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Default Bayesian testing for scale parameters in the log-logistic distributions

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Abstract

This paper deals with the problem of testing on the equality of the scale parameters in the log-logistic distributions. We propose default Bayesian testing procedures for the scale parameters under the reference priors. The reference prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. Therefore, we propose the default Bayesian testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor under the reference priors. To justify proposed procedures, a simulation study is provided and also, an example is given.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, log-logistic distribution, reference prior

1. Introduction

The log-logistic distribution is very useful in survival analysis of data sets such as survival times of cancer patients in which the hazard increases initially and decreases later (Bennett, 1983). Also in economics, the distributions of wealth or income are commonly distributed as Fisk distribution (Fisk, 1961). This Fisk distribution is an another form of a log-logistic distribution. For further details on the importance and applications of a log-logistic distribution, one may refer to Shoukri *et al.* (1988), Geskus (2001), Robson and Reed (1999) and Ahmad *et al.* (1988).

The present paper focuses on Bayesian testing of the equality of scale parameters in the log-logistic distributions. The log-logistic distribution can be used as the basis of an accelerated failure time model by allowing scale parameter to differ between groups. An accelerated life tests assume that the changes of the stress level make the scale parameter different between groups but the shape parameter is fixed. In this case, one may want to know whether a change of stress level makes a change of scale parameter or not. The equality of scale parameters means that the stress is not strong enough to change the scale parameter.

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In Bayesian model selection or testing problem, the Bayes factor plays an important role in this problem. This Bayes factor under improper priors contains arbitrary constants. For objective Bayesian inference, the noninformaive priors such as reference prior (Berger and Bernardo, 1989, 1992) or Jeffreys' prior are typically improper. The use of the noninformative priors for Bayesian model selection problem makes the Bayes factor be defined up to arbitrary constants. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have studied this problem.

To deal with this problem, Spiegelhalter and Smith (1982) have used the imaginary training sample. O'Hagan (1995) proposed the fractional Bayes factor for removing the constants. He used a portion of the likelihood. Berger and Pericchi (1996) proposed the use of training sample. And they called the their Bayes factor as the intrinsic Bayes factor.

The fractional Bayes factor and intrinsic Bayes factor have shown to be very useful in various model selection problem (Kang *et al.*, 2013, 2014b). So, we are interested in developing the intrinsic Bayes factor and the fractional Bayes factor for solving the equality of scale parameters in two log-logistic distributions.

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference prior, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor. In Section 4, to justify proposed procedures, a simulation study is provided and also, an example is given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses H_1, H_2, $\cdots$, H_q are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ of hypothesis H_i. Assume that the parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis H_i, $i = 1, 2, \cdots, q$. Then the posterior probability of the hypothesis H_i being true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}, \tag{2.2}$$

where $m_j(\mathbf{x})$ and $m_i(\mathbf{x})$ are posterior marginal distributions of hypotheses H_j and H_i, respectively.

The B_{ji} is interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. The use of noninformative prior π_i^N which is improper in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be used as a training sample and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \cdots, q, \tag{2.3}$$

where $m_i^N(\cdot)$ is a posterior marginal with a noninformative prior π_i^N.

In view (2.3), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor $B_{ji}(l)$ with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training samples $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample, which is the minimum of the training sample, to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all possible combination of the minimal training samples is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L}\sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all the training sample Bayes factors.

Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.5) and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b,of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}, \tag{2.7}$$

where m_i^b and m_j^b are posterior marginal distributions using the b fraction of likelihood and the noninformative priors π_i^N and π_j^N, respectively.

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. See O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995).

3. Bayesian hypothesis testing procedures

Suppose that the non-negative random variable X distributes as the log-logistic distribution with the scale parameter α and the shape parameter β. Then the probability density function of the random variable X is given by

$$f(x|\alpha,\beta)=\beta\alpha^{\beta}\frac{x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^2},\alpha,\beta>0,x>0.$$

Denote it as $X\sim\mathcal{LL}(\alpha,\beta)$.

