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PAIRWISE FUZZY REGULAR VOLTERRA SPACES

G. THANGARAJ* AND V. CHANDIRAN

Abstract. In this paper the concepts of pairwise fuzzy regular Volterra spaces and pairwise fuzzy weakly regular Volterra spaces are introduced. Several characterizations of pairwise fuzzy regular Volterra spaces and pairwise fuzzy weakly regular Volterra spaces are investigated.

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1. Introduction

The usual notion of set topology was generalized with the introduction of fuzzy topology by C.L.Chang [4] in 1968, based on the concept of fuzzy sets invented by L.A.Zadeh [20] in 1965. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. Today fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. In 1989, A.Kandil [9] introduced the concept of fuzzy bitopological spaces. The concepts of Volterra spaces have been studied extensively in classical topology in [5], [6], [7] and [8]. In 1992, G.Balasubramanian [2] introduced the concept of fuzzy G_δ-set in fuzzy topological spaces. The concept of Volterra spaces in fuzzy setting was introduced and studied by G.Thangaraj and S.Soundararajan in [18]. The concept of pairwise Volterra spaces in fuzzy setting was introduced in [12] and studied by the authors in [13] and [14]. In this paper, the concepts of pairwise fuzzy regular G_δ-set and pairwise fuzzy regular F_σ-set are introduced and studied. By means of pairwise fuzzy regular G_δ-set, the concept of pairwise fuzzy regular Volterra spaces and

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*Corresponding author.

pairwise fuzzy weakly regular Volterra spaces are introduced and several characterizations of pairwise fuzzy regular Volterra spaces and pairwise fuzzy weakly regular Volterra spaces are studied.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are fuzzy topologies on the non-empty set X.

Definition 2.1. A fuzzy set λ in a set X is a function from X to $[0, 1]$, that is, $\lambda : X \longrightarrow [0, 1]$.

Definition 2.2. Let λ and μ be fuzzy sets in X. Then for all $x \in X$,

(i). $\lambda = \mu \Leftrightarrow \lambda(x) = \mu(x)$
(ii). $\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x)$
(iii). $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = max\{\lambda(x), \mu(x)\}$
(iv). $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = min\{\lambda(x), \mu(x)\}$
(v). $\eta = \lambda^c \Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as

(vi). $\psi(x) = sup_i\{\lambda_i(x) \ / \ x \in X\}$
(vii). $\delta(x) = inf_i\{\lambda_i(x) \ / \ x \in X\}$.

Definition 2.3. The closure and interior of a fuzzy set λ in a fuzzy topological space (X, T) are defined as

(i). $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$
(ii). $cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.4 ([1])**.** *For a fuzzy set λ of a fuzzy topological space X,*

(i). $1 - int(\lambda) = cl(1 - \lambda)$
(ii). $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.5 ([2])**.** Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$ for each $\lambda_i \in T$.

Definition 2.6 ([2])**.** Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty} \lambda_i$ for each $1 - \lambda_i \in T$.

Lemma 2.7 ([1])**.** *For a family $\mathcal{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leq cl\big(\vee(\lambda_\alpha)\big)$. In case $\mathcal{A}$ is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leq int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.8 ([12])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set if $\lambda \in T_i$, $(i = 1, 2)$. The complement of pairwise fuzzy open set in (X, T_1, T_2) is called a pairwise fuzzy closed set.

Definition 2.9 ([12]). A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy G_δ-set if $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.10 ([12]). A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy F_σ-set if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy closed sets in (X, T_1, T_2).

Definition 2.11 ([11]). A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy dense set if $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$.

Definition 2.12 ([11]). A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy nowhere dense set if $int_{T_1}cl_{T_2}(\lambda) = 0 = int_{T_2}cl_{T_1}(\lambda)$.

Definition 2.13 ([10]). Let (X, T_1, T_2) be a fuzzy bitopological space and λ be any fuzzy set in (X, T_1, T_2). Then λ is called a pairwise fuzzy β-open set if $\lambda \leq cl_{T_1}int_{T_2}cl_{T_1}(\lambda)$ and $\lambda \leq cl_{T_2}int_{T_1}cl_{T_2}(\lambda)$.

Definition 2.14 ([13]). A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy σ-nowhere dense set if λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2) such that $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$.

Definition 2.15 ([12]). A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy Volterra space if $cl_{T_i}\left(\wedge_{k=1}^{N}(\lambda_k)\right) = 1$, $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Definition 2.16 ([12]). A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy weakly Volterra space if $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Definition 2.17 ([15]). Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a pairwise fuzzy first category set if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Any other fuzzy set in (X, T_1, T_2) is said to be a pairwise fuzzy second category set in (X, T_1, T_2).

Definition 2.18 ([15]). If λ is a pairwise fuzzy first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a pairwise fuzzy residual set in (X, T_1, T_2).

