Journal of the Korean Data & Information Science Society
2015, **26**(6), 1583–1592

http://dx.doi.org/10.7465/jkdi.2015.26.6.1583
한국데이터정보과학회지

Default Bayesian one sided testing for the shape parameter in the log-logistic distribution†

Sang Gil Kang[1]

[1]Department of Computer and Data Information, Sangji University

Received 15 September 2015, revised 14 October 2015, accepted 23 October 2015

Abstract

This paper deals with the problem of testing on the shape parameter in the log-logistic distribution. We propose default Bayesian testing procedures for the shape parameter under the reference priors. The reference prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. We can solve the this problem by the intrinsic Bayes factor and the fractional Bayes factor. Therefore we propose the default Bayesian testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors. Simulation study and an example are provided.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, log-logistic distribution, reference prior.

1. Introduction

The log-logistic distribution is well known in survival analysis of data sets such as survival times of cancer patients in which the hazard increases initially and decreases later (Bennett, 1983). Also in economic studies of distributions of wealth or income, it is known as Fisk distribution (Fisk, 1961) and is considered as an equivalent alternative to a log-normal distribution. The density function of the log-logistic distribution is given by

$$f(x|\alpha,\beta) = \frac{\beta\alpha^{\beta}x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^2},\ x>0,\ \alpha>0,\ \beta>0, \tag{1.1}$$

where α is the scale parameter and β is the shape parameter.

The failure rate function of the log-logistic distribution can be decreasing or upside-down bath tub, depending on the choice of the shape parameter. Chen (1997) proposed the exact confidence interval and exact test about the shape parameter. Also the properties of the order statistics of the log-logistic distribution discussed by Ragab and Green (1984), Ali and Khan (1987), and Balakrishnan and Malik (1987). For further details on the importance and applications of a log-logistic distribution one may refer to Franco (1984), Shoukri *et al.* (1988), Ahmad *et al.* (1988), Robson and Reed (1999) and Geskus (2001).

† This research was supported by Sangji University Research Fund, 2014.

[1] Professor, Department of Computer and Data Information, Sangji University, Wonju 220-702, Korea. E-mail: sangkg@sangji.ac.kr

The present paper focuses on Bayesian testing the shape parameter in the log-logistic distribution. When the shape parameter is say $\beta > 1$, the hazard rate function becomes unimodal and when $\beta \leq 1$, the hazard decreases monotonically.

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training sample in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang *et al.*, 2013; Kang *et al.*, 2014b). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

In this paper, we propose the objective Bayesian one-sided hypothesis testing procedures for the shape parameter in the log-logistic distribution based on the Bayes factors. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference prior, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors. In Section 4, simulation study and an example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses $H_1, H_2, \cdots, H_q$ are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i. The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis H_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be so used and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, \; i = 1, \cdots, q. \tag{2.3}$$

In view (2.3), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor $B_{ji}(l)$ with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training samples $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L}\sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all the training sample Bayes factors. However the AIBF are often not suitable for nonnested situations, especially when testing one-sided hypotheses as here (Dmochowski, 1996). An attractive alternative, given by Berger and Pericchi (1996) is to embed the competing models in a larger encompassing model H_0 so that all of the H_i are nested within H_0. The encompassing arithmetic intrinsic Bayes factor (EIBF) is then defined as

$$B_{ji}^{EI} = B_{ji}^N \times \frac{\sum_{l=1}^{L} B_{i0}^N(\mathbf{x}(l))}{\sum_{l=1}^{L} B_{j0}^N(\mathbf{x}(l))}, \tag{2.7}$$

where $B_{i0}^N(\mathbf{x}(l)) = m_i^N(\mathbf{x}(l))/m_0^N(\mathbf{x}(l))$. Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{MI} and B_{ji}^{EI} from (2.6) and (2.7), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction,b,of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.8}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995)).