Let $X_i, i=1,\cdots,n$ denote observations from the log-logistic distribution $\mathcal{LL}(\alpha_1,\beta_1)$ with the scale parameter α_1 and the shape parameter β_1, and let $Y_i, i=1,\cdots,m$ denote observations from $\mathcal{LL}(\alpha_2,\beta_2)$. Then likelihood function is given by

$$L(\alpha_1,\alpha_2,\beta_1,\beta_2|\mathbf{x},\mathbf{y}) \quad = \quad \beta_1^n\beta_2^m\alpha_1^{n\beta_1}\alpha_2^{m\beta_2}\prod_{i=1}^{n}\frac{x_i^{\beta_1-1}}{(\alpha_1^{\beta_1}+x_i^{\beta_1})^2}\prod_{i=1}^{m}\frac{y_i^{\beta_2-1}}{(\alpha_2^{\beta_2}+y_i^{\beta_2})^2}, \tag{3.1}$$

where $\mathbf{x}=(x_1,\cdots,x_n)$, $\mathbf{y}=(y_1,\cdots,y_m)$, $\alpha_1>0$, $\beta_1>0$, $\alpha_2>0$ and $\beta_2>0$. We are interested in testing the hypotheses $H_1:\alpha_1=\alpha_2$ versus $H_2:\alpha_1\neq\alpha_2$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing procedure based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1:\alpha_1=\alpha_2$ is

$$L_1(\alpha,\beta_1,\beta_2|\mathbf{x},\mathbf{y}) \quad = \quad \beta_1^n\beta_2^m\alpha^{n\beta_1+m\beta_2}\prod_{i=1}^{n}\frac{x_i^{\beta_1-1}}{(\alpha^{\beta_1}+x_i^{\beta_1})^2}\prod_{i=1}^{m}\frac{y_i^{\beta_2-1}}{(\alpha^{\beta_2}+y_i^{\beta_2})^2}. \tag{3.2}$$

And under the hypothesis H_1, the reference prior for (α,β_1,β_2) is

$$\pi_1^N(\alpha,\beta_1,\beta_2)\propto\alpha^{-1}\beta_1^{-1}\beta_2^{-1} \tag{3.3}$$

by Kang *et al.* (2014a). Then from the likelihood (3.2) and the reference prior (3.3), the element $m_1^b(\mathbf{x},\mathbf{y})$ of the FBF under H_1 is given by

$$\begin{aligned} & m_1^b(\mathbf{x},\mathbf{y}) \\ = & \int_0^\infty\int_0^\infty\int_0^\infty L_1^b(\alpha,\beta_1,\beta_2|\mathbf{x},\mathbf{y})\pi_1^N(\alpha,\beta_1,\beta_2)d\alpha d\beta_1 d\beta_2 \qquad (3.4) \\ = & \int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{bn-1}\beta_2^{bm-1}\alpha^{b(n\beta_1+m\beta_2)-1}\prod_{i=1}^{n}\frac{x_i^{b(\beta_1-1)}}{(\alpha^{\beta_1}+x_i^{\beta_1})^{2b}}\prod_{i=1}^{m}\frac{y_i^{b(\beta_2-1)}}{(\alpha^{\beta_2}+y_i^{\beta_2})^{2b}}d\alpha d\beta_1 d\beta_2. \end{aligned}$$

For the hypothesis H_2, the reference prior for $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ is

$$\pi_2^N(\alpha_1,\alpha_2,\beta_1,\beta_2)\propto\alpha_1^{-1}\alpha_2^{-1}\beta_1^{-1}\beta_2^{-1} \tag{3.5}$$

by Kang *et al.* (2014a). The likelihood function under the hypothesis H_2 is

$$L_2(\alpha_1,\alpha_2,\beta_1,\beta_2|\mathbf{x},\mathbf{y}) \quad = \quad \beta_1^n\beta_2^m\alpha_1^{n\beta_1}\alpha_2^{m\beta_2}\prod_{i=1}^{n}\frac{x_i^{\beta_1-1}}{(\alpha_1^{\beta_1}+x_i^{\beta_1})^2}\prod_{i=1}^{m}\frac{y_i^{\beta_2-1}}{(\alpha_2^{\beta_2}+y_i^{\beta_2})^2}. \tag{3.6}$$

Thus from the likelihood (3.6) and the reference prior (3.5), the element $m_2^b(\mathbf{x}, \mathbf{y})$ of FBF under H_2 is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L_2^b(\alpha_1, \alpha_2, \beta_1, \beta_2 | \mathbf{x}, \mathbf{y}) \pi_2^N(\alpha_1, \alpha_2, \beta_1, \beta_2) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{bn-1} \beta_2^{bm-1} \alpha_1^{bn\beta_1 - 1} \alpha_2^{bm\beta_2 - 1} \\ &\times \prod_{i=1}^n \frac{x_i^{b(\beta_1 - 1)}}{(\alpha_1^{\beta_1} + x_i^{\beta_1})^{2b}} \prod_{i=1}^m \frac{y_i^{b(\beta_2 - 1)}}{(\alpha_2^{\beta_2} + y_i^{\beta_2})^{2b}} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2. \end{aligned} \tag{3.7}$$

Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{m_2^N(\mathbf{x}, \mathbf{y})}{m_1^N(\mathbf{x}, \mathbf{y})} = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})}, \tag{3.8}$$

where

$$\begin{aligned} S_1(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{n-1} \beta_2^{m-1} \alpha^{n\beta_1 + m\beta_2 - 1} \\ &\times \prod_{i=1}^n \frac{x_i^{\beta_1}}{(\alpha^{\beta_1} + x_i^{\beta_1})^2} \prod_{i=1}^m \frac{y_i^{\beta_2}}{(\alpha^{\beta_2} + y_i^{\beta_2})^2} d\alpha d\beta_1 d\beta_2 \end{aligned}$$

and

$$\begin{aligned} S_2(\mathbf{x}, \mathbf{y}) &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{n-1} \beta_2^{m-1} \alpha_1^{n\beta_1 - 1} \alpha_2^{m\beta_2 - 1} \\ &\times \prod_{i=1}^n \frac{x_i^{\beta_1}}{(\alpha_1^{\beta_1} + x_i^{\beta_1})^2} \prod_{i=1}^m \frac{y_i^{\beta_2}}{(\alpha_2^{\beta_2} + y_i^{\beta_2})^2} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2. \end{aligned}$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x}, \mathbf{y})}{m_2^b(\mathbf{x}, \mathbf{y})} = \frac{S_1(\mathbf{x}, \mathbf{y}; b)}{S_2(\mathbf{x}, \mathbf{y}; b)}, \tag{3.9}$$

where

$$\begin{aligned} S_1(\mathbf{x}, \mathbf{y}; b) &= \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{bn-1} \beta_2^{bm-1} \alpha^{b(n\beta_1 + m\beta_2) - 1} \\ &\times \prod_{i=1}^n \frac{x_i^{b\beta_1}}{(\alpha^{\beta_1} + x_i^{\beta_1})^{2b}} \prod_{i=1}^m \frac{y_i^{b\beta_2}}{(\alpha^{\beta_2} + y_i^{\beta_2})^{2b}} d\alpha d\beta_1 d\beta_2 \end{aligned}$$

and

$$\begin{aligned} S_2(\mathbf{x}, \mathbf{y}; b) &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{bn-1} \beta_2^{bm-1} \alpha_1^{bn\beta_1 - 1} \alpha_2^{bm\beta_2 - 1} \\ &\times \prod_{i=1}^n \frac{x_i^{b\beta_1}}{(\alpha_1^{\beta_1} + x_i^{\beta_1})^{2b}} \prod_{i=1}^m \frac{y_i^{b\beta_2}}{(\alpha_2^{\beta_2} + y_i^{\beta_2})^{2b}} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2. \end{aligned}$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_1(\mathbf{x}, \mathbf{y}; b) S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y}) S_2(\mathbf{x}, \mathbf{y}; b)}. \tag{3.10}$$

Note that the calculations of the FBF of H_2 versus H_1 require four dimensional integration.

3.2. Bayesian hypothesis testing procedure based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal densities of $(X_{j_1}, X_{j_2}, Y_{k_1}, Y_{k_2})$ are finite for all $1 \leq j_1 < j_2 \leq n$ and $1 \leq k_1 < k_2 \leq m$ under each hypothesis (see Theorem 3.1 of Kang *et al.* (2014a)). Thus we conclude that any training sample of size 4 is a minimal training sample.

The marginal density $m_1^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})$ under H_1 is given by

$$\begin{aligned}
& m_1^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) \\
= & \int_0^\infty \int_0^\infty \int_0^\infty f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2} | \alpha, \beta_1, \beta_2) \pi_1^N(\alpha, \beta_1, \beta_2) d\alpha d\beta_1 d\beta_2 \\
= & \int_0^\infty \int_0^\infty \int_0^\infty \beta_1\beta_2 \alpha^{2(\beta_1+\beta_2)-1} \frac{(x_{j_1}x_{j_2})^{(\beta_1-1)}}{(\alpha^{\beta_1} + x_{j_1}^{\beta_1})^2} \frac{(y_{k_1}y_{k_2})^{(\beta_2-1)}}{(\alpha^{\beta_2} + y_{k_1}^{\beta_2})^2} \frac{(\alpha^{\beta_1} + x_{j_2}^{\beta_1})^{-2}}{(\alpha^{\beta_2} + y_{k_2}^{\beta_2})^2} d\alpha d\beta_1 d\beta_2.
\end{aligned}$$

