Switching properties of bivariate Shewhart control charts for monitoring the covariance matrix

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Abstract

A control chart is very useful in monitoring various production process. There are many situations in which the simultaneous control of two or more related quality variables is necessary. We construct bivariate Shewhart control charts based on the trace of the product of the estimated variance-covariance matrix and the inverse of the in-control matrix and investigate the properties of bivariate Shewart control charts with VSI procedure for monitoring covariance matrix in term of ATS (Average time to signal) and ANSW (Average number of switch) and probability of switch, ASI (Average sampling interval). Numerical results show that ATS is smaller than ARL. From examining the properties of switching in changing covariances and variances in Σ, ANSW values show that it does not switch frequently and does not matter to use VSI procedure.

Keywords: Average run length, average number of switches, average sampling interval, average time to signal, switching property.

1. Introduction

The purpose of a control chart is to detect assignable causes of variation that occur in the process. During the control process, we want to detect any departure from in-control state and find this change as quickly as possible. For a good control chart, it can be aware of a shift quickly in the process when the process is out-of-control state and generate few false alarms when the process is in-control state.

There are many problems in the quality control that imply a vector of measurements of several quality variables rather than a single quality variable. If the quality variables are correlated, it has better sensitivity by using multivariate control charts.

For control charts, it takes samples from the process at fixed sampling interval (FSI). FSI procedures are simple and very popular, but we can change time interval as a function of what is observed from the process. Strategy of changing time interval is called variable sampling interval (VSI). Under the VSI procedures, when there is an indication of the process change, the time interval would be short.

VSI procedures were first investigated by Arnold (1970) and his works were extended by Smeach and Jernigan (1977). Also Reynolds and Arnold (1989) developed general expressions

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such as ARL and ATS for VSI control charts. But VSI procedures have a disadvantage that is frequent switching between different sampling intervals which requires more cost and effort to administer the process than corresponding FSI procedures. Amin and Letsinger (1991) studied switching behavior and run rules for switching between different sampling intervals. Amin and Hemasinha (1993) developed expression for the ANSW and ATS for VSI $\bar{X}$ -chart for evaluating the switching behavior.

In this paper, we investigate the properties of bivariate Shewart charts with VSI procedure for monitoring covariance matrix in term of ATS (Average Time to Signal) and ANSW (Average number of Switch) and probability of switch, ASI (Average sampling interval). In Section 2, we introduce notation, assumptions and properties of VSI procedures. And we construct bivariate Shewhart control charts based on the trace of the product of the estimated variance-covariance matrix and the inverse of the in-control matrix. In Section 3, we simulate bivariate control charts with three types of sifts in the covariance matrix.

2. Description of control procedures

Suppose that the process of interest has p quality variables presented by the random vector $\underline{X}' = (X_1, X_2, \cdots, X_p)$ and we take a sequence of samples of size n at each sampling occasion i $(i = 1, 2, \cdots)$. It will be assumed that the successive observation vectors are independent and have bivariate normal distribution with $N_p(\mu, \Sigma)$ where the mean vector $\mu = \mu_0$ is known.

2.1. Evaluating sample statistic

Suppose that the process of interest has p quality variables whose distribution is bivariate normal with mean vector μ and covariance matrix Σ where the mean vector $\mu = \mu_0$ is known. It will be convenient to let σ represent the vector of standard deviations of the variable. Let μ_0, Σ_0 and σ_0 be the in-control values of μ, Σ and σ. We will usually refer to Σ_0 as target, even though, in practice, some of the components of Σ_0 may correspond to estimated values rather than specified target value.

We take a sequence of random samples of size $n \geq p$ at each sampling point, where the sampling points are d time units apart. Let X_{kij} be the jth observation for variable i at sampling point k for $k = 1, 2, \cdots,$ $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, n$, and let the corresponding standardized observation be

$$Z_{kij} = (X_{kij} - \mu_{0i})/\sigma_{0i}$$

where μ_{0i} is the ith component of μ_0 and σ_{0i} is the ith component of σ_0. Also let

$$\boldsymbol{Z_{kj}} = (Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})', \; j = 1, 2, \cdots, n$$

be the vector of standardized observations for observation vector j at sampling point k. Let Σ_Z be the covariance matrix of $\boldsymbol{Z}_{kj}$, and let Σ_{Z0} be the in-control value of Σ_Z. The in-control distribution of Z_{kij} is standard normal, so Σ_{Z0} is also the in-control correlation matrix of the unstandardized observations. When $n \geq p$, some control statistics used for monitoring Σ are functions of the sample estimates $\widehat{\Sigma}_Z$. At sampling point k, let $\widehat{\Sigma}_{Zk}$ be the maximum likelihood estimator of Σ_Z, where the (i, i') element of $\widehat{\Sigma}_{Zk}$ is $\sum_{j=1}^{n} Z_{kij}Z_{ki'j}/n$.