Definition 2.19 ([19]). Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a pairwise fuzzy σ-first category set if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Any other fuzzy set in (X, T_1, T_2) is said to be a pairwise fuzzy σ-second category set in (X, T_1, T_2).

Definition 2.20 ([19]). A fuzzy bitopological space (X, T_1, T_2) is called pairwise fuzzy σ-first category space if the fuzzy set 1_X is a pairwise fuzzy σ-first category set in (X, T_1, T_2). That is., $1_X = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Otherwise, (X, T_1, T_2) will be called a pairwise fuzzy σ-second category space.

3. Pairwise fuzzy regular G_δ-sets and pairwise fuzzy regular F_σ-sets

Definition 3.1. Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a pairwise fuzzy regular G_δ-set if $\lambda = \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(\lambda_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$, where (λ_k)'s are fuzzy sets in (X, T_1, T_2).

Definition 3.2. Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set μ in (X, T_1, T_2) is called a pairwise fuzzy regular F_σ-set if $\mu = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$, where (μ_k)'s are fuzzy sets in (X, T_1, T_2).

Proposition 3.3. *If λ is a pairwise fuzzy regular G_δ-set in a fuzzy bitopological space (X, T_1, T_2) if and only if $1 - \lambda$ is a pairwise fuzzy regular F_σ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy regular G_δ-set in (X, T_1, T_2). Then $\lambda = \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(\lambda_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$, where (λ_k)'s are fuzzy sets in (X, T_1, T_2). Now $1 - \lambda = 1 - \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(\lambda_k)\big) = \vee_{k=1}^{\infty}\big(1 - int_{T_i} cl_{T_j}(\lambda_k)\big) = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(1 - \lambda_k)\big)$. Let $\mu_k = 1 - \lambda_k$. Hence $1 - \lambda = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$ implies that $1 - \lambda$ is a pairwise fuzzy regular F_σ-set in (X, T_1, T_2).

Conversely, let λ be a pairwise fuzzy regular F_σ-set in (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$ where (μ_k)'s are fuzzy sets in (X, T_1, T_2). Now $1 - \lambda = 1 - \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big) = \wedge_{k=1}^{\infty}\big(1 - cl_{T_i} int_{T_j}(\mu_k)\big) = \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(1 - \mu_k)\big)$. Let $1 - \mu_k = \lambda_k$. Hence $1 - \lambda = \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(\lambda_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$ implies that $1 - \lambda$ is a pairwise fuzzy regular G_δ-set in (X, T_1, T_2). □

Definition 3.4 ([3])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy regular open set in (X, T_1, T_2) if $int_{T_1} cl_{T_2}(\lambda) = \lambda = int_{T_2} cl_{T_1}(\lambda)$.

Definition 3.5 ([3])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy regular closed set in (X, T_1, T_2) if $cl_{T_1} int_{T_2}(\lambda) = \lambda = cl_{T_2} int_{T_1}(\lambda)$.

Proposition 3.6. *Let (X, T_1, T_2) be a fuzzy bitopological space.*

(a). *If λ is a pairwise fuzzy open set in (X, T_1, T_2), then $cl_{T_i}(\lambda)$, $(i = 1, 2)$ is a pairwise fuzzy regular closed set in (X, T_1, T_2).*

(b). *If μ is a pairwise fuzzy closed set in (X, T_1, T_2), then $int_{T_i}(\mu)$, $(i = 1, 2)$ is a pairwise fuzzy regular open set in (X, T_1, T_2).*

Proof. (a). Let λ be a pairwise fuzzy open set in (X, T_1, T_2) and $int_{T_j} cl_{T_i}(\lambda) \leq cl_{T_i}(\lambda)$, $(i \neq j$ and $i, j = 1, 2)$ implies that $cl_{T_i} int_{T_j} cl_{T_i}(\lambda) \leq cl_{T_i} cl_{T_i}(\lambda) = cl_{T_i}(\lambda)$. Hence $cl_{T_i} int_{T_j}\big(cl_{T_i}(\lambda)\big) \leq cl_{T_i}(\lambda) \longrightarrow (1)$. Since λ is a pairwise fuzzy open set, we have $\lambda = int_{T_j}(\lambda)$, $(j = 1, 2)$. Now $\lambda = int_{T_j}(\lambda) \leq int_{T_j} cl_{T_i}(\lambda)$ implies that $\lambda \leq int_{T_j} cl_{T_i}(\lambda)$. Hence $cl_{T_i}(\lambda) \leq cl_{T_i} int_{T_j}\big(cl_{T_i}(\lambda)\big) \longrightarrow (2)$. From (1) and (2) we have $cl_{T_i}int_{T_j}\big(cl_{T_i}(\lambda)\big) = cl_{T_i}(\lambda)$, $(i \neq j$ and $i, j = 1, 2)$. Therefore $cl_{T_i}(\lambda)$ is a pairwise fuzzy regular closed set in (X, T_1, T_2).