And the marginal density $m_2^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})$ under H_2 is given by

$$\begin{aligned}
& m_2^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) \\
= & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2} | \alpha_1, \alpha_2, \beta_1, \beta_2) \pi_2^N(\alpha_1, \alpha_2, \beta_1, \beta_2) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \\
= & \int_0^\infty \int_0^\infty (x_{j_1}x_{j_2})^{(\beta_1-1)} (y_{k_1}y_{k_2})^{(\beta_2-1)} \left[\frac{\beta_1(x_{j_1}^{\beta_1} + x_{j_2}^{\beta_1}) \log \frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^{\beta_1} - x_{j_1}^{\beta_1})^3} - \frac{2}{(x_{j_2}^{\beta_1} - x_{j_1}^{\beta_1})^2} \right] \\
\times & \left[\frac{\beta_2(y_{k_1}^{\beta_2} + y_{k_2}^{\beta_2}) \log \frac{y_{k_2}}{y_{k_1}}}{(y_{k_2}^{\beta_2} - y_{k_1}^{\beta_2})^3} - \frac{2}{(y_{k_2}^{\beta_2} - y_{k_1}^{\beta_2})^2} \right] d\beta_1 d\beta_2.
\end{aligned}$$

Note that in the above marginal density m_2^N, integration with respect to α_1 and α_2 is given by

$$\begin{aligned}
& \int_0^\infty \int_0^\infty f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2} | \alpha_1, \alpha_2, \beta_1, \beta_2) \pi_2^N(\alpha_1, \alpha_2, \beta_1, \beta_2) d\alpha_1 d\alpha_2 \\
= & \int_0^\infty \int_0^\infty \beta_1\beta_2 \alpha_1^{2\beta_1-1} \alpha_2^{2\beta_2-1} \frac{(x_{j_1}x_{j_2})^{(\beta_1-1)}}{(\alpha_1^{\beta_1} + x_{j_1}^{\beta_1})^2 (\alpha_1^{\beta_1} + x_{j_2}^{\beta_1})^2} \frac{(y_{k_1}y_{k_2})^{(\beta_2-1)}}{(\alpha_2^{\beta_2} + y_{k_1}^{\beta_2})^2 (\alpha_2^{\beta_2} + y_{k_2}^{\beta_2})^2} d\alpha_1 d\alpha_2. \\
= & (x_{j_1}x_{j_2})^{(\beta_1-1)} \left[\frac{1}{(x_{j_2}^{\beta_1} - x_{j_1}^{\beta_1})^2} \left(\frac{x_{j_1}^{\beta_1}}{\alpha_1^{\beta_1} + x_{j_1}^{\beta_1}} + \frac{x_{j_2}^{\beta_1}}{\alpha_1^{\beta_1} + x_{j_2}^{\beta_1}} \right) - \frac{(x_{j_1}^{\beta_1} + x_{j_2}^{\beta_1})}{(x_{j_2}^{\beta_1} - x_{j_1}^{\beta_1})^3} \log \frac{\alpha_1^{\beta_1} + x_{j_2}^{\beta_1}}{\alpha_1^{\beta_1} + x_{j_1}^{\beta_1}} \right]_0^\infty \\
\times & (y_{k_1}y_{k_2})^{(\beta_2-1)} \left[\frac{1}{(y_{k_2}^{\beta_2} - y_{k_1}^{\beta_2})^2} \left(\frac{y_{k_1}^{\beta_2}}{\alpha_2^{\beta_2} + y_{k_1}^{\beta_2}} + \frac{y_{j_2}^{\beta_2}}{\alpha_2^{\beta_2} + y_{k_2}^{\beta_2}} \right) - \frac{(y_{k_1}^{\beta_2} + y_{k_2}^{\beta_2})}{(y_{k_2}^{\beta_2} - y_{k_1}^{\beta_2})^3} \log \frac{\alpha_2^{\beta_2} + y_{k_2}^{\beta_2}}{\alpha_2^{\beta_2} + y_{k_1}^{\beta_2}} \right]_0^\infty \\
= & (x_{j_1}x_{j_2})^{(\beta_1-1)} (y_{k_1}y_{k_2})^{(\beta_2-1)} \left[\frac{\beta_1(x_{j_1}^{\beta_1} + x_{j_2}^{\beta_1}) \log \frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^{\beta_1} - x_{j_1}^{\beta_1})^3} - \frac{2}{(x_{j_2}^{\beta_1} - x_{j_1}^{\beta_1})^2} \right] \\
\times & \left[\frac{\beta_2(y_{k_1}^{\beta_2} + y_{k_2}^{\beta_2}) \log \frac{y_{k_2}}{y_{k_1}}}{(y_{k_2}^{\beta_2} - y_{k_1}^{\beta_2})^3} - \frac{2}{(y_{k_2}^{\beta_2} - y_{k_1}^{\beta_2})^2} \right].
\end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\mathbf{x},\mathbf{y})}{S_1(\mathbf{x},\mathbf{y})}\left[\frac{1}{L}\sum_{j_1<j_2,k_1<k_2}\frac{T_1(x_{j_1},x_{j_2},y_{k_1},y_{k_2})}{T_2(x_{j_1},x_{j_2},y_{k_1},y_{k_2})}\right], \tag{3.11}$$