2.2. Properties of VSI procedures

The basic idea of VSI control charts is that if there is some indication of a process change, the time interval would be short and it would be long if not. For VSI charts, the sampling intervals are random variable and the sampling interval depends on the past sample informations of $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_i$. Reynolds (1989) investigated the theoretical aspects of a VSI two-sided control charts. Reynolds and Arnold (1989) investigated the theoretical aspects of a VSI one-sided Shewhart control charts. Reynolds (1995) evaluated properties of variable sampling interval control charts. Chang and Heo (2012) investgated switching properties of CUSUM charts for controlling mean vector. Reynolds and Stoumbos (2001) studied on monitoring the process mean and variance using individual observations and variable sampling intervals. Chang and Cho (2005) studied CUSUM charts for monitoring mean vector with variable sampling intervals. Reynolds and Cho (2006), Reynolds and Cho (2011), Jeong and Cho (2012) studied multivariate control charts for the mean vector or covariance matrix.

To implement two sampling interval control charts, there are two disjoint regions I_1, I_2 that divide the in-control region. I_i is the region in which the sampling interval d_i is used ($i = 1, 2$). In this paper, we assume that the VSI chart starts at time 0 and the interval used before the first sample, is a fixed constant, say d_0. Then the ARL and ATS can be expressed as

$$\begin{cases} ARL = 1 + \psi_1 + \psi_2 \\ ATS = d_0 + d_1\psi_1 + d_2\psi_2 \end{cases}$$

where ψ_i is the expected number of samples before the signal. Also ATS can be expressed as $ATS = d \cdot ARL$ and d can be interpreted as the average sampling interval (ASI) of the charts to signal. And ρ_1 can be interpreted as the long-run proportion of sampling interval that d_1 is used where

$$d = d_1\rho_1 + d_2(1 - \rho_1)$$

The VSI procedures are substantially more efficient than FSI procedures in the term of ARL and ATS. But, the ARL and ATS do not show any switching information between the different sampling intervals d_1 and d_2. Therefore, it is necessary to define the number of switches (NSW) as the number of switches made from the start of the process until the chart signals, and let average number of switches (ANSW) be the expected value of the NSW. The ANSW can be obtained as follows

$$ANSW = NSW \cdot P(\text{switches})$$

And, the probability of switch is given by

$$P(\text{switch}) = P(d_1) \cdot P(d_2|d_1) + P(d_2) \cdot P(d_1|d_2)$$

where $p(d_i)$ is the probability of using sampling interval d_i, and $p(d_i|d_j)$ is the conditional probability of using sampling interval d_i in the current sample given that the sampling interval d_j ($d_i \neq d_j$) was used in the previous sample.

By Amin and Lestinger (1991), the number of samples taken from the time the process starts using the sampling interval d_i until a switch is made to sampling interval d_j ($i \neq j$), and let average number of samples until a switch (ANSSW) be the expected value of the number of samples until a switch.

2.3. Bivariate Shewhart control chart

Reynolds and Stoumbos (2004a, 2004b) suggested that Shewhart control charts are very sensitive to the choice of the sample size. So, Shewhart control chart is one of the most widely used control charts for monitoring the production process. A Shewhart control chart can detect large changes in monitored parameter quickly. But Shewhart control chart is relatively inefficient in detecting small shift in control parameter, because it uses only the information in the current sample. We can obtain a sample statistic for monitoring covariance matrix by using the statistic for testing

$$H_0 : \Sigma = \Sigma_0 \text{ vs } H_1 : \Sigma \neq \Sigma_0.$$

The Shewhart-type control chart proposed by Hotelling (1947) for monitoring mean vector μ (frequently called Hotelling's T^2 chart) was originally developed for the case in which Σ_0 is unknown. If Σ_0 is assumed to be known, then this control chart is equivalent to a control chart based on the statistic used with an upper control limit (UCL).