(b). Let μ be a pairwise fuzzy closed set in (X, T_1, T_2). Then $1 - \mu$ is a pairwise fuzzy open set in (X, T_1, T_2). By (a), $cl_{T_i}(1 - \mu)$ is a pairwise fuzzy regular closed set in (X, T_1, T_2). Then $1 - int_{T_i}(\mu)$ is a pairwise fuzzy regular closed set in (X, T_1, T_2). Hence $int_{T_i}(\mu)$ is a pairwise fuzzy regular open set in (X, T_1, T_2). □

Proposition 3.7. *Let (X, T_1, T_2) be a fuzzy bitopological space.*

(1). *If λ is a pairwise fuzzy regular G_δ-set in (X, T_1, T_2), then $\lambda = \wedge_{k=1}^{\infty}(\delta_k)$, where (δ_k)'s are pairwise fuzzy regular open sets in (X, T_1, T_2).*

(2). *If μ is a pairwise fuzzy regular F_σ-set in (X, T_1, T_2), then $\mu = \vee_{k=1}^{\infty}(\eta_k)$, where (η_k)'s are pairwise fuzzy regular closed sets in (X, T_1, T_2).*

Proof. (1). Let λ be a pairwise fuzzy regular G_δ-set in (X, T_1, T_2). Then $\lambda = \wedge_{k=1}^{\infty}\big(int_{T_i}cl_{T_j}(\lambda_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$, where (λ_k)'s are in (X, T_1, T_2). Take $\delta_k = int_{T_i}cl_{T_j}(\lambda_k)$. Now $int_{T_i}cl_{T_j}(\delta_k) = int_{T_i}cl_{T_j}[int_{T_i}cl_{T_j}(\lambda_k)] \leq int_{T_i}cl_{T_j}cl_{T_j}(\lambda_k) = int_{T_i}cl_{T_j}(\lambda_k) = \delta_k$. Hence $int_{T_i}cl_{T_j}(\delta_k) \leq \delta_k \longrightarrow (A)$. Also, $int_{T_i}cl_{T_j}(\delta_k) = int_{T_i}cl_{T_j}[int_{T_i}cl_{T_j}(\lambda_k)] \geq int_{T_i}int_{T_i}cl_{T_j}(\lambda_k) = int_{T_i}cl_{T_j}(\lambda_k) = \delta_k$. Hence $int_{T_i}cl_{T_j}(\delta_k) \geq \delta_k \longrightarrow (B)$. From (A) and (B), we have $int_{T_i}cl_{T_j}(\delta_k) = \delta_k$. Hence (δ_k)'s are pairwise fuzzy regular open sets in (X, T_1, T_2). Therefore $\lambda = \wedge_{k=1}^{\infty}(\delta_k)$, where the fuzzy sets (δ_k)'s are pairwise fuzzy regular open sets in (X, T_1, T_2).

(2). Let μ be a pairwise fuzzy regular F_σ-set in (X, T_1, T_2). Then, by proposition 3.3, $1 - \mu$ is a pairwise fuzzy regular G_δ-set in (X, T_1, T_2). By (1), $1 - \mu = \wedge_{k=1}^{\infty}(\delta_k)$, where the fuzzy sets (δ_k)'s are pairwise fuzzy regular open sets in (X, T_1, T_2). Now $\mu = \vee_{k=1}^{\infty}(1 - \delta_k)$. Let $1 - \delta_k = \eta_k$. Hence $\mu = \vee_{k=1}^{\infty}(\eta_k)$, where the fuzzy sets (η_k)'s are pairwise fuzzy regular closed sets in (X, T_1, T_2). □

Proposition 3.8. *If λ is a pairwise fuzzy regular G_δ-set in a fuzzy bitopological space (X, T_1, T_2), then λ is a pairwise fuzzy G_δ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy regular G_δ-set in (X, T_1, T_2). Then by proposition 3.7, $\lambda = \wedge_{k=1}^{\infty}(\delta_k)$ where the fuzzy sets (δ_k)'s are pairwise fuzzy regular open sets in (X, T_1, T_2). Since every pairwise fuzzy regular open set is a pairwise fuzzy open set in (X, T_1, T_2), (δ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2). Hence $\lambda = \wedge_{k=1}^{\infty}(\delta_k)$, where (δ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2), implies that λ is a pairwise fuzzy G_δ-set in (X, T_1, T_2). □

Proposition 3.9. *If μ is a pairwise fuzzy regular F_σ-set in a fuzzy bitopological space (X, T_1, T_2), then μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2).*

Proof. Let μ be a pairwise fuzzy regular F_σ-set in (X, T_1, T_2). Then by proposition 3.7, $\mu = \vee_{k=1}^{\infty}(\eta_k)$ where the fuzzy sets (η_k)'s are pairwise fuzzy regular closed sets in (X, T_1, T_2). Since every pairwise fuzzy regular closed set is a pairwise fuzzy closed set in (X, T_1, T_2), (η_k)'s are pairwise fuzzy closed sets in

(X, T_1, T_2). Hence $\mu = \vee_{k=1}^{\infty}(\eta_k)$, where (η_k)'s are pairwise fuzzy closed sets in (X, T_1, T_2), implies that μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). □

Proposition 3.10. *If λ is a pairwise fuzzy regular F_σ-set in a fuzzy bitopological space (X, T_1, T_2), then $\vee_{k=1}^{\infty} int_{T_i} int_{T_j}(\lambda_k) \leq \lambda$,* $(i \neq j$ and $i, j = 1, 2)$.