where $L = nm(n-1)(m-1)/4$,

$$\begin{aligned} &T_1(x_{j_1},x_{j_2},y_{k_1},y_{k_2})\\ &= \int_0^\infty\int_0^\infty\int_0^\infty \beta_1\beta_2\alpha^{2(\beta_1+\beta_2)-1}\frac{(x_{j_1}x_{j_2})^{\beta_1-1}(y_{k_1}y_{k_2})^{\beta_2-1}}{(\alpha^{\beta_1}+x_{j_1}^{\beta_1})^2(\alpha^{\beta_1}+x_{j_2}^{\beta_1})^2}\frac{(\alpha^{\beta_2}+y_{k_2}^{\beta_2})^{-2}}{(\alpha^{\beta_2}+y_{k_1}^{\beta_2})^2}d\alpha d\beta_1 d\beta_2 \end{aligned}$$

and

$$\begin{aligned} &T_2(x_{j_1},x_{j_2},y_{k_1},y_{k_2})\\ &= \int_0^\infty\int_0^\infty (x_{j_1}x_{j_2})^{\beta_1-1}(y_{k_1}y_{k_2})^{\beta_2-1}\left[\frac{\beta_1(x_{j_1}^{\beta_1}+x_{j_2}^{\beta_1})\log\frac{x_{j_2}}{x_{j_1}}}{(x_{j_2}^{\beta_1}-x_{j_1}^{\beta_1})^3}-\frac{2}{(x_{j_2}^{\beta_1}-x_{j_1}^{\beta_1})^2}\right]\\ &\times\left[\frac{\beta_2(y_{k_1}^{\beta_2}+y_{k_2}^{\beta_2})\log\frac{y_{k_2}}{y_{k_1}}}{(y_{k_2}^{\beta_2}-y_{k_1}^{\beta_2})^3}-\frac{2}{(y_{k_2}^{\beta_2}-y_{k_1}^{\beta_2})^2}\right]d\beta_1 d\beta_2. \end{aligned}$$

Also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x},\mathbf{y})}{S_1(\mathbf{x},\mathbf{y})}ME\left[\frac{T_1(x_{j_1},x_{j_2},y_{k_1},y_{k_2})}{T_2(x_{j_1},x_{j_2},y_{k_1},y_{k_2})}\right]. \tag{3.12}$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require three dimensional integration.

4. Numerical studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ and (n,m). In particular, for fixed $(\alpha_1,\beta_1,\alpha_2,\beta_2)$, we take 500 independent random samples of $\mathbf{X}_i$ and $\mathbf{Y}_i$ with sample sizes n and m from the log-logistic distributions, respectively.

In this simulation study, we assume that when the scale parameter $\alpha_1 = 1$, α_2=0.3, 0.8, 1, 1.6, 3 and when $\alpha_1 = 3$, α_2=1.8, 2.5, 3, 3.5, 5. Also, for the shape parameters, we set $(\beta_1,\beta_2) = (1,1), (1,3), (3,3)$.

We want to test the hypotheses $H_1 : \alpha_1 = \alpha_2$ versus $H_2 : \alpha_1 \neq \alpha_2$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities.

Tables 4.1 and 4.2 show the results of the averages and the standard deviations in parentheses of posterior probabilities. In Tables 4.1 and 4.2, $P^F(\cdot)$, $P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, AIBF and MIBF, respectively. From Tables 4.1 and 4.2, the FBF, the AIBF and the MIBF accept the hypothesis H_1 when the values of α_2 are close to values of α_1, whereas reject the hypothesis

H_1 when the values of α_2 are far from values of α_1. Also the AIBF and the MIBF give a similar behavior for all sample sizes. However the AIBF and the MIBF favor the hypothesis H_1 than the FBF.