$$(Z_{k1}, Z_{k2}, \cdots Z_{kp})\Sigma_{ZO}^{-1}(Z_{k1}, Z_{k2}, \cdots Z_{kp})'$$

Hotelling(1947) proposed a control chart for monitoring Σ based on

$$\sum_{j=1}^{n}(Z_{kij}, Z_{k2j}, \cdots, Z_{kpj})\Sigma_{Z0}^{-1}(Z_{kij}, Z_{k2j}, \cdots, Z_{kpj})' = ntr(\widehat{\Sigma_{Zk}}\Sigma_{Z0}^{-1}) = Y_k.$$

This control chart has both a lower control limit (LCL) and an UCL.

For the VSI Shewhart control chart based on Y_k, suppose that the sampling interval;

$$d_1 \text{ is used when } Y_k \in (g_{Y_k}, h_{Y_k}],$$
$$d_2 \text{ is used when } Y_k \in (0, g_{Y_k}],$$

where $g_{Y_k} \leq h_{Y_k}$ and $d_1 < d_2$.

The percentage point of Y_k can be obtained from the chi-square distribution when the process is in-control. If the process is in-control, the statistic Y_k has a chi-squared distribution with p degree of freedom. Hence, the design parameters g_{Y_k} and h_{Y_k} can be obtained to satisfy a desired ARL and ATS.

3. Numerical result and concluding remarks

The ability of a control chart to detect any shifts in the production process is determined by the length of time required to signal. Thus, a good control chart detects shifts as quickly as possible at out-of-control state and produce few alarms at in-control state.

In evaluating the properties of the VSI procedure, we compare the performance of the VSI to the same procedure using FSI procedure. Also, we will use process parameters based on ATS and ARL.

In order to evaluate the performances and compare the proposed bivariate Shewhart control charts fairly, some kinds of standards for comparison are necessary. In our computation, each control chart has been set so that on-target ARL and ATS were approximately equal to 800.0 and the sample size for each control chart was 2, 4 for $p = 2$. And we used $d_0 = 0, d_1 = 1.9, d_2 = 0.1$. The performance of the charts for monitoring the covariance matrix depends on the value of Σ. The following types of shifts were considered

(1) variances are changed and covariances are not changed,

(2) covariances are changed and variances are not changed,

(3) variances and covariances are simultaneously changed.

Tables 3.1-3.2 give the values of and ARL and ATS, ANSW, P(switch), ANSSW, ASI for $n = 2, 4$ and $p = 2$ and three different in-control correlation coefficients $\rho_0 = 0.9, 0.5, 0.3$ when covariances are changed and variance are not changed. Here the changed value considered in Tables 3.1-3.2 are those of decreased by 10%, 50%, 70% and 90% of ρ_0 values, respectively. As shown in Tables 3.1-3.2, the bivariate Shewharts control chart proposed by Hotelling (1947) for monitoring the variance-covariance matrix are effective in detecting changes of covariances in Σ.

Tables 3.3-3.4 give the values of ARL, ATS, ANSW, P(switch), ANSSW, ASI for $n = 2, 4$ and $p = 2$ when variances are changed and covariances are not changed, respectively. Here the variances are changed from $\sigma = \sqrt{c}\sigma_0$, for $c =$1.21, 1.44, 1.69, 4. As shown in Tables 3.3-3.4, the bivariate Shewharts control chart proposed by Hotelling (1947) for monitoring the variance-covariance matrix are effective in detecting changes of variances in Σ.

For $n = 2, 4$ and $p = 2$, Tables 3.5-3.6 give ARL, ATS, ANSW, P(switch), ANSSW, ASI in each cell when only 2 variances and covariances are simultaneously changed respectively. Here the variances are changed from $\sigma = \sqrt{c}\sigma_0$, for c=1.21, 1.44, 1.69, 4. And covariances are also changed from $\rho_0 = 0.9$ to $\rho_0 = 0.72, 0.54$, $\rho_0 = 0.5$ to $\rho_0 = 0.4, 0.3$ and $\rho_0 = 0.3$ to $\rho_0 = 0.21, 0.8$. As shown in Tables 3.5-3.6, the bivariate Shewharts control chart proposed by Hotelling (1947) for monitoring the variance-covariance matrix are effective in detecting simultaneously changes of variance and covariance in Σ.