Proof. Let λ be a pairwise fuzzy regular F_σ-set in (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\lambda_k)\big)$, where (λ_k)'s are in (X, T_1, T_2). Now $\lambda = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\lambda_k)\big) \geq \vee_{k=1}^{\infty} int_{T_j}(\lambda_k)$. Hence $\vee_{k=1}^{\infty} int_{T_j}(\lambda_k) \leq \lambda$. Then $\vee_{k=1}^{\infty} int_{T_i} int_{T_j}(\lambda_k) \leq \vee_{k=1}^{\infty} int_{T_j}(\lambda_k) \leq \lambda$. Hence $\vee_{k=1}^{\infty} int_{T_i} int_{T_j}(\lambda_k) \leq \lambda$. □

Proposition 3.11. *If λ is a pairwise fuzzy regular G_δ-set in a fuzzy bitopological space (X, T_1, T_2), then $\lambda \leq \wedge_{k=1}^{\infty} cl_{T_i} cl_{T_j}(\lambda_k)$,* $(i \neq j$ and $i, j = 1, 2)$.

Proof. Let λ be a pairwise fuzzy regular G_δ-set in (X, T_1, T_2). Then $\lambda = \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(\lambda_k)\big)$, where (λ_k)'s are in (X, T_1, T_2). Now $\lambda = \wedge_{k=1}^{\infty}\big(int_{T_i} cl_{T_j}(\lambda_k)\big) \leq \wedge_{k=1}^{\infty} cl_{T_j}(\lambda_k)$. Hence $\lambda \leq \wedge_{k=1}^{\infty} cl_{T_j}(\lambda_k) \leq \wedge_{k=1}^{\infty} cl_{T_i} cl_{T_j}(\lambda_k)$. Hence $\lambda \leq \wedge_{k=1}^{\infty} cl_{T_i} cl_{T_j}(\lambda_k)$. □

Proposition 3.12. *If $cl_{T_j}[\wedge_{k=1}^{\infty} int_{T_i} cl_{T_j}(\lambda_k)] = 1$,* $(i \neq j$ and $i, j = 1, 2)$, *where (λ_k)'s are fuzzy sets in a fuzzy bitopological space (X, T_1, T_2), then (λ_k)'s are pairwise fuzzy β-open sets in (X, T_1, T_2).*

Proof. Let $cl_{T_j}[\wedge_{k=1}^{\infty} int_{T_i} cl_{T_j}(\lambda_k)] = 1$, $(i \neq j$ and $i, j = 1, 2)$, where (λ_k)'s are fuzzy sets in (X, T_1, T_2). Since $cl_{T_j}[\wedge_{k=1}^{\infty} int_{T_i} cl_{T_j}(\lambda_k)] \leq \wedge_{k=1}^{\infty} cl_{T_j}[int_{T_i} cl_{T_j}(\lambda_k)]$, we have $1 \leq \wedge_{k=1}^{\infty} cl_{T_j}[int_{T_i} cl_{T_j}(\lambda_k)]$. That is, $\wedge_{k=1}^{\infty} cl_{T_j}[int_{T_i} cl_{T_j}(\lambda_k)] = 1$. This implies that $cl_{T_j}[int_{T_i} cl_{T_j}(\lambda_k)] = 1$ and hence $\lambda_k \leq cl_{T_j}[int_{T_i} cl_{T_j}(\lambda_k)]$. Therefore (λ_k)'s are pairwise fuzzy β-open sets in (X, T_1, T_2). □

Proposition 3.13. *If $int_{T_j}(\lambda) = 0$,* $(j = 1, 2)$ *for a pairwise fuzzy regular F_σ-set λ in a fuzzy bitopological space (X, T_1, T_2), then λ is a pairwise fuzzy first category set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy regular F_σ-set in (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big)$, $(i \neq j$ and $i, j = 1, 2)$, where (μ_k)'s are fuzzy sets in (X, T_1, T_2). Now $int_{T_j}(\lambda) = 0$ implies that $int_{T_j}\Big(\vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big)\Big) = 0$. But $\vee_{k=1}^{\infty}\Big(int_{T_j}\big(cl_{T_i} int_{T_j}(\mu_k)\big)\Big) \leq int_{T_j}\Big(\vee_{k=1}^{\infty}\big(cl_{T_i} int_{T_j}(\mu_k)\big)\Big)$. Then we have $\vee_{k=1}^{\infty}\Big(int_{T_j}\big(cl_{T_i} int_{T_j}(\mu_k)\big)\Big) = 0$. This implies that $int_{T_j} cl_{T_i}\big(int_{T_j}(\mu_k)\big) = 0$. Also, $int_{T_j} cl_{T_i}\Big(cl_{T_i}\big(int_{T_j}(\mu_k)\big)\Big) = int_{T_j} cl_{T_i}\big(int_{T_j}(\mu_k)\big) = 0$ and hence $cl_{T_i} int_{T_j}(\mu_k)$ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2). Therefore, λ is a pairwise fuzzy first category set in (X, T_1, T_2). □