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

β_1	β_2	α_1	α_2	n, m	$P^F(H_1\|\mathbf{x},\mathbf{y})$	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
1.0	1.0	1.0	0.3	5,5	0.503 (0.157)	0.565 (0.166)	0.603 (0.161)
				5,10	0.516 (0.192)	0.562 (0.190)	0.603 (0.186)
				10,10	0.473 (0.230)	0.520 (0.238)	0.558 (0.241)
				10,20	0.441 (0.263)	0.473 (0.266)	0.510 (0.271)
			0.8	5,5	0.586 (0.099)	0.653 (0.105)	0.688 (0.098)
				5,10	0.624 (0.121)	0.673 (0.116)	0.711 (0.111)
				10,10	0.647 (0.137)	0.702 (0.134)	0.738 (0.129)
				10,20	0.688 (0.140)	0.727 (0.134)	0.761 (0.129)
			1.0	5,5	0.573 (0.116)	0.636 (0.159)	0.677 (0.118)
				5,10	0.628 (0.124)	0.678 (0.123)	0.716 (0.116)
				10,10	0.652 (0.126)	0.709 (0.124)	0.745 (0.117)
				10,20	0.689 (0.144)	0.728 (0.140)	0.762 (0.134)
			1.6	5,5	0.558 (0.117)	0.641 (0.125)	0.677 (0.119)
				5,10	0.602 (0.141)	0.658 (0.143)	0.695 (0.138)
				10,10	0.610 (0.165)	0.677 (0.166)	0.713 (0.162)
				10,20	0.653 (0.173)	0.698 (0.172)	0.734 (0.167)
			3.0	5,5	0.512 (0.148)	0.609 (0.153)	0.653 (0.147)
				5,10	0.501 (0.194)	0.575 (0.202)	0.618 (0.196)
				10,10	0.491 (0.213)	0.577 (0.219)	0.617 (0.217)
				10,20	0.476 (0.242)	0.538 (0.252)	0.577 (0.253)
1.0	3.0	1.0	0.3	5,5	0.450 (0.188)	0.492 (0.195)	0.535 (0.190)
				5,10	0.435 (0.224)	0.457 (0.215)	0.499 (0.214)
				10,10	0.362 (0.249)	0.388 (0.254)	0.428 (0.260)
				10,20	0.361 (0.248)	0.372 (0.247)	0.411 (0.255)
			0.8	5,5	0.576 (0.125)	0.619 (0.130)	0.652 (0.125)
				5,10	0.611 (0.148)	0.621 (0.144)	0.661 (0.135)
				10,10	0.654 (0.142)	0.688 (0.144)	0.725 (0.139)
				10,20	0.677 (0.170)	0.682 (0.167)	0.718 (0.164)
			1.0	5,5	0.577 (0.120)	0.624 (0.129)	0.660 (0.125)
				5,10	0.628 (0.122)	0.640 (0.123)	0.678 (0.116)
				10,10	0.656 (0.140)	0.691 (0.140)	0.729 (0.135)
				10,20	0.690 (0.148)	0.695 (0.145)	0.733 (0.140)
			1.6	5,5	0.551 (0.146)	0.598 (0.161)	0.637 (0.156)
				5,10	0.591 (0.146)	0.606 (0.152)	0.643 (0.145)
				10,10	0.613 (0.166)	0.653 (0.174)	0.691 (0.170)
				10,20	0.640 (0.187)	0.646 (0.191)	0.685 (0.188)
			3.0	5,5	0.457 (0.172)	0.510 (0.192)	0.552 (0.187)
				5,10	0.478 (0.199)	0.504 (0.209)	0.549 (0.204)
				10,10	0.380 (0.240)	0.420 (0.259)	0.462 (0.265)
				10,20	0.414 (0.253)	0.431 (0.261)	0.472 (0.267)
3.0	3.0	1.0	0.3	5,5	0.233 (0.169)	0.246 (0.172)	0.273 (0.181)
				5,10	0.167 (0.170)	0.168 (0.163)	0.191 (0.175)
				10,10	0.042 (0.078)	0.043 (0.078)	0.051 (0.087)
				10,20	0.024 (0.067)	0.023 (0.062)	0.027 (0.071)
			0.8	5,5	0.620 (0.113)	0.628 (0.117)	0.652 (0.113)
				5,10	0.654 (0.148)	0.632 (0.147)	0.658 (0.144)
				10,10	0.650 (0.188)	0.654 (0.190)	0.678 (0.187)
				10,20	0.675 (0.204)	0.651 (0.204)	0.675 (0.202)
			1.0	5,5	0.625 (0.113)	0.633 (0.118)	0.655 (0.115)
				5,10	0.703 (0.112)	0.682 (0.113)	0.704 (0.111)
				10,10	0.707 (0.133)	0.710 (0.133)	0.733 (0.131)
				10,20	0.753 (0.133)	0.728 (0.145)	0.752 (0.133)
			1.6	5,5	0.511 (0.184)	0.523 (0.186)	0.552 (0.184)
				5,10	0.528 (0.223)	0.512 (0.220)	0.541 (0.218)
				10,10	0.464 (0.244)	0.469 (0.246)	0.496 (0.248)
				10,20	0.438 (0.269)	0.420 (0.263)	0.444 (0.268)
			3.0	5,5	0.264 (0.182)	0.287 (0.188)	0.317 (0.196)
				5,10	0.182 (0.171)	0.187 (0.168)	0.210 (0.178)
				10,10	0.058 (0.100)	0.064 (0.106)	0.074 (0.116)
				10,20	0.035 (0.078)	0.035 (0.077)	0.041 (0.086)