Numerical results for various tables give ARL, ATS, ANSW, P(switch), ANSSW, ASI. First, we realize that ATS is smaller than ARL. That means VSI control chart is faster than FSI control chart for detecting variations in the production process. Second, we examine the properties of switching in changing covariances and variances in Σ. ANSW values show that it does not switch frequently and does not matter to use VSI procedure.

Table 3.1 ARL and ATS, ANSW, P(switch), ANSSW, ASI when covariances are changed and variance are not changed (n=2, p=2)

	FSI			VSI		
	ARL	ATS	ANSW	P(switch)	ANSSW	ASI
$\rho_0 = 0.9$						
$\rho = 0.90$	799.72	799.87	398.39	0.50	2.01	1.00
$\rho = 0.81$	57.32	41.93	24.75	0.43	2.32	0.72
$\rho = 0.45$	4.44	2.64	0.72	0.16	6.18	0.39
$\rho = 0.27$	3.12	1.96	0.35	0.11	9.04	0.34
$\rho = 0.09$	2.51	1.67	0.20	0.08	12.44	0.31
$\rho_0 = 0.5$						
$\rho = 0.50$	799.72	799.87	398.39	0.50	2.01	1.00
$\rho = 0.45$	599.65	581.00	298.02	0.50	2.01	0.97
$\rho = 0.25$	177.92	156.79	86.52	0.49	2.07	0.88
$\rho = 0.15$	109.20	92.91	51.69	0.47	2.11	0.84
$\rho = 0.05$	71.54	59.25	33.51	0.46	2.17	0.82
$\rho_0 = 0.3$						
$\rho = 0.30$	799.72	799.87	398.39	0.50	2.01	1.00
$\rho = 0.21$	596.15	579.30	296.37	0.50	2.01	0.97
$\rho = 0.15$	463.89	444.83	230.18	0.50	2.02	0.96
$\rho = 0.09$	360.18	341.84	178.13	0.50	2.02	0.95
$\rho = 0.03$	280.52	262.47	138.21	0.49	2.03	0.94

Table 3.2 ARL and ATS, ANSW, P(switch), ANSSW, ASI when covariances are changed and variance are not changed ($n=4$, $p=2$)

	FSI			VSI		
	ARL	ATS	ANSW	P(switch)	ANSSW	ASI
			$\rho_0 = 0.9$			
$\rho = 0.90$	799.72	799.87	398.39	0.50	2.01	1.00
$\rho = 0.81$	36.11	22.21	13.39	0.37	2.70	0.59
$\rho = 0.45$	2.54	1.48	0.14	0.06	18.07	0.23
$\rho = 0.27$	1.86	1.25	0.05	0.03	38.23	0.19
$\rho = 0.09$	1.57	1.16	0.02	0.01	72.52	0.16
			$\rho_0 = 0.5$			
$\rho = 0.50$	799.72	799.87	398.39	0.50	2.01	1.00
$\rho = 0.45$	551.93	527.57	273.81	0.50	2.02	0.95
$\rho = 0.25$	141.12	114.59	65.76	0.47	2.13	0.81
$\rho = 0.15$	80.35	61.81	36.16	0.45	2.22	0.76
$\rho = 0.05$	50.69	37.11	21.70	0.43	2.34	0.71
			$\rho_0 = 0.3$			
$\rho = 0.30$	799.72	799.87	398.39	0.50	2.01	1.00
$\rho = 0.21$	595.15	580.30	295.88	0.50	2.01	0.50
$\rho = 0.15$	429.96	404.00	212.48	0.49	2.02	0.94
$\rho = 0.09$	327.54	301.15	160.99	0.49	2.04	0.92
$\rho = 0.03$	248.17	223.83	121.17	0.49	2.05	0.90

Table 3.3 ARL and ATS, ANSW, P(switch), ANSSW, ASI when variances are changed and covariances are not changed ($n=2$, $p=2$)