3. Bayesian hypothesis testing procedures

Let $X_i, i = 1, \cdots, n$ denote observations from the log-logistic distribution $\mathcal{LL}(\alpha, \beta)$ with the scale parameter α and the shape parameter β. Then likelihood function is given by

$$f(\mathbf{x}, \mathbf{y}|\alpha, \beta) = \beta^n \alpha^{n\beta} \prod_{i=1}^n \frac{x_i^{\beta-1}}{(\alpha^\beta + x_i^\beta)^2}, \tag{3.1}$$

where $\mathbf{x} = (x_1, \cdots, x_n)$, $\alpha > 0$ and $\beta > 0$.

The shape of the failure rate function of the log-logistic distribution depends on the shape parameter β. When $\beta \le 1$, the hazard rate function is decreasing. When $\beta > 1$, the shape of the hazard rate function is an upside-down bath tub. Thus we are interested in testing the hypotheses $H_1 : \beta \le \beta_0$ versus $H_2 : \beta > \beta_0$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing procedure based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \beta \le \beta_0$ is

$$L_1(\alpha, \beta|\mathbf{x}) = \beta^n \alpha^{n\beta} \prod_{i=1}^n \frac{x_i^{\beta-1}}{(\alpha^{\beta_1} + x_i^\beta)^2}, \tag{3.2}$$

where $\beta \le \beta_0$. And under the hypothesis H_1, the reference prior for (α, β) is

$$\pi_1^N(\alpha, \beta) \propto \alpha^{-1}\beta^{-1}, \tag{3.3}$$

where $\beta \le \beta_0$. The reference prior (3.3) derived by Kang *et al.* (2014a). Then from the likelihood (3.2) and the reference prior (3.3), the element $m_1^b(\mathbf{x}, \mathbf{y})$ of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}) &= \int_0^{\beta_0}\int_0^\infty L_1^b(\alpha, \beta|\mathbf{x})\pi_1^N(\alpha, \beta)d\alpha d\beta \\ &= \int_0^{\beta_0}\int_0^\infty \beta^{bn-1}\alpha^{bn\beta-1}\prod_{i=1}^n \frac{x_i^{b(\beta-1)}}{(\alpha^\beta + x_i^\beta)^{2b}} d\alpha d\beta. \end{aligned} \tag{3.4}$$

For the hypothesis H_2, the reference prior for (α, β) is

$$\pi_2^N(\alpha, \beta) \propto \alpha^{-1}\beta^{-1}, \tag{3.5}$$

where $\beta > \beta_0$. The likelihood function under the hypothesis H_2 is

$$L_2(\alpha, \beta|\mathbf{x}) = \beta^n \alpha^{n\beta} \prod_{i=1}^{n} \frac{x_i^{\beta-1}}{(\alpha^\beta + x_i^\beta)^2}, \tag{3.6}$$

where $\beta > \beta_0$. Thus from the likelihood (3.6) and the reference prior (3.5), the element $m_2^b(\mathbf{x}, \mathbf{y})$ of FBF under H_2 is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x}) &= \int_{\beta_0}^{\infty}\int_0^{\infty} L_2^b(\alpha, \beta|\mathbf{x})\pi_2^N(\alpha, \beta) d\alpha d\beta \\ &= \int_{\beta_0}^{\infty}\int_0^{\infty} \beta^{bn-1}\alpha^{bn\beta-1} \prod_{i=1}^{n} \frac{x_i^{b(\beta-1)}}{(\alpha^\beta + x_i^\beta)^{2b}} d\alpha d\beta. \end{aligned} \tag{3.7}$$

Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}, \tag{3.8}$$

where

$$S_1(\mathbf{x}) = \int_0^{\beta_0}\int_0^{\infty} \beta^{n-1}\alpha^{n\beta-1} \prod_{i=1}^{n} \frac{x_i^\beta}{(\alpha^\beta + x_i^\beta)^2} d\alpha d\beta$$

and

$$S_2(\mathbf{x}) = \int_{\beta_0}^{\infty}\int_0^{\infty} \beta^{n-1}\alpha^{n\beta-1} \prod_{i=1}^{n} \frac{x_i^\beta}{(\alpha^\beta + x_i^\beta)^2} d\alpha d\beta.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})} = \frac{S_1(\mathbf{x}; b)}{S_2(\mathbf{x}; b)}, \tag{3.9}$$

where

$$S_1(\mathbf{x}; b) = \int_0^{\beta_0}\int_0^{\infty} \beta^{bn-1}\alpha^{bn\beta-1} \prod_{i=1}^{n} \frac{x_i^{b\beta}}{(\alpha^\beta + x_i^\beta)^{2b}} d\alpha d\beta$$

and

$$S_2(\mathbf{x}; b) = \int_{\beta_0}^{\infty}\int_0^{\infty} \beta^{bn-1}\alpha^{bn\beta-1} \prod_{i=1}^{n} \frac{x_i^{b\beta}}{(\alpha^\beta + x_i^\beta)^{2b}} d\alpha d\beta.$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_1(\mathbf{x}; b) S_2(\mathbf{x})}{S_1(\mathbf{x}) S_2(\mathbf{x}; b)}. \tag{3.10}$$

Note that the calculations of the FBF of H_2 versus H_1 require two dimensional integration.

3.2. Bayesian hypothesis testing procedure based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal densities of (X_{j_1}, X_{j_2}) are finite for all $1 \le j_1 < j_2 \le n$ under each hypothesis (Kang *et al.*, 2014a). Thus we conclude that any training sample of size 2 is a minimal training sample.

We consider the encompassing model $H_0(= H_1 \cup H_2) : \beta > 0$. Then the hypotheses H_1 and H_2 are nested within H_0. Therefore under the encompassing model, the marginal density $m_0^N(x_{j_1}, x_{j_2})$ is given by

$$\begin{aligned} m_0^N(x_{j_1}, x_{j_2}) &= \int_0^\infty \int_0^\infty f(x_{j_1}, x_{j_2})|\alpha, \beta)\pi_1^N(\alpha, \beta) d\alpha d\beta \\ &= \int_0^\infty (x_{j_1} x_{j_2})^{(\beta-1)} \left[\frac{\beta(x_{j_1}^\beta + x_{j_2}^\beta)\log\frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^\beta - x_{j_1}^\beta)^3} - \frac{2}{(x_{j_2}^\beta - x_{j_1}^\beta)^2} \right] d\beta. \end{aligned}$$

Next for each hypothesis, the marginal density is as in the following. Under H_1 the marginal density $m_1^N(x_{j_1}, x_{j_2})$ is given by

$$\begin{aligned} m_1^N(x_{j_1}, x_{j_2}) &= \int_0^{\beta_0} \int_0^\infty f(x_{j_1}, x_{j_2})|\alpha, \beta)\pi_1^N(\alpha, \beta) d\alpha d\beta \\ &= \int_0^{\beta_0} (x_{j_1} x_{j_2})^{(\beta-1)} \left[\frac{\beta(x_{j_1}^\beta + x_{j_2}^\beta)\log\frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^\beta - x_{j_1}^\beta)^3} - \frac{2}{(x_{j_2}^\beta - x_{j_1}^\beta)^2} \right] d\beta. \end{aligned}$$

And the marginal density $m_2^N(x_{j_1}, x_{j_2})$ under H_2 is given by

$$\begin{aligned} m_2^N(x_{j_1}, x_{j_2}) &= \int_{\beta_0}^\infty \int_0^\infty f(x_{j_1}, x_{j_2})|\alpha, \beta)\pi_2^N(\alpha, \beta) d\alpha d\beta \\ &= \int_{\beta_0}^\infty (x_{j_1} x_{j_2})^{(\beta-1)} \left[\frac{\beta(x_{j_1}^\beta + x_{j_2}^\beta)\log\frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^\beta - x_{j_1}^\beta)^3} - \frac{2}{(x_{j_2}^\beta - x_{j_1}^\beta)^2} \right] d\beta. \end{aligned}$$

