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On the maximum and minimum in a bivariate uniform distribution

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Abstract

We obtain means and variances of $\max\{X,Y\}$ and $\min\{X,Y\}$ in the underlying Morgenstern type bivariate uniform variables X and Y with same scale parameters and different scale parameters respectively. And we obtain the conditional expectations in the underlying Morgenstern type bivariate uniform variables. Here, we shall consider the conditional expectations to know the dependence of one variable on the other variable and we consider the behaviors of means and variances of $\max\{X,Y\}$ and $\min\{X,Y\}$ with respect to changes in means, variances, and the correlation coefficient of the underlying Morgenstern type bivariate uniform variables

Keywords: Correlation coefficient, maximum, minimum, monotone function, Morgenstern bivariate uniform distribution.

1. Introduction

Tahmasebi and Jafari (2012) studied estimations of a scale parameter of Morgenstern type bivariate uniform distribution. A biological study on purslane plants (portulaca oleracea) has done in the research recently. The applications of this study show that the shoot height of the plant is a correlated character with the shoot diameter in Tahmasebi and Jafari (2012).

There are similar situations in the engineering science whenever subsystems are considered having two components with life times (X,Y) being independent of the life time of the entire system. In many applications in a biology, X and Y could be the shoot height and the shoot diameter of a plant in Tahmasebi and Jafari (2012).

And it is important for us to know which of two variables, X and Y, is larger or smaller. Exact distributions for $M = \max\{X,Y\}$ and $m = \min\{X,Y\}$ are useful in lots of the application including a biology, economics, an operation research and genetics.

And we consider the behaviors of means and variances of $M = \max\{X,Y\}$ and $m = \min\{X,Y\}$ with respect to changes in means, variances, and the correlation coefficient of underlying bivariate variables X and Y. In the extreme value literature, two papers have studied

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the distributions of $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$ with respect to the means, variances and the correlation coefficient in Ker (2001) and Lien (2005), and Hakamipour et al. (2011) studied extremes of a bivariate Pareto distribution. Woo and Nadarajah (2013) studied maximum and minimum of multivariate Pareto random variables. Lee and Kang (2014) studied some properties for maximum and minimum in a bivariate exponential distribution with a dependence parameter

In this paper, we obtain means and variances of $\max\{X, Y\}$ and $\min\{X, Y\}$ in the underlying Morgenstern type bivariate uniform variables X and Y with same scale parameters and different scale parameters respectively. And we obtain the conditional expectations in the underlying Morgenstern type bivariate uniform variables. We shall consider the conditional expectations to know the dependence of one variable on the other variable and we shall consider the behaviors of means and variances of $\max\{X, Y\}$ and $\min\{X, Y\}$ with respect to changes in means, variances, and the correlation coefficient of the underlying Morgenstern type bivariate uniform variables

2. Property of maximum and minimum

A general family of bivariate distributions is proposed by Morgenstern (1956) with specified marginal distributions $F_X(x)$ and $F_Y(x)$ as

$$F_{X,Y}(x, y) = F_x(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))], \ \ |\alpha| \le 1,$$

where α is the association parameter between X and Y. A member of this family is the Morgenstern type bivariate uniform distribution with the probability density function,

$$f_{X,Y}(x, y) = \frac{1}{\theta_1\theta_2}\left[1 + \alpha\left(1 - \frac{2x}{\theta_1}\right)\left(1 - \frac{2y}{\theta_2}\right)\right], \text{ for } 0 < x < \theta_1 \text{ and } 0 < y < \theta_2, \quad (2.1)$$

where $|\alpha| \le 1$.

The marginal density functions of X and Y are the uniform density functions over $(0, \theta_1)$ and $(0, \theta_2)$, respectively.

From the density function (2.1), we can obtain the product moment and the correlation coefficient between X and Y as follows :

$$E(XY) = \theta_1 \cdot \theta_2 \left(\frac{1}{4} + \frac{1}{36}\alpha\right) \text{ and } \rho_{X,Y} \equiv \rho = \frac{1}{3}\alpha. \quad (2.2)$$

From the marginal density function of Y and the Morgenstern type bivariate uniform density function (2.1), the conditional density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{1}{\theta_1}\left[1 + \alpha\left(1 - \frac{2}{\theta_1}x\right)\left(1 - \frac{2}{\theta_2}y\right)\right] \text{ for } 0 < x < \theta_1, \quad (2.3)$$

and similarly, the conditional density function of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{1}{\theta_2}\left[1 + \alpha\left(1 - \frac{2}{\theta_1}x\right)\left(1 - \frac{2}{\theta_2}y\right)\right] \text{ for } 0 < y < \theta_2. \quad (2.4)$$

In the reliability of the engineering and sciences, it is a useful for us to consider the conditional expectation to know dependence of one variable on the other variable as the following. From the conditional density functions in (2.3) and (2.4), we can obtain conditional expectations as follows :

$$E(X|Y=y) = -\rho \cdot \frac{\theta_1}{\theta_2} y + \frac{\theta_1}{2}\left(1+\frac{\rho}{3}\right),$$

$$E(Y|X=x) = -\rho \cdot \frac{\theta_2}{\theta_1} x + \frac{\theta_2}{2}\left(1+\frac{\rho}{3}\right).$$

Therefore, $E(X|Y=y)$ and $E(Y|X=y)$ increasing functions of y and x respectively if $\rho < 0$, elsewhere they are decreasing functions.

Here, we consider the behaviors of means and variances of $\max\{X,Y\}$ and $\min\{X,Y\}$ with respect to changes in means, variances, and the correlation coefficient of the underlying Morgenstern type bivariate uniform variables.

Let (X,Y) have the Morgenstern type bivariate uniform density function (2.1).

Then, we can obtain the k-th moments of $M = \max\{X,Y\}$ and $m = \min\{X,Y\}$ as follows :

(a) If $\theta_1 \geq \theta_2$, then

$$\begin{aligned} E(M^k) =& \frac{1}{k+1}\left(\theta_1^k - \frac{\theta_2^{k+1}}{\theta_1}