Definition 3.14 ([16])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy strongly irresolvable space if $cl_{T_1} int_{T_2}(\lambda) = 1 = cl_{T_2} int_{T_1}(\lambda)$ for each pairwise fuzzy dense set λ in (X, T_1, T_2).

Theorem 3.15 ([17]). *If $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$ for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2).*

Proposition 3.16. *If the pairwise fuzzy regular G_δ-set λ is pairwise fuzzy dense in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then λ is a pairwise fuzzy residual set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy regular G_δ-set with $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$. Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space and by theorem 3.15, $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2). That is, $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$. Now $1 - \lambda$ is a pairwise fuzzy regular F_σ-set with $1 - cl_{T_i}(\lambda) = 0$. That is., $1 - \lambda$ is a pairwise fuzzy regular F_σ-set with $int_{T_i}(1 - \lambda) = 0$. Then by proposition 3.13, $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2). Therefore λ is a pairwise fuzzy residual set in (X, T_1, T_2). □

4. Pairwise Fuzzy regular Volterra Spaces

Definition 4.1. A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy regular Volterra space if $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1,\ (i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2).

Proposition 4.2. *If $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0,\ (i = 1, 2)$ where the fuzzy sets (μ_k)'s are pairwise fuzzy regular F_σ-sets with $int_{T_i}(\mu_k) = 0,\ (i = 1, 2)$ in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy regular Volterra space.*

Proof. Suppose that $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0,\ (i = 1, 2)$ where the fuzzy sets (μ_k)'s are pairwise fuzzy regular F_σ-sets with $int_{T_i}(\mu_k) = 0$. Now $1 - int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 1$. Then we have $cl_{T_i}\big(1 - \vee_{k=1}^{N}(\mu_k)\big) = 1$. This implies that $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \mu_k)\big) = 1$. Since (μ_k)'s are pairwise fuzzy regular F_σ-sets in (X, T_1, T_2), by proposition 3.3, $(1 - \mu_k)$'s are pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Also, $int_{T_i}(\mu_k) = 0$ implies that $1 - int_{T_i}(\mu_k) = 1$. Then we have $cl_{T_i}(1 - \mu_k) = 1,\ (1, 2)$. Then $cl_{T_1}cl_{T_2}(1 - \mu_k) = cl_{T_1}(1) = 1$ and $cl_{T_2}cl_{T_1}(1 - \mu_k) = cl_{T_2}(1) = 1$. Hence $(1 - \mu_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Let $\lambda_k = 1 - \mu_k$. Then (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Hence we have $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$ where the (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy regular Volterra space. □

Proposition 4.3. *A fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Volterra space, then (X, T_1, T_2) is a pairwise fuzzy regular Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy Volterra space. Now, consider $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big),\ (i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). By proposition 3.8, the pairwise fuzzy regular G_δ-sets (λ_k)'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence in

$cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big)$, (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Volterra space, $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$. Hence we have $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy regular Volterra space. □

Proposition 4.4. *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy regular Volterra and pairwise fuzzy strongly irresolvable space, then $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0$, $(i = 1, 2)$ where (μ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2).*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy regular Volterra space. Then $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$, $(i = 1, 2)$, where the fuzzy sets (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Now $1 - cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 0$ implies that $int_{T_i}\Big(\vee_{k=1}^{N}(1-\lambda_k)\Big) = 0$. Since the fuzzy sets (λ_k)'s are pairwise fuzzy regular G_δ-sets, $(1-\lambda_k)$'s are pairwise fuzzy regular F_σ-sets in (X, T_1, T_2). By proposition 3.9, $(1-\lambda_k)$'s are pairwise fuzzy F_σ-sets in (X, T_1, T_2). Also, $cl_{T_i}(\lambda_k) = 1$ implies that $1 - cl_{T_i}(\lambda_k) = 0$ and hence $int_{T_i}(1-\lambda_k) = 0$. Let $\mu_k = 1 - \lambda_k$. Then $int_{T_i} int_{T_j}(\mu_k) \leq int_{T_i}(\mu_k) = 0$ implies that $int_{T_i} int_{T_j}(\mu_k) = 0$. Hence (μ_k)'s are pairwise fuzzy F_σ-sets with $int_{T_i} int_{T_j}(\mu_k) = 0$. Then (μ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Therefore $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0$, $(i = 1, 2)$, where (μ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). □