Therefore the EIBF of H_2 versus H_1 is given by

$$B_{21}^{EI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})} \left[\frac{\sum_{j_1<j_2}^n T_1(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}{\sum_{j_1<j_2}^n T_2(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})} \right], \tag{3.11}$$

where

$$\begin{aligned} T_0(x_{j_1}, x_{j_2}) &= \int_0^\infty (x_{j_1} x_{j_2})^{\beta} \left[\frac{\beta(x_{j_1}^\beta + x_{j_2}^\beta)\log\frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^\beta - x_{j_1}^\beta)^3} - \frac{2}{(x_{j_2}^\beta - x_{j_1}^\beta)^2} \right] d\beta, \\ T_1(x_{j_1}, x_{j_2}) &= \int_0^{\beta_0} (x_{j_1} x_{j_2})^{\beta} \left[\frac{\beta(x_{j_1}^\beta + x_{j_2}^\beta)\log\frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^\beta - x_{j_1}^\beta)^3} - \frac{2}{(x_{j_2}^\beta - x_{j_1}^\beta)^2} \right] d\beta \end{aligned}$$

and

$$T_2(x_{j_1}, x_{j_2}) = \int_{\beta_0}^{\infty} (x_{j_1} x_{j_2})^{\beta} \left[\frac{\beta(x_{j_1}^{\beta} + x_{j_2}^{\beta}) \log \frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^{\beta} - x_{j_1}^{\beta})^3} - \frac{2}{(x_{j_2}^{\beta} - x_{j_1}^{\beta})^2} \right] d\beta.$$

Also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})} ME \left[\frac{T_1(x_{j_1}, x_{j_2})}{T_2(x_{j_1}, x_{j_2})} \right]. \tag{3.12}$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require two dimensional integration.

4. Numerical studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of (α, β) and n. In particular, for fixed (α, β), we take 1,000 independent random samples of $\mathbf{X}_i$ with sample sizes n from the log-logistic distribution. We want to test the hypotheses $H_1 : \beta \leq 1$ versus $H_2 : \beta > 1$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities.

Table 4.1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities. In Table 4.1, $P^F(\cdot)$, $P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, EIBF and MIBF, respectively. From Table 4.1, the FBF, the EIBF and the MIBF accept the hypothesis H_1 when the values of α_2 are close to values of α_1, whereas reject the hypothesis H_1 when the values of α_2 are far from values of α_1. Also the EIBF and the MIBF give a similar behavior for all sample sizes. But the FBF favors the hypothesis H_2 than the EIBF and the MIBF when the values of α are close to 1 and are far from 1, respectively. That is, the FBF has considerable bias toward the hypothesis H_2. This fact does not a surprising result. Berger and Mortera (1999) showed that the FBF has considerable bias toward one of the hypotheses in nonsymmetric situations, and so the FBF should not be used in clearly nonsymmetric testing situations.

Example 4.1 This example is taken from Dey and Kundu (2010). The data is obtained from Lawless (1982), and it represents the number of revolution before failure of each 23 ball bearings in the life tests and they are as follows: 17.88, 28.92, 33.0, 41.52, 42.12, 45.60, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

Dey and Kundu (2010) concluded that the log-normal distribution and the log-logistic distribution have good fit for this data in terms of the log-likelihood values, Kolmogorov-Smirnov distances and the χ^2 values. For this data sets, the maximum likelihood estimates of β and α are 3.349 and 64.011, respectively.

We want to test the hypotheses $H_1 : \beta \leq \beta_0$ versus $H_2 : \beta > \beta_0$. The values of the Bayes factors and the posterior probabilities of H_1 are given in Table 4.2. From the results of Table 4.2, the posterior probabilities based on various Bayes factors give the same answer. The the EIBF and the MIBF give almost the same results. But the FBF favors the hypothesis H_2 than the EIBF and the MIBF.