Table 4.2 The averages and the standard deviations in parentheses of posterior probabilities

β_1	β_2	α_1	α_2	n, m	$P^F(H_1\|\mathbf{x},\mathbf{y})$	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
1.0	1.0	3.0	1.8	5,5	0.567 (0.119)	0.678 (0.131)	0.715 (0.125)
				5,10	0.606 (0.126)	0.695 (0.125)	0.734 (0.117)
				10,10	0.602 (0.154)	0.697 (0.151)	0.735 (0.141)
				10,20	0.639 (0.170)	0.712 (0.161)	0.749 (0.154)
			2.5	5,5	0.579 (0.098)	0.703 (0.113)	0.738 (0.104)
				5,10	0.632 (0.103)	0.724 (0.107)	0.762 (0.099)
				10,10	0.640 (0.123)	0.740 (0.120)	0.777 (0.113)
				10,20	0.674 (0.130)	0.753 (0.122)	0.790 (0.109)
			3.0	5,5	0.593 (0.086)	0.719 (0.099)	0.754 (0.090)
				5,10	0.638 (0.104)	0.734 (0.108)	0.771 (0.099)
				10,10	0.647 (0.110)	0.755 (0.107)	0.790 (0.100)
				10,20	0.679 (0.124)	0.761 (0.115)	0.796 (0.108)
			3.5	5,5	0.575 (0.106)	0.703 (0.119)	0.737 (0.113)
				5,10	0.621 (0.118)	0.724 (0.120)	0.760 (0.113)
				10,10	0.638 (0.119)	0.754 (0.115)	0.790 (0.107)
				10,20	0.682 (0.136)	0.764 (0.135)	0.799 (0.128)
			5.0	5,5	0.582 (0.103)	0.730 (0.118)	0.765 (0.111)
				5,10	0.619 (0.123)	0.734 (0.133)	0.772 (0.122)
				10,10	0.621 (0.141)	0.745 (0.141)	0.784 (0.131)
				10,20	0.628 (0.181)	0.728 (0.183)	0.766 (0.176)
1.0	3.0	3.0	1.8	5,5	0.551 (0.133)	0.631 (0.130)	0.668 (0.124)
				5,10	0.587 (0.148)	0.631 (0.138)	0.671 (0.130)
				10,10	0.581 (0.183)	0.644 (0.177)	0.685 (0.173)
				10,20	0.618 (0.193)	0.650 (0.185)	0.689 (0.181)
			2.5	5,5	0.577 (0.103)	0.665 (0.110)	0.700 (0.101)
				5,10	0.618 (0.124)	0.660 (0.125)	0.698 (0.119)
				10,10	0.626 (0.154)	0.695 (0.146)	0.732 (0.139)
				10,20	0.670 (0.154)	0.702 (0.148)	0.741 (0.142)
			3.0	5,5	0.579 (0.096)	0.664 (0.115)	0.700 (0.106)
				5,10	0.621 (0.126)	0.667 (0.131)	0.703 (0.124)
				10,10	0.638 (0.128)	0.709 (0.127)	0.748 (0.122)
				10,20	0.688 (0.132)	0.719 (0.131)	0.755 (0.128)
			3.5	5,5	0.574 (0.107)	0.661 (0.127)	0.698 (0.116)
				5,10	0.615 (0.130)	0.661 (0.140)	0.701 (0.133)
				10,10	0.634 (0.130)	0.708 (0.135)	0.746 (0.130)
				10,20	0.666 (0.153)	0.700 (0.159)	0.737 (0.155)
			5.0	5,5	0.540 (0.135)	0.637 (0.161)	0.675 (0.155)
				5,10	0.570 (0.172)	0.624 (0.192)	0.662 (0.187)
				10,10	0.558 (0.181)	0.641 (0.196)	0.683 (0.191)
				10,20	0.605 (0.195)	0.650 (0.205)	0.691 (0.202)
3.0	3.0	3.0	1.8	5,5	0.481 (0.181)	0.505 (0.185)	0.536 (0.180)
				5,10	0.495 (0.232)	0.493 (0.228)	0.521 (0.228)
				10,10	0.424 (0.251)	0.441 (0.256)	0.468 (0.259)
				10,20	0.390 (0.270)	0.383 (0.272)	0.410 (0.272)
			2.5	5,5	0.597 (0.125)	0.628 (0.145)	0.651 (0.125)
				5,10	0.641 (0.150)	0.637 (0.160)	0.665 (0.144)
				10,10	0.660 (0.159)	0.682 (0.159)	0.706 (0.156)
				10,20	0.702 (0.170)	0.695 (0.170)	0.719 (0.166)
			3.0	5,5	0.616 (0.103)	0.648 (0.108)	0.671 (0.104)
				5,10	0.680 (0.122)	0.679 (0.121)	0.703 (0.118)
				10,10	0.691 (0.137)	0.713 (0.138)	0.736 (0.136)
				10,20	0.746 (0.132)	0.741 (0.133)	0.762 (0.130)
			3.5	5,5	0.606 (0.109)	0.642 (0.114)	0.665 (0.110)
				5,10	0.651 (0.139)	0.652 (0.139)	0.677 (0.134)
				10,10	0.667 (0.149)	0.694 (0.148)	0.718 (0.146)
				10,20	0.699 (0.171)	0.694 (0.181)	0.719 (0.168)
			5.0	5,5	0.483 (0.178)	0.532 (0.183)	0.560 (0.180)
				5,10	0.503 (0.204)	0.517 (0.206)	0.547 (0.203)
				10,10	0.420 (0.246)	0.454 (0.255)	0.481 (0.258)
				10,20	0.371 (0.265)	0.376 (0.267)	0.401 (0.273)

