = \ln R_n(\omega) = \sum_{k=0}^{n}\sum_{t\in L_k} \ln[C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t}] - \ln \mu(X^{T^{(n)}}). \tag{5}$$

for the likelihood ratio and the logarithmic likelihood ratio of $\{X_t, t \in T\}$, relative to the product binomial distribution:

$$\prod_{k=0}^{n}\prod_{t\in L_k} C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t},\ X_t \in S,\ t \in L_k,\ 0 \le k \le n.$$

where log is natural logarithm. ω is the sample point. Let

$$r(\omega) = \limsup_{n\to\infty} (1\Big/|T^{(n)}|) r_n(\omega).$$

Then it will be shown in (31) that $r(\omega) \le 0$ $a.s.$ in any case. Hence, $r_n(\omega)$ can be used as a random measure of the deviation between the true joint distribution function $\mu(X^{T^{(n)}})$ and the reference product binomial distribution function $\prod_{k=0}^{n}\prod_{t\in L_k} C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t}$. Roughly speaking, this deviation may be regarded as the one between $X^{T^{(n)}}$ and the independent case. The smaller $r_n(\omega)$ is, the smaller the deviation is.

There have been some works on limit theorems for tree-indexed stochastic processes. Benjamini and Peres have given the notion of the tree-indexed homogeneous Markov chains and studied the recurrence and ray-recurrence for them (see[1]).Liu and Ma have studied strong limit theorem for the average of ternary functions of Markov chains in bi-infinite random environments. (see[2]). Yang proved some strong laws of large numbers for asymptotic even-odd Markov chains indexed by a homogeneous tree (see[5]). Li and Yang have studied strong convergence properties of pairwise NQD random sequences (see[6]). Ye and Berger, by using Pemantle's result and a combinatorial approach, have studied the asymptotic equipartition property (AEP) in the sense of convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree(see[9-10]). Peng and Yang have studied a class of small deviation theorems for functionals of random field and asymptotic equipartition property (AEP) for arbitrary random field on a homogeneous trees (see[8]). Recently, Yang have studied some limit theorems for countable homogeneous Markov chains indexed by a homogeneous tree and strong law of large numbers and the asymptotic equipartition property (AEP) for finite homogeneous Markov chains indexed by a homogeneous tree (see[7] and [11]). Wang has also studied some Shannon-McMillan approximation theorems for arbitrary random field on the generalized Bethe tree (see[12]). Zhong and Yang (see[14]) have studied some asymptotic equipartition properties

(AEP) for asymptotic circular Markov chains. Wang (see[15]) has also discussed some small deviation theorems for stochastic truncated function sequence for arbitrary random field indexed by a homogeneous tree.

It is known to all that the binomial distribution is one of the classical probability distributions. It has comprehensive applications in all fields of the economical life. In this paper, our aim is to establish a class of strong limit theorems represented by the inequalities for the arbitrary random field with respect to the product binomial distributions indexed by the infinite tree by constructing the consistent distribution and a nonnegative martingale with pure analytical methods. As corollaries, some limit theorems for the Markov chain field and the random field which obeys binomial distributions indexed by the infinite tree are generalized.

2. Main results

Lemma 1 ([15]). *Let μ_1 and μ_2 be two probability measures on $(\Omega, \mathcal{F})$, $D \in \mathcal{F}$, denote $\alpha > 0$. Let $\{\sigma_n, n \geq 0\}$ be a nonnegative stochastic sequence such that*

$$\liminf_{n} \frac{\sigma_n}{n^\alpha} > 0, \ \mu_1 - a.s. \ \omega \in D, \tag{6}$$

then

$$\limsup_{n\to\infty} \frac{1}{\sigma_n} \ln \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \leq 0. \ \mu_1 - a.s. \ \omega \in D. \tag{7}$$

In particular, if $\sigma_n = |T^{(n)}|$, we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \ln \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \leq 0, \ \mu_1 - a.s. \tag{8}$$

Proof. See reference [15]. □

Theorem 1. *Let $X = \{X_t, t \in T\}$ be an arbitrary random field defined by (1) taking values in $S = \{0, 1, 2, 3, \cdots, N\}$ indexed by the infinite tree T. We put*

$$D(c) = \{\omega : \liminf_{n\to\infty} \left(\sum_{k=0}^{n} \sum_{t\in L_k} Y_t \Big/ n^\beta \right) > 0, \ \liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n} \sum\limits_{t\in L_k} Y_t} r_n(\omega) \geq -c\} \tag{9}$$