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

β_0	α	β	n	$P^F(H_1\|\mathbf{x})$	$P^{EI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.0	0.5	0.5	5	0.670 (0.237)	0.857 (0.185)	0.812 (0.192)
			10	0.870 (0.183)	0.960 (0.094)	0.949 (0.102)
			20	0.976 (0.071)	0.995 (0.024)	0.993 (0.028)
		0.7	5	0.491 (0.211)	0.721 (0.228)	0.678 (0.217)
			10	0.613 (0.253)	0.826 (0.196)	0.803 (0.194)
			20	0.769 (0.236)	0.912 (0.141)	0.902 (0.144)
		0.9	5	0.379 (0.184)	0.593 (0.244)	0.563 (0.224)
			10	0.397 (0.217)	0.646 (0.246)	0.627 (0.232)
			20	0.436 (0.256)	0.683 (0.253)	0.669 (0.246)
		1.0	5	0.339 (0.165)	0.543 (0.234)	0.519 (0.215)
			10	0.315 (0.202)	0.547 (0.261)	0.534 (0.246)
			20	0.307 (0.225)	0.544 (0.270)	0.535 (0.261)
		1.1	5	0.301 (0.147)	0.492 (0.227)	0.475 (0.209)
			10	0.243 (0.164)	0.453 (0.243)	0.445 (0.230)
			20	0.203 (0.173)	0.406 (0.258)	0.400 (0.247)
		1.3	5	0.243 (0.128)	0.402 (0.210)	0.399 (0.199)
			10	0.162 (0.122)	0.325 (0.217)	0.328 (0.210)
			20	0.095 (0.104)	0.216 (0.202)	0.218 (0.195)
		1.5	5	0.202 (0.117)	0.333 (0.196)	0.334 (0.189)
			10	0.102 (0.091)	0.211 (0.181)	0.217 (0.177)
			20	0.040 (0.052)	0.100 (0.124)	0.105 (0.124)
1.0	1.0	0.5	5	0.682 (0.233)	0.866 (0.179)	0.818 (0.188)
			10	0.856 (0.186)	0.956 (0.092)	0.941 (0.104)
			20	0.973 (0.085)	0.993 (0.031)	0.991 (0.036)
		0.7	5	0.486 (0.205)	0.715 (0.225)	0.667 (0.212)
			10	0.614 (0.257)	0.822 (0.203)	0.796 (0.201)
			20	0.756 (0.244)	0.904 (0.149)	0.890 (0.154)
		0.9	5	0.387 (0.175)	0.608 (0.230)	0.574 (0.211)
			10	0.406 (0.225)	0.651 (0.248)	0.629 (0.235)
			20	0.445 (0.259)	0.689 (0.259)	0.671 (0.246)
		1.0	5	0.338 (0.156)	0.545 (0.227)	0.517 (0.208)
			10	0.323 (0.201)	0.555 (0.255)	0.539 (0.240)
			20	0.315 (0.224)	0.555 (0.264)	0.543 (0.254)
		1.1	5	0.304 (0.146)	0.493 (0.224)	0.476 (0.206)
			10	0.255 (0.171)	0.468 (0.250)	0.456 (0.233)
			20	0.214 (0.182)	0.418 (0.264)	0.411 (0.253)
		1.3	5	0.251 (0.131)	0.414 (0.224)	0.404 (0.203)
			10	0.166 (0.134)	0.325 (0.231)	0.322 (0.219)
			20	0.097 (0.103)	0.221 (0.201)	0.222 (0.194)
		1.5	5	0.213 (0.114)	0.351 (0.192)	0.352 (0.184)
			10	0.098 (0.084)	0.205 (0.171)	0.212 (0.169)
			20	0.043 (0.067)	0.103 (0.137)	0.106 (0.135)
1.0	5.0	0.5	5	0.672 (0.236)	0.856 (0.188)	0.778 (0.209)
			10	0.866 (0.185)	0.957 (0.102)	0.925 (0.135)
			20	0.976 (0.080)	0.994 (0.033)	0.988 (0.048)
		0.7	5	0.489 (0.212)	0.713 (0.231)	0.632 (0.224)
			10	0.606 (0.256)	0.816 (0.203)	0.752 (0.220)
			20	0.769 (0.240)	0.908 (0.147)	0.873 (0.171)
		0.9	5	0.379 (0.174)	0.596 (0.237)	0.529 (0.212)
			10	0.400 (0.222)	0.642 (0.247)	0.574 (0.236)
			20	0.451 (0.262)	0.687 (0.250)	0.635 (0.251)
		1.0	5	0.325 (0.152)	0.527 (0.229)	0.475 (0.204)
			10	0.310 (0.198)	0.533 (0.260)	0.481 (0.242)
			20	0.312 (0.228)	0.541 (0.269)	0.492 (0.257)
		1.1	5	0.296 (0.144)	0.479 (0.222)	0.436 (0.199)
			10	0.239 (0.165)	0.441 (0.253)	0.394 (0.227)
			20	0.204 (0.168)	0.401 (0.254)	0.363 (0.233)
		1.3	5	0.240 (0.121)	0.392 (0.200)	0.371 (0.185)
			10	0.161 (0.122)	0.314 (0.215)	0.286 (0.192)
			20	0.096 (0.099)	0.214 (0.193)	0.195 (0.171)
		1.5	5	0.202 (0.110)	0.332 (0.187)	0.316 (0.175)
			10	0.113 (0.097)	0.228 (0.186)	0.213 (0.168)
			20	0.045 (0.062)	0.106 (0.132)	0.100 (0.121)