	FSI					
	ARL	ATS	ANSW	P(switch)	ANSSW	ASI
			$\rho_0 = 0.90$			
$c = 1$	799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	253.97	229.31	124.10	0.49	2.05	0.90
	198.31	164.53	94.25	0.48	2.10	0.83
$c = 1.69$	31.48	23.04	12.95	0.41	2.43	0.70
	32.47	19.30	11.59	0.36	2.80	0.57
$c = 4$	2.64	1.77	0.24	0.09	11.14	0.33
	2.93	1.54	0.17	0.06	16.83	0.22
			$\rho_0 = 0.5$			
$c = 1$	799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	338.68	308.25	166.24	0.49	2.04	0.50
	198.31	164.53	94.25	0.48	2.10	0.51
$c = 1.69$	75.96	57.70	33.85	0.45	2.24	0.75
	32.47	19.30	11.59	0.36	2.80	0.57
$c = 4$	6.10	3.48	1.24	0.20	4.91	0.42
	2.93	1.54	0.17	0.06	16.83	0.22
			$\rho_0 = 0.3$			
$c = 1$	799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	344.89	314.08	169.38	0.49	2.04	0.91
	198.31	164.53	94.25	0.48	2.10	0.83
$c = 1.69$	81.14	61.80	36.32	0.45	2.23	0.75
	32.47	19.30	11.59	0.36	2.80	0.58
$c = 4$	6.73	3.79	1.44	0.21	4.68	0.43
	2.93	1.54	0.17	0.06	16.83	0.22

Table 3.4 ARL and ATS, ANSW, P(switch), ANSSW, ASI when variances are changed and covariances are not changed ($n=4$, $p=2$)

	FSI			VSI		
	ARL	ATS	ANSW	P(switch)	ANSSW	ASI
			$\rho_0 = 0.90$			
$c = 1$	799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	212.99	181.56	102.22	0.48	2.08	0.85
	145.16	109.41	65.74	0.45	2.21	0.75
$c = 1.69$	20.04	12.12	6.78	0.34	2.95	0.56
	17.56	8.16	4.44	0.25	3.96	0.41
$c = 4$	1.64	1.19	0.03	0.02	55.42	0.18
	1.66	1.11	0.01	0.01	130.33	0.13
			$\rho_0 = 0.5$			
$c = 1$	799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	290.27	251.67	140.47	0.48	2.06	0.51
	145.16	109.41	65.74	0.45	2.20	0.53
$c = 1.69$	49.94	32.51	19.86	0.40	2.51	0.63
	17.56	8.16	4.44	0.25	3.96	0.41
$c = 4$	3.37	1.77	0.28	0.08	11.95	0.26
	1.66	1.11	0.01	0.01	130.33	0.13
			$\rho_0 = 0.3$			
$c = 1$	799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	384.18	345.54	188.18	0.49	2.04	0.90
	145.16	109.41	65.74	0.45	2.21	0.75
$c = 1.69$	54.21	35.49	21.72	0.40	2.50	0.64
	15.64	7.10	3.73	0.24	0.00	0.00
$c = 4$	3.71	1.88	0.34	0.09	10.75	0.26
	1.66	1.11	0.01	0.01	130.33	0.13

Table 3.5 ARL and ATS, ANSW, P(switch), ANSSW, ASI when variances and covariances are changed ($n=2$, $p=2$)