Denote $\alpha > 1$, $\beta > 0$, $0 \leq c < (\alpha - 1)^2 \alpha^b bN$, then

$$\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n} \sum\limits_{t\in L_k} Y_t} \sum_{k=0}^{n} \sum_{t\in L_k} (X_t - Np_t) Y_t \leq 2\sqrt{\alpha^b bNc} + c. \ \mu - a.s. \ \omega \in D(c) \tag{10}$$

$$\liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n} \sum\limits_{t\in L_k} Y_t} \sum_{k=0}^{n} \sum_{t\in L_k} (X_t - Np_t) Y_t \geq -2\sqrt{\alpha^{b+1} bNc}. \ \mu - a.s. \ \omega \in D(c). \tag{11}$$

Proof. Consider the probability space $(\Omega, \mathcal{F}, \mu)$, let λ be an arbitrary real number, $\delta_i(j)$ be Kronecker function. We construct the following product distribution:

$$\begin{aligned}
&\mu_Q(X^{T^{(n)}};\lambda)\\
=&\lambda^{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t X_t} \prod_{k=0}^{n}\prod_{t\in L_k}\left(\frac{1}{1+(\lambda^{Y_t}-1)p_t}\right)^N \cdot \prod_{k=0}^{n}\prod_{t\in L_k} C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t},\\
&n\geq 0.
\end{aligned} \tag{12}$$

By (12) we can write

$$\begin{aligned}
&\sum_{x^{L_n}\in S}\mu_Q(X^{T^{(n)}};\lambda)\\
=&\sum_{x^{L_n}\in S}\prod_{k=0}^{n}\prod_{t\in L_k}\lambda^{X_tY_t}\left(\frac{1}{1+(\lambda^{Y_t}-1)p_t}\right)^N C_N^{X_t}p_t^{X_t}(1-p_t)^{N-X_t}\\
=&\mu_Q(X^{T^{(n-1)}};\lambda)\cdot\sum_{x^{L_n}\in S}\prod_{t\in L_n}\lambda^{x_ty_t}\left(\frac{1}{1+(\lambda^{y_t}-1)p_t}\right)^N C_N^{x_t}p_t^{x_t}(1-p_t)^{N-x_t}\\
=&\mu_Q(X^{T^{(n-1)}};\lambda)\cdot\prod_{t\in L_n}\sum_{x_t\in S}\left(\frac{1}{1+(\lambda^{y_t}-1)p_t}\right)^N C_N^{x_t}(\lambda^{y_t}p_t)^{x_t}(1-p_t)^{N-x_t}\\
=&\mu_Q(X^{T^{(n-1)}};\lambda)\cdot\prod_{t\in L_n}\left(\frac{1}{1+(\lambda^{y_t}-1)p_t}\right)^N(1+(\lambda^{y_t}-1)p_t)^N\\
=&\mu_Q(X^{T^{(n-1)}};\lambda).
\end{aligned} \tag{13}$$

Therefore, we know $\mu_Q(x^{T^{(n)}};\lambda),\ n\geq 0$ are a family of consistent distribution functions defined on $S^{T^{(n)}}$. Denote

$$U_n(\lambda,\omega)=\frac{\mu_Q(X^{T^{(n)}};\lambda)}{\mu(X^{T^{(n)}})}. \tag{14}$$

By (4) and (12), we can rewrite (14) as

$$\begin{aligned}
U_n(\lambda,\omega)=&\frac{1}{\mu(X^{T^{(n)}})}\lambda^{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} X_tY_t}\prod_{k=0}^{n}\prod_{t\in L_k}\left(\frac{1}{1+(\lambda^{Y_t}-1)p_t}\right)^N\\
&\cdot\prod_{k=0}^{n}\prod_{t\in L_k}C_N^{X_t}p_t^{X_t}(1-p_t)^{N-X_t}\\
=&\lambda^{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} X_tY_t}\prod_{k=0}^{n}\prod_{t\in L_k}\left(\frac{1}{1+(\lambda^{Y_t}-1)p_t}\right)^N R_n(\omega).
\end{aligned} \tag{15}$$