Table 4.2 Bayes factor and posterior probabilities of $H_1 : \beta \leq \beta_0$

β_0	B_{21}^F	$P^F(H_1\|\mathbf{x})$	B_{21}^{EI}	$P^{AI}(H_1\|\mathbf{x})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x})$
2.5	17.7126	0.0534	6.6621	0.1305	6.2107	0.1387
3.0	4.7936	0.1726	1.5282	0.3955	1.5307	0.3951
3.3	2.5870	0.2788	0.7410	0.5744	0.7656	0.5664
3.6	1.4355	0.4106	0.3674	0.7313	0.3909	0.7190
4.0	0.6239	0.6158	0.1362	0.8801	0.1518	0.8682
4.5	0.1824	0.8457	0.0323	0.9688	0.0386	0.9629
5.0	0.0406	0.9610	0.0057	0.9943	0.0075	0.9926

5. Concluding remarks

In this paper, we developed the objective Bayesian one-sided hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the shape parameter of the log-logistic distribution under the reference priors. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. The EIBF and the MIBF give the similar results. However the FBF clearly favors the hypothesis H_2 than the EIBF and the MIBF, and has considerable biased to the hypothesis H_2. Therefore from resutls of our simulation and example, we recommend the use of the EIBF and the MIBF than the FBF for practical application in view of its simplicity and ease of implementation.

References

Ahmad, M. I., Sinclair, C. D. and Werritty, A. (1988). Log-logistic flood frequency analysis. *Journal of Hydrology*, **98**, 205-212.

Ali, M. M. and Khan, A. H. (1987). On order statistics from the log-logistic distribution. *Journal of Statistical Planning and Inference*, **17**, 103-108.

Balakrishnan, N. and Malik, H. J. (1987). Moments of order statistics form truncated log-logistic distribution. *Journal of Statistical Planning and Inference*, **17**, 251-267.

Bennett, S. (1983). Log-logistic regression models for survival data. *Journal of Royal Statistical Society*, C, **32**, 165-171.