Proposition 4.5. *If the pairwise fuzzy strongly irresolvable space (X, T_1, T_2) is a pairwise fuzzy regular Volterra space, then $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$, $(i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy residual sets in (X, T_1, T_2).*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy regular Volterra space. Then $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$, $(i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). By proposition 3.16, (λ_k)'s are pairwise fuzzy residual sets in (X, T_1, T_2). Hence $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$, $(i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy residual sets in (X, T_1, T_2). □

Proposition 4.6. *If the pairwise fuzzy strongly irresolvable space (X, T_1, T_2) is a pairwise fuzzy regular Volterra space, then $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0$, $(i = 1, 2)$ where the fuzzy sets (μ_k)'s are pairwise fuzzy first category sets in (X, T_1, T_2).*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy regular Volterra space. Then $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$, $(i = 1, 2)$ where the fuzzy sets (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Now $1 - cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 0$ implies that $int_{T_i}\Big(1 - \wedge_{k=1}^{N}(\lambda_k)\Big) = 0$. Then we have $int_{T_i}\Big(\vee_{k=1}^{N}(1-\lambda_k)\Big) = 0$, $(i = 1, 2)$.

By proposition 3.3, the fuzzy sets (λ_k)'s are pairwise fuzzy regular G_δ-sets implies that $(1-\lambda_k)$'s are pairwise fuzzy F_σ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space and by theorem 3.15, $cl_{T_1}(\lambda_k) = 1$ and $cl_{T_2}(\lambda_k) = 1$ in (X, T_1, T_2). That is, $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. Also $cl_{T_i}(\lambda_k) = 1$ implies that $1 - cl_{T_i}(\lambda_k) = 0$. Then $int_{T_i}(1 - \lambda_k) = 0$. Hence the fuzzy sets $(1 - \lambda_k)$'s are pairwise fuzzy F_σ-sets with $int_{T_i}(1 - \lambda_k) = 0$. Therefore by proposition 3.13, $(1 - \lambda_k)$'s are pairwise fuzzy first category sets in (X, T_1, T_2). Let $\mu_k = 1 - \lambda_k$. Hence we have $int_{T_i}\big(\vee_{k=1}^{N} (\mu_k)\big) = 0,\ (i = 1, 2)$ where the fuzzy sets (μ_k)'s are pairwise fuzzy first category sets in (X, T_1, T_2). □

Definition 4.7. A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy regular σ-nowhere dense set if λ is a pairwise fuzzy regular F_σ-set in (X, T_1, T_2) such that $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$.

Proposition 4.8. *If λ is a pairwise fuzzy regular σ-nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then λ is a pairwise fuzzy regular F_σ-set in (X, T_1, T_2).*

Proof. The proof follows from definition 4.7. □

Proposition 4.9. *If λ is a pairwise fuzzy regular σ-nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy regular G_δ-set in (X, T_1, T_2).*

Proof. The proof follows from proposition 4.8. □

Proposition 4.10. *If $int_{T_i}[\vee_{k=1}^{N}(\lambda_k)] = 0,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy regular σ-nowhere dense sets in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy regular Volterra space.*

Proof. Let (λ_k)'s be pairwise fuzzy regular σ-nowhere dense sets in (X, T_1, T_2) such that $int_{T_i}[\vee_{k=1}^{N}(\lambda_k)] = 0$. Now $\vee_{k=1}^{N}[int_{T_i}(\lambda_k)] \le int_{T_i}[\vee_{k=1}^{N}(\lambda_k)]$ implies that $\vee_{k=1}^{N}[int_{T_i}(\lambda_k)] \le 0$. That is, $\vee_{k=1}^{N}[int_{T_i}(\lambda_k)] = 0$. This implies that $int_{T_i}(\lambda_k) = 0$, for each i. Then, $cl_{T_i}(1-\lambda_k) = 1 - int_{T_i}(\lambda_k) = 1 - 0 = 1$. Hence $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Since (λ_k)'s be pairwise fuzzy regular σ-nowhere dense sets in (X, T_1, T_2), we have by proposition 4.9, $(1 - \lambda_k)$'s are pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Thus, $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Now $cl_{T_i}\big(\wedge_{k=1}^{N} (1 - \lambda_k)\big) = cl_{T_i}\big(1 - \vee_{k=1}^{N}(\lambda_k)\big) = 1 - int_{T_i}\big(\vee_{k=1}^{N} (\lambda_k)\big) = 1 - 0 = 1$ and hence (X, T_1, T_2) is a pairwise fuzzy regular Volterra space. □

5. Pairwise fuzzy weakly regular Volterra spaces

Definition 5.1. A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy weakly regular Volterra space if $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2).