Since μ and μ_Q are two probability measures, it is easy to see that $\{U_n(\lambda,\omega),\mathcal{F}_n,n\geq 1\}$ is a nonnegative martingale according to Doob's martingale convergence theorem(see[12]). Hence, we have

$$\lim_{n\to\infty}U_n(\lambda,\omega)=U_\infty(\lambda,\omega)<\infty.\ \mu-a.s. \tag{16}$$

By the first inequality of (9), Lemma 1 and (14), we can write

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \ln U_n(\lambda,\omega) \le 0. \ \mu - a.s. \ \omega \in D(c) \tag{17}$$

According to (5) and (15), we can rewrite (17) as

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \left[\sum_{k=0}^{n}\sum_{t\in L_k} X_tY_t \ln\lambda - \sum_{k=0}^{n}\sum_{t\in L_k} N\ln((\lambda^{Y_t}-1)p_t+1) + r_n(\omega)\right] \\ &\le 0. \qquad \mu - a.s. \ \omega \in D(c)\end{aligned} \tag{18}$$

By the limit property of superior limit, we can obtain by the second inequality of (9) and (18) that

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} [\sum_{k=0}^{n}\sum_{t\in L_k} X_tY_t \ln\lambda - \sum_{k=0}^{n}\sum_{t\in L_k} N\ln[(\lambda^{Y_t}-1)p_t+1)] \\ &\le \limsup_{n\to\infty} \frac{-1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} r_n(\omega) \le c. \qquad \mu - a.s. \ \omega \in D(c)\end{aligned} \tag{19}$$

Letting $\lambda \in (1,\alpha)$ and dividing both sides of (19) by $\ln\lambda$, we have

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \left[\sum_{k=0}^{n}\sum_{t\in L_k} X_tY_t - \sum_{k=0}^{n}\sum_{t\in L_k} \frac{N\ln\left((\lambda^{Y_t}-1)p_t+1\right)}{\ln\lambda}\right] \\ &\le \frac{c}{\ln\lambda}. \qquad \mu - a.s. \ \omega \in D(c)\end{aligned} \tag{20}$$

According to the property of superior limit

$$\limsup_{n\to\infty}(a_n-b_n) \le 0 \Rightarrow \limsup_{n\to\infty}(a_n-c_n) \le \limsup_{n\to\infty}(b_n-c_n),$$

By (20) and the inequality

$$1-1/x \le \ln x \le x-1\,(x>0),\lambda^x - 1 - x\ln\lambda \le (x\ln\lambda)^2 e^{|x\ln\lambda|},$$

noticing that $0 \le Y_t \le b, \ t \in T$, we can write

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \sum_{k=0}^{n}\sum_{t\in L_k} Y_t(X_t - Np_t) \\ &\le \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \left[\sum_{k=0}^{n}\sum_{t\in L_k} \frac{N\ln((\lambda^{Y_t}-1)p_t+1)}{\ln\lambda} - \sum_{k=0}^{n}\sum_{t\in L_k} Y_tNp_t\right] + \frac{c}{\ln\lambda}\end{aligned}$$

$$
\begin{aligned}
&=\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}N\left[\frac{\ln((\lambda^{Y_t}-1)p_t+1)}{\ln\lambda}-Y_tp_t\right]+\frac{c}{\ln\lambda}\\
&\le\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}N\left[\frac{(\lambda^{Y_t}-1)p_t}{\ln\lambda}-Y_tp_t\right]+\frac{c}{\ln\lambda}\\
&=\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}N\left[\frac{\lambda^{Y_t}-1-Y_t\ln\lambda}{\ln\lambda}\right]p_t+\frac{c}{\ln\lambda} \qquad (21)\\
&\le\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}NY_t^2\ln\lambda\cdot e^{Y_t\ln\lambda}p_t+\frac{c}{\ln\lambda}\\
&\le\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}NY_tb\,(\lambda-1)\,\lambda^{Y_t}p_t+\frac{c}{1-1/\lambda}\\
&\le\lambda^bbN\,(\lambda-1)\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}Y_t+\frac{c}{\lambda-1}+c\\
&\le\alpha^bbN\,(\lambda-1)+\frac{c}{\lambda-1}+c. \qquad \mu\_a.s.\ \omega\in D(c)
\end{aligned}
$$

It is easy to show that in the case $0<c<(\alpha-1)^2\alpha^bbN$, the function $f(\lambda)=\alpha^bbN\,(\lambda-1)+\frac{c}{\lambda-1}+c$ attains its smallest value $f(\lambda=1+\sqrt{c/(\alpha^bbN)})=2\sqrt{\alpha^bbNc}+c$. Hence by (21), we have

$$
\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}Y_t(X_t-Np_t)\le 2\sqrt{\alpha^bbNc}+c. \quad \mu-a.s.\ \omega\in D(c) \tag{22}
$$