Berger, J. O. and Bernardo, J. M. (1989). Estimating a product of means : Bayesian analysis with reference priors. *Journal of the American Statistical Association*, **84**, 200-207.

Berger, J. O. and Bernardo, J. M. (1992). On the development of reference priors (with discussion). *Bayesian Statistics IV*, edited by J. M. Bernardo, J. O. Berger, A. P. Dawid and A. F. M. Smith, Oxford University Press, Oxford, 35-60.

Berger, J. O. and Mortera, J. (1999). Defaut Bayes factor for one-sided hypothesis testing. *Journal of the American Statistical Association*, **94**, 542-554.

Berger, J. O. and Pericchi, L. R. (1996). The intrinsic Bayes factor for model selection and prediction. *Journal of the American Statistical Association*, **91**, 109-122.

Berger, J. O. and Pericchi, L. R. (1998). Accurate and stable Bayesian model selection: The median intrinsic Bayes factor. *Sankya B*, **60**, 1-18.

Berger, J. O. and Pericchi, L. R. (2001). Objective Bayesian methods for model selection: Introduction and comparison (with discussion). In *Model Selection, Institute of Mathematical Statistics Lecture Notes-Monograph Series*, **38**, 135-207.

Chen, Z. (1997). Exact confidence interval for the shape parameter of a log-logistic distribution. *Journal of Statistical Computation and Simulation*, **56**, 193-211.

Dmochowski, J. (1996). Intrinsic priors via Kullback-Leibler geometry. *Bayesian Statistics V*, edited by J. M. Bernardo, J. O. Berger, A. P. Dawid and A. F. M. Smith, Oxford University Press, Oxford, 543-549.

Dey, A. K. and Kundu, D. (2010). Discriminating between the log-normal and log-logistic distributions. *Communications in Statistics-Theory and Methods*, **39**, 280-292.

Fisk P. R. (1961). The graduation of income distributions. *Econometrica*, **29**, 171-185.

Franco, M. A. P. (1984). A log logistic model for survival time with covariates. *Biometrika*, **71**, 621-623.

Geskus, R. B. (2001). Methods for estimating the AIDS incubation time distribution when data of seroconversion is censored. *Statistics in Medicine*, **20**, 795-812.

Kang, S. G., Kim, D. H. and Lee, W. D. (2013). Default Bayesian testing for the scale parameters in two parameter exponential distributions. *Journal of the Korean Data & Information Science Society*, **24**, 949-957.

Kang, S. G., Kim, D. H. and Lee, W. D. (2014a). Noninformative priors for the log-logistic distribution. *Journal of the Korean Data & Information Science Society*, **25**, 227-235.

Kang, S. G., Kim, D. H. and Lee, W. D. (2014b). Default Bayesian testing for the scale parameters in the half logistic distributions. *Journal of the Korean Data & Information Science Society*, **25**, 465-472.

Lawless, J. F. (1982). *Statistical models and methods for lifetime data*, John Wiley and Sons, New York.

O'Hagan, A. (1995). Fractional Bayes factors for model comparison (with discussion). *Journal of Royal Statistical Society B*, **57**, 99-118.

O'Hagan, A. (1997). Properties of intrinsic and fractional Bayes factors. *Test*, **6**, 101-118.

Ragab, A. and Green, J. (1984). On order statistics from the log-logistic distribution and their properties. *Communication in Statistics-Theory and Methods*, **13**, 2713-2724.

Robson, A. and Reed, D. (1999). Statistical procedures for flood frequency estimation. *Flood estimation handbook, 3*, Institute of Hydrology, Wallingford, UK.

Shoukri, M. M., Mian I. U. M. and Tracy, C. (1988). Sampling properties of estimators of log-logistic distribution with application to Canadian precipitation data. *Canadian Journal of Statistics*, **16**, 223-236.

Spiegelhalter, D. J. and Smith, A. F. M. (1982). Bayes factors for linear and log-linear models with vague prior information. *Journal of Royal Statistical Society B*, **44**, 377-387.