Proposition 5.2. *If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy regular Volterra space, then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy regular Volterra space. Then, $cl_{T_i}\Big(\wedge_{k=1}^{N} (\lambda_k)\Big) = 1,\ (i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). This implies that $\wedge_{k=1}^{N}(\lambda_k) \neq 0$ in (X, T_1, T_2). [Otherwise if $\wedge_{k=1}^{N}(\lambda_k) = 0$, then $cl_{T_i}\Big(\wedge_{k=1}^{N} (\lambda_k)\Big) = cl_{T_i}(0) = 0 \neq 1$, a contradiction]. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

Proposition 5.3. *If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space, then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Let (λ_k)'s $(k = 1\ to\ N)$ be pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in a pairwise fuzzy weakly Volterra space (X, T_1, T_2). Then, by proposition 3.8, the pairwise fuzzy regular G_δ-sets (λ_k)'s in (X, T_1, T_2), are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence (λ_k)'s are pairwise fuzzy dense and pairwise G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space, $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

Theorem 5.4 ([15])**.** *If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proposition 5.5. *If $\vee_{k=1}^{N}(\lambda_k) \neq 1$, where (λ_k)'s are pairwise fuzzy nowhere dense and pairwise fuzzy regular F_σ-sets in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Let (λ_k)'s $(k = 1\ to\ N)$ be pairwise fuzzy nowhere dense and pairwise fuzzy regular F_σ-sets in (X, T_1, T_2) such that $\vee_{k=1}^{N}(\lambda_k) \neq 1$. Then, we have $1 - \vee_{k=1}^{N}(\lambda_k) \neq 0$. This implies that $\wedge_{k=1}^{N}(1 - \lambda_k) \neq 0$. Since (λ_k)'s are pairwise fuzzy nowhere dense sets, we have by theorem 5.4, $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Also, since (λ_k)'s are pairwise fuzzy regular F_σ-sets, by proposition 3.3, $(1 - \lambda_k)$'s are pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Hence $\wedge_{k=1}^{N}(1 - \lambda_k) \neq 0$, where $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

Proposition 5.6. *If each pairwise fuzzy nowhere dense set is a pairwise fuzzy regular F_σ-set in a pairwise fuzzy second category space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy second category space in which each pairwise fuzzy nowhere dense set is a pairwise fuzzy regular F_σ-set. Since

(X, T_1, T_2) is a pairwise fuzzy second category space, $\vee_{\alpha=1}^{\infty}(\mu_\alpha) \neq 1$, where (μ_α)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). By hypothesis, (μ_α)'s are pairwise fuzzy regular F_σ-sets in (X, T_1, T_2). Let us take the first $N(\mu_\alpha)$'s as (λ_k)'s in (X, T_1, T_2). Then $\vee_{k=1}^{N}(\lambda_k) \leq \vee_{\alpha=1}^{\infty}(\mu_\alpha)$ implies that $\vee_{k=1}^{N}(\lambda_k) \neq 1$. Thus $\vee_{k=1}^{N}(\lambda_k) \neq 1$, where (λ_k)'s are pairwise fuzzy nowhere dense and pairwise fuzzy regular F_σ-sets in (X, T_1, T_2). Therefore proposition 5.5, (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

Proposition 5.7. *If $\vee_{k=1}^{N}(\lambda_k) = 1$, where (λ_k)'s are pairwise fuzzy regular F_σ-sets in a pairwise fuzzy weakly regular Volterra space (X, T_1, T_2), then there exists atleast one λ_k in (X, T_1, T_2) with $int_{T_i}(\lambda_k) \neq 0$, $(i = 1, 2)$.*

Proof. Suppose that $int_{T_i}(\lambda_k) = 0$, for all $k = 1$ *to* N in (X, T_1, T_2). Then, $1 - int_{T_i}(\lambda_k) = 1$. This will imply that $cl_{T_i}(1 - \lambda_k) = 1$. Since (λ_k)'s are pairwise fuzzy regular F_σ-sets in (X, T_1, T_2), $(1 - \lambda_k)$'s are pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Then, $\wedge_{k=1}^{N}(1 - \lambda_k) = 1 - (\vee_{k=1}^{N}(\lambda_k)) = 1 - 1 = 0$. Hence we will have $\wedge_{k=1}^{N}(1 - \lambda_k) = 0$, where $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2) and this will imply that (X, T_1, T_2) will not be a fuzzy weakly regular Volterra space, a contradiction to the hypothesis. Hence there must be atleast one λ_k in (X, T_1, T_2) with $int_{T_i}(\lambda_k) \neq 0$. □

Proposition 5.8. *If $\vee_{k=1}^{N}(\lambda_k) = 1$, where (λ_k)'s are pairwise fuzzy regular F_σ-sets such that $int_{T_i}(\lambda_k) \neq 0$, $(i = 1, 2)$ for atleast one λ_k in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Suppose that $\wedge_{k=1}^{N}(\lambda_k) = 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). Then $1 - \wedge_{k=1}^{N}(\lambda_k) = 1$ and hence $\vee_{k=1}^{N}(1 - \lambda_k) = 1$, where $(1 - \lambda_k)$'s are pairwise fuzzy regular F_σ-sets in (X, T_1, T_2) such that $int_{T_i}(1 - \lambda_k) = 0$, for all $k = 1$ *to* N in (X, T_1, T_2), a contradiction to the hypothesis. Hence $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