		FSI					
		ARL	ATS	ANSW	P(switch)	ANSSW	ASI
				$\rho_0 = 0.90$			
$c = 1$		799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	$\rho = 0.72$	11.85	6.92	3.41	0.29	3.47	0.51
		9.64	5.34	2.43	0.25	3.97	0.46
	$\rho = 0.54$	4.82	2.78	0.81	0.17	5.94	0.39
		4.18	2.39	0.59	0.14	7.05	0.36
$c = 1.69$	$\rho = 0.72$	6.21	3.46	1.24	0.20	5.02	0.41
		4.55	2.38	0.61	0.13	7.45	0.32
	$\rho = 0.54$	3.41	2.01	0.39	0.11	8.84	0.32
		2.61	1.56	0.17	0.06	15.43	0.25
$c = 4$	$\rho = 0.72$	1.98	1.36	0.08	0.04	24.03	0.23
		1.60	1.14	0.02	0.01	77.99	0.15
	$\rho = 0.54$	1.70	1.23	0.04	0.02	41.68	0.20
		1.37	1.08	0.01	0.01	157.9	0.13
				$\rho_0 = 0.5$			
$c = 1$		799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	$\rho = 0.4$	11.85	6.92	3.41	0.29	3.47	0.51
		9.64	5.34	2.43	0.25	3.97	0.46
	$\rho = 0.3$	4.82	2.78	0.81	0.17	5.94	0.39
		4.18	2.39	0.59	0.14	7.05	0.36
$c = 1.69$	$\rho = 0.4$	6.21	3.46	1.24	0.20	5.02	0.41
		4.55	2.38	0.61	0.13	7.45	0.32
	$\rho = 0.3$	3.41	2.01	0.39	0.11	8.84	0.32
		2.61	1.56	0.17	0.06	15.43	0.25
$c = 4$	$\rho = 0.4$	1.98	1.36	0.08	0.04	24.03	0.23
		1.60	1.14	0.02	0.01	77.99	0.15
	$\rho = 0.3$	1.70	1.23	0.04	0.02	41.68	0.20
		1.37	1.08	0.01	0.01	157.90	0.13
				$\rho_0 = 0.3$			
$c = 1$		799.72	799.87	398.39	0.50	2.00	0.50
$c = 1.21$	$\rho = 0.21$	269.65	239.22	131.28	0.49	2.05	0.88
		158.71	128.21	74.38	0.47	2.13	0.80
	$\rho = 0.18$	246.03	216.57	119.37	0.49	2.06	0.88
		158.79	128.21	74.38	0.47	2.13	0.52
$c = 1.69$	$\rho = 0.21$	69.67	51.75	30.57	0.44	2.28	0.73
		28.02	16.33	9.67	0.35	2.90	0.55
	$\rho = 0.18$	65.43	48.26	28.52	0.44	2.29	0.72
		26.77	15.51	9.13	0.34	2.93	0.55
$c = 4$	$\rho = 0.21$	6.45	3.60	1.33	0.21	4.86	0.42
		2.80	1.50	0.15	0.05	18.30	0.21
	$\rho = 0.18$	6.36	3.55	1.29	0.20	4.92	0.42
		2.77	1.49	0.15	0.05	18.59	0.21

Table 3.6 ARL and ATS, ANSW, P(switch), ANSSW, ASI when variances and covariances are changed ($n=4$, $p=2$)

		FSI					
		ARL	ATS	ANSW	P(switch)	ANSSW	ASI
				$\rho_0 = 0.90$			
$c = 1.00$		799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	$\rho = 0.72$	6.59	3.15	1.09	0.17	6.03	0.34
		5.20	2.42	0.65	0.13	8.00	0.29
	$\rho = 0.54$	2.72	1.52	0.16	0.06	16.83	0.23
		2.37	1.37	0.10	0.04	23.82	0.20
$c = 1.69$	$\rho = 0.72$	3.43	1.76	0.28	0.08	12.40	0.25
		2.51	1.36	0.09	0.04	26.47	0.18
	$\rho = 0.54$	1.96	1.25	0.05	0.03	37.50	0.18
		1.58	1.12	0.02	0.01	99.19	0.14
$c = 4$	$\rho = 0.72$	1.32	1.07	0.01	0.01	196.44	0.13
		1.14	1.02	0.00	0.00	4275.80	0.11
	$\rho = 0.54$	1.19	1.04	0.00	0.00	534.59	0.12
		1.07	1.01	0.00	0.00	20660.0	0.10
				$\rho_0 = 0.5$			
$c = 1$		799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	$\rho = 0.4$	157.22	124.26	72.91	0.46	2.16	0.78
		85.01	58.27	35.91	0.42	2.37	0.67
	$\rho = 0.3$	89.59	65.42	39.36	0.44	2.28	0.72
		51.07	32.43	19.95	0.39	2.56	0.62
$c = 1.69$	$\rho = 0.4$	34.57	20.65	12.48	0.36	2.77	0.57
		12.79	5.64	2.74	0.21	4.67	0.37
	$\rho = 0.3$	24.27	13.65	7.95	0.33	3.05	0.53
		9.61	4.18	1.77	0.18	5.44	0.34
$c = 4$	$\rho = 0.4$	3.06	1.63	0.21	0.07	14.22	0.24
		1.54	1.09	0.01	0.01	177.48	0.12
	$\rho = 0.3$	2.80	1.53	0.17	0.06	16.83	0.22
		1.42	1.08	0.01	0.01	179.52	0.10
				$\rho_0 = 0.3$			
$c = 1$		799.72	799.87	398.39	0.50	2.01	1.00
$c = 1.21$	$\rho = 0.21$	221.60	184.45	105.58	0.48	2.10	0.83
		112.47	81.25	49.47	0.44	2.27	0.71
	$\rho = 0.18$	198.33	163.10	93.91	0.47	2.11	0.82
		102.78	73.35	44.78	0.44	2.30	0.70
$c = 1.69$	$\rho = 0.21$	45.00	28.30	17.32	0.38	2.60	0.61
		15.13	6.85	3.55	0.23	4.26	0.39
	$\rho = 0.18$	42.32	26.32	16.08	0.38	2.63	0.60
		14.39	6.48	3.30	0.23	4.36	0.39
$c = 4$	$\rho = 0.21$	3.53	1.80	0.30	0.09	11.64	0.25
		1.60	1.10	0.01	0.01	146.11	0.12
	$\rho = 0.18$	3.46	1.77	0.29	0.08	11.96	0.25
		1.59	1.10	0.01	0.01	151.22	0.12