Example 4.1 This example is taken from Dey and Kundu (2010). The data is obtained from Lawless (1982), and it represents the number of revolution before failure of each 23 ball bearings in the life tests. Dey and Kundu (2010) concluded that the log-normal distribution and the log-logistic distribution have good fit for this data in terms of the log-likelihood

values, Kolmogorov-Smirnov distances and the χ^2 values. For testing the equality of the scale parameters, we randomly divided this data into two groups. The data sets are given by

Group 1 : 33.0, 41.52, 42.12, 48.8, 51.84, 54.12, 55.56, 68.44, 68.88, 84.12, 93.12, 105.12, 128.04

Group 2 : 173.4, 45.6, 98.64, 68.64, 105.84, 28.92, 127.92, 51.96, 17.88, 67.8

For this data sets, the maximum likelihood estimates of (α_1, β_1) and (α_2, β_2) are (67.47, 2.60) and (61.39, 4.39), respectively.

We want to test the hypotheses $H_1 : \alpha_1 = \alpha_2$ versus $H_2 : \alpha_1 \neq \alpha_2$. The values of the Bayes factors and the posterior probabilities of H_1 are given in Table 4.3. From the results of Table 4.3, the posterior probabilities based on various Bayes factors give the same answer. The FBF, the AIBF and the MIBF select the hypothesis H_1 and the values of AIBF and MIBF are almost the same.

Table 4.3 Bayes factor and posterior probabilities of $H_1 : \alpha_1 = \alpha_2$

B_{21}^F	$P^F(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{AI}	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
0.09745	0.91120	0.00047	0.99953	0.00029	0.99971

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the scale parameters of the log-logistic distributions under the reference priors. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. The FBF favours the hypothesis H_2 rather than the AIBF and the MIBF, and the FBF and the AIBF give the similar results. From our simulation and example, we recommend the use of the FBF rather than the AIBF and MIBF for practical application in view of its simplicity and ease of implementation.

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