In the case $c=0$, we choose $\lambda_i\in(1,\alpha)(i=1,2,\cdots)$ such that $\lambda_i\to1^+$(as $i\to\infty$), by (21) we have

$$
\limsup_{n\to\infty}\frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k}Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}Y_t(X_t-Np_t)\le 0. \qquad \mu-a.s.\ \omega\in D(0) \tag{23}
$$

It implies that (22) is also valid when $c=0$.

Letting $\lambda\in(\frac{1}{\alpha},1)$, dividing both sides of (19) by $\ln\lambda$, we get

$$\liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\left[\sum_{k=0}^{n}\sum_{t\in L_k} X_tY_t - \sum_{k=0}^{n}\sum_{t\in L_k}\frac{N\ln((\lambda^{Y_t}-1)p_t+1)}{\ln\lambda}\right] \\ \geq \frac{c}{\ln\lambda}. \qquad \mu - a.s.\ \omega\in D(c) \tag{24}$$

In virtue of the property of inferior limit,

$$\liminf_{n\to\infty}(a_n-b_n)\geq 0 \Rightarrow \liminf_{n\to\infty}(a_n-c_n)\geq \liminf_{n\to\infty}(b_n-c_n),$$

By (24) and inequality $1-1/x\leq \ln x\leq x-1, (x>0)$, we can write

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k} Y_t(X_t-Np_t)\\
&\geq \liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\left(\sum_{k=0}^{n}\sum_{t\in L_k}\frac{N\ln((\lambda^{Y_t}-1)p_t+1)}{\ln\lambda} - \sum_{k=0}^{n}\sum_{t\in L_k} Y_tNp_t\right)+\frac{c}{\ln\lambda}\\
&\geq \liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k} N\left[\frac{(\lambda^{Y_t}-1)p_t}{\ln\lambda}-Y_tp_t\right]+\frac{c}{\ln\lambda}\\
&\geq \liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k} N\left(\frac{\lambda^{Y_t}-1-Y_t\ln\lambda}{\ln\lambda}\right)p_t+\frac{c}{\lambda-1}\\
&\geq \liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k} NY_t^2\ln\lambda\cdot e^{-Y_t\ln\lambda}p_t+\frac{c}{\lambda-1} \qquad (25)\\
&\geq \left(\frac{\lambda-1}{\lambda}\right) bN\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k} Y_t\cdot\lambda^{-Y_t}p_t+\frac{c}{\lambda-1}\\
&\geq (\lambda-1)\,\lambda^{-b-1}bN\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k} Y_t+\frac{c}{\lambda-1}\\
&\geq (\lambda-1)\,\alpha^{b+1}bN+\frac{c}{\lambda-1}. \qquad \mu - a.s.\ \omega\in D(c)
\end{aligned}$$

When $0<c<(\alpha-1)^2\alpha^b bN$, we can get the function $h(\lambda)=(\lambda-1)\,\alpha^{b+1}bN+\frac{c}{\lambda-1}$ attains its smallest value $h(\lambda=1-\sqrt{c/(\alpha^{b+1}bN)})=-2\sqrt{\alpha^{b+1}bNc}$. Hence, it can follows from (25) that

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \sum_{k=0}^{n}\sum_{t\in L_k} Y_t(X_t - Np_t) \geq -2\sqrt{\alpha^{b+1}bNc}. \quad \mu - a.s.\ \omega \in D(c) \tag{26}$$

In the case $c = 0$, we select $\lambda_i \in (\frac{1}{\alpha}\ , 1)(i = 1, 2, \cdots)$ such that $\lambda_i \to 1^-$(as $i \to \infty$), by (25) we attain

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \sum_{k=0}^{n}\sum_{t\in L_k} Y_t(X_t - Np_t) \geq 0. \qquad \mu - a.s.\ \omega \in D(0) \tag{27}$$

It means that (26) also holds in the case $c = 0$. $\square$

Corollary 1. *Under the assumption of Theorem 1, we have*

$$\lim_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \sum_{k=0}^{n}\sum_{t\in L_k} Y_t(X_t - Np_t) = 0. \qquad \mu - a.s.\ \omega \in D(0) \tag{28}$$

Proof. Letting $c = 0$ in Theorem 1, (28) follows from (10), (11) immediately. $\square$

Corollary 2. *Let $X = \{X_t, t \in T\}$ be a random field taking values in $S = \{0, 1, 2, 3, \cdots, N\}$ which obeys the product binomial distributions (2) indexed by the infinite tree T. Then*

$$\lim_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} \sum_{k=0}^{n}\sum_{t\in L_k} Y_t(X_t - Np_t) = 0. \qquad \mu_P - a.s.\ \omega \in D(\omega) \tag{29}$$

where $D(\omega)$ is defined as (30).