Proposition 5.9. *If, (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in a fuzzy bitopological space (X, T_1, T_2), such that $\wedge_{k=1}^{N}(\lambda_k)$ is not a pairwise fuzzy nowhere dense set in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Suppose that the fuzzy bitopological space (X, T_1, T_2) is not a pairwise fuzzy weakly regular Volterra space. Then we have $\wedge_{k=1}^{N}(\lambda_k) = 0$. This will imply that $int_{T_i} cl_{T_j}[\wedge_{k=1}^{N}(\lambda_k)] = 0$, $(i \neq j$ and $i, j = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2), a contradiction. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

Proposition 5.10. *If the pairwise fuzzy strongly irresolvable space (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space, then $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy residual sets in (X, T_1, T_2).*

Proof. The proof follows from propositions 4.5 and 5.2. □

Proposition 5.11. *If the pairwise fuzzy strongly irresolvable space (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space, then*

(1) $\vee_{k=1}^{N}(\mu_k) \neq 1$, *where (μ_k)'s are pairwise fuzzy first category sets in (X, T_1, T_2).*
(2) $\vee_{k=1}^{N}(\mu_k) \neq 1$, *where (μ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2).*

Proof. (1). Let the pairwise fuzzy strongly irresolvable space (X, T_1, T_2) be a pairwise fuzzy weakly regular Volterra space. Then $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in (X, T_1, T_2). This implies that $1 - \wedge_{k=1}^{N}(\lambda_k) \neq 1$. Then $\vee_{k=1}^{N}(1 - \lambda_k) \neq 1$. Since (λ_k)'s are pairwise fuzzy regular G_δ-sets in (X, T_1, T_2), by proposition 3.3, $(1 - \lambda_k)$'s are pairwise fuzzy regular F_σ-sets in (X, T_1, T_2). Also, since (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(\lambda_k) = 1 = cl_{T_2}cl_{T_1}(\lambda_k)$. Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space and by theorem 3.15, $cl_{T_1}(\lambda_k) = 1$ and $cl_{T_2}(\lambda_k) = 1$ in (X, T_1, T_2). That is, $cl_{T_i}(\lambda_k) = 1$, $(i = 1, 2)$. Now $1 - cl_{T_i}(\lambda_k) = 0$. This implies that $int_{T_i}(1 - \lambda_k) = 0$. Then, by proposition 3.13, $(1 - \lambda_k)$'s are pairwise fuzzy first category sets in (X, T_1, T_2). Let $1 - \lambda_k = \mu_k$. Hence $\vee_{k=1}^{N}(\mu_k) \neq 1$, where (μ_k)'s are pairwise fuzzy first category sets in (X, T_1, T_2).

(2). By (1), $int_{T_i}(1 - \lambda_k) = 0$ and hence $int_{T_i}int_{T_j}(1 - \lambda_k) = 0$, $(i \neq j$ and $i, j = 1, 2)$. Then $(1 - \lambda_k)$'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Let $1 - \lambda_k = \mu_k$. Hence $\vee_{k=1}^{N}(\mu_k) \neq 1$, where (μ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). □

Theorem 5.12 ([13])**.** *In a fuzzy bitopological space (X, T_1, T_2), a fuzzy set λ is pairwise fuzzy σ-nowhere dense in (X, T_1, T_2) if and only if $1 - \lambda$ is a pairwise fuzzy dense and pairwise fuzzy G_δ-set in (X, T_1, T_2).*

Proposition 5.13. *If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-second category space, then (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space.*

Proof. Let (λ_k)'s $(k = 1\ to\ \infty)$ be pairwise fuzzy dense and pairwise fuzzy regular G_δ-sets in a pairwise fuzzy σ-second category space (X, T_1, T_2). Then, by proposition 3.8, the pairwise fuzzy regular G_δ-sets (λ_k)'s in (X, T_1, T_2), are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Now, by theorem 5.12, $(1 - \lambda_k)$'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy σ-second category space, $\vee_{k=1}^{\infty}(1 - \lambda_k) \neq 1$. This implies that $\wedge_{k=1}^{\infty}(\lambda_k) \neq 0$. But $\wedge_{k=1}^{\infty}(\lambda_k) \leq \wedge_{k=1}^{N}(\lambda_k)$ implies that $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly regular Volterra space. □

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G. Thangaraj, Department of Mathematics, Thiruvalluvar University, Vellore 632 115, India.
e-mail: `g.thangaraj@rediffmail.com`

V. Chandiran, Research Scholar, Department of Mathematics, Thiruvalluvar University, Vellore - 632 115, India.
e-mail: `profvcmaths@gmail.com`