References

Amin, R. W. and Lestinger, W. C. (1991) Improved switching rules in control procedures using variable sampling interval. *Communications in Statistics-Simulation and Computation*, **20**, 205-203.

Amin, R. W. and Hemasinha, R. (1993). the switching behavior of X charts with variable sampling intervals. *Theory and Methods*, **22**, 2081-2102.

Arnold, J. C. (1970). A Markovian sampling policy applied to quality monitoring of streams. *Biometrics*, **26**, 739-747.

Chang, D. J. and Heo S. Y. (2012). Switching properties of CUSUM charts for controlling mean vector. *Journal of the korean Data & Information Science Society*, **23**, 859-866.

Chang, D. J. and Cho G. Y. (2005). CUSUM charts for monitoring mean vector with variable sampling intervals. *Journal of the korean Data Analysis Society*, **7**, 1133-1143.

Hotelling, H. (1947). *Multivariate quality control, techniques of statistical analysis*, McGraw-Hill, New York, 111-184.

Jeong, J. I. and Cho G. Y. (2012). Multivariate Shewhart control charts for monitoring the vatiance-covariance matrix. *Journal of the Korean Data & Information Science Society*, **23**, 617-626.

Reynolds, Jr, M. R. (1989). Optimal two-sided variable sampling interval control charts for the exponential family. *Sequential Analysis*, **8**, 361-379.

Reynolds, Jr, M. R. (1995). Evaluating properties of variable sampling interval control charts. *Sequential Analysis*, **14**, 59-97.

Reynolds, Jr, M. R. and Arnold, J. C. (1989). Optimal one-sided Shewhart control charts with variable sampling intervals between samples. *Sequential Analysis*, **8**, 51-77.

Reynolds, Jr, M. R. and Cho, G. Y. (2006). Multivariate control charts for monitoring the mean vector and covariance Matrix. *Journal of Quality Technology*, **38**, 230-253.

Reynolds, Jr, M. R. and Cho. G. Y. (2011). Multivariate control charts for Monitoring the mean vector and covariance matrix with variable sampling intervals. *Sequential Analysis*, **30**, 1-40.

Reynolds, Jr, M. R. and Stoumbos, Z. G. (2001). Monitoring the process mean and variance using individual observations and variable sampling intervals. *Journal of Quality Technology*, **33**, 181-205.

Reynolds, Jr, M. R. and Stoumbos, Z. G. (2004a). Control charts and the optimal allocation of sampling resources. *Technometrics*, **46**, 200-214.

Reynolds, Jr, M. R. and Stoumbos, Z. G. (2004b). Should observations be grouped for effective process monitoring?. *Journal of Quality Technology*, **36**, 343-366.

Smeach, S. C. and Jernigan, R. W. (1977). Further aspects of a Markovian sampling policy for water quality monitoring. *Biometrics*, **33**, 41-46.