Proof. At the moment, we know that $\mu_P \equiv \mu$. Therefore, we obtain $r_n(\omega) \equiv 0$, $D(0) = D(\omega)$. (29) follows from (28) immediately. $\square$

Corollary 3. *Let $X = \{X_t, t \in T\}$ be an arbitrary random field indexed by an infinite tree. $r_n(\omega)$ is defined by (5). Denote $\beta > 0$,*

$$D(\omega) = \{\omega : \liminf_{n\to\infty} \left(\sum_{k=0}^{n}\sum_{t\in L_k} Y_t \Big/ n^{\beta} \right) > 0\}. \tag{30}$$

Then

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n}\sum_{t\in L_k} Y_t} r_n(\omega) \leq 0. \qquad \mu - a.s.\ \omega \in D(\omega) \tag{31}$$

Proof. Letting $\lambda = 1$ in (17), we get by (18)

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \ln U_n(1,\omega) &= \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \ln R_n(\omega) \\ &= \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} r_n(\omega) \\ &\le 0.\end{aligned} \tag{32}$$

(31) follows from (32) immediately. □

Corollary 4. *Under the assumption of Theorem 1, we have*

$$\lim_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} r_n(\omega) = 0. \qquad \mu - a.s.\ \omega \in D(0) \tag{33}$$

Proof. Letting $c = 0$ in Theorem 1, we obtain

$$D(0) = \{\omega : \liminf_{n\to\infty} \left(\sum_{k=0}^{n}\sum_{t\in L_k} Y_t \Big/ n^{\beta} \right) > 0,\ \liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} r_n(\omega) \ge 0\}.$$

(33) follows from (31) and $D(0)$ directly. □

3. Some limit theorems for Markov chain field with respect to the binomial distributions.

Definition 5 (see [7])**.** Let T be an infinite tree, $S = \{0, 1, 2, \cdots, N\}$ be a finite state space, $\{X_t, t \in T\}$ be a collection of $S-$valued random variables defined on the measurable space $\{\Omega, \mathcal{F}\}$. Let

$$q = \{q(x), x \in S\} \tag{34}$$

be a distribution on S, and

$$Q = (Q(y|x)),\ x, y \in S \tag{35}$$

be a strictly positive stochastic matrix on S^2. If for any vertices t, τ,

$$Q(X_t = y | X_{1_t} = x, X_\tau\ for\ t \wedge \tau \le 1_t) = Q(X_t = y | X_{1_t} = x) = Q(y|x),\ \ \forall x, y \in S,$$
$$Q(X_0 = x) = q(x),\ \ \forall x \in S.$$

$\{X_t, t \in T\}$ will be called $S-$valued Markov chains indexed by an infinite tree with the initial distribution (34) and transition matrix (35).

Definition 6. Let $Q = Q(j|i)$ and $q = (q(0), q(1) \cdots, q(N))$ be defined as before, μ_Q be another probability measure on $(\Omega, \mathcal{F})$. If

$$\mu_Q(x_0) = q(x_0), \tag{36}$$

$$\mu_Q(x^{T^{(n)}}) = q(x_0)\prod_{k=1}^{n}\prod_{t\in L_k} Q(x_t|x_{1_t}), \quad n \geq 1. \tag{37}$$

then μ_Q will be called a Markov chain field on the infinite tree T determined by the stochastic matrix Q and the initial distribution q.

Theorem 2. *Let $X = \{X_t, t \in T\}$ be a Markov chain field indexed by an infinite tree with the initial distribution (36) and the joint distribution (37). Denote $Y_t \in [a,b], (a > 0)$ $t \in T$. If*

$$\sum_{i\in S}\sum_{j\in S}\left[\frac{Q(j|i)}{C_N^j p_t^j (1-p_t)^{N-j}} - 1\right]^+ \leq a \cdot c. \tag{38}$$

Then

$$\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}(X_t - Np_t)Y_t \leq 2\sqrt{\alpha^b bNc} + c.\ \mu_Q - a.s.\ \omega \in D(\omega) \tag{39}$$

$$\liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=0}^{n}\sum_{t\in L_k}(X_t - Np_t)Y_t \geq -2\sqrt{\alpha^{b+1} bNc}.\ \mu_Q - a.s.\ \omega \in D(\omega) \tag{40}$$

Proof. Let $\mu = \mu_Q$, by (4), (5) and (38) we can write

$$\limsup_{n\to\infty} \frac{-1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} r_n(\omega) = \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \ln \frac{\mu_Q(X^{T^{(n)}})}{\mu_P(X^{T^{(n)}})}$$

$$= \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \ln \frac{q(X_0)\prod\limits_{k=1}^{n}\prod\limits_{t\in L_k} Q(X_t|X_{1_t})}{\prod\limits_{k=0}^{n}\prod\limits_{t\in L_k} C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t}}$$

$$\leq \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \ln \frac{q(X_0)}{C_N^{X_0} p_t^{X_0}(1-p_t)^{N-X_0}}$$

$$+ \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=1}^{n}\sum_{t\in L_k} \ln \frac{Q(X_t|X_{1_t})}{C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t}}$$

$$\leq \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=1}^{n}\sum_{t\in L_k}\left[\frac{Q(X_t|X_{1_t})}{C_N^{X_t} p_t^{X_t}(1-p_t)^{N-X_t}} - 1\right]^+$$

$$= \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t}\sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i\in S}\sum_{j\in S}\delta_i(X_{1_t})\delta_j(X_t)\left[\frac{Q(j|i)}{C_N^j p_t^j(1-p_t)^{N-j}} - 1\right]^+$$

$$\le \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i\in S}\sum_{j\in S}\left[\frac{Q(j|i)}{C_N^j p_t^j(1-p_t)^{N-j}} - 1\right]^+$$

$$\le \sum_{i\in S}\sum_{j\in S}\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} \sum_{k=1}^{n}\sum_{t\in L_k}\left[\frac{Q_k(j|i)}{C_N^j p_t^j(1-p_t)^{N-j}} - 1\right]^+$$

$$\le \sum_{i\in S}\sum_{j\in S}\limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} a} \sum_{k=1}^{n}\sum_{t\in L_k}\left[\frac{Q_k(j|i)}{C_N^j p_t^j(1-p_t)^{N-j}} - 1\right]^+$$

$$= \sum_{i\in S}\sum_{j\in S}\limsup_{n\to\infty} \frac{|T^{(n)}|-1}{a\cdot|T^{(n)}|}\left[\frac{Q(j|i)}{C_N^j p_t^j(1-p_t)^{N-j}} - 1\right]^+$$

$$= \sum_{i\in S}\sum_{j\in S}\frac{1}{a}\left[\frac{Q(j|i)}{C_N^j p_t^j(1-p_t)^{N-j}} - 1\right]^+ \le c.\ a.s. \tag{41}$$

Hence, we obtain by (41)

$$\liminf_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{n}\sum\limits_{t\in L_k} Y_t} r_n(\omega) \ge -c.\ a.s.\ \omega \in D(\omega) \tag{42}$$

It means that $D(c) = D(\omega)$. (39), (40) follow from (10), (11) immediately. $\square$

4. Conclusion.

In this paper, we mainly investigate a kind of small deviation theorems, represented by inequalities, for the arbitrary random field with respect to the product binomial distributions indexed by the infinite tree on the generalized random selection system by constructing a series of consistent distributions and a nonnegative martingale with pure analytical methods. As results, some limit properties for the Markov chain field with respect to the binomial distributions indexed by the infinite tree on the generalized random selection system are obtained.

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Fang Li received M.Sc. from Jiangsu University. Since 2005 she has been at Anhui Normal University. Her research interests include limit theory in probability theory and information theory.

College of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, China.
e-mail: `lifangahnu@aliyun.com`

Kangkang Wang received M.Sc. from Jiangsu University and Ph.D at Nanjing University of Aeronautics and Astronautics. Since 2005 he has been at Jiangsu University of Science and Technology. His research interests include limit theory in probability theory and information theory.

School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, China.
e-mail: `wkk.cn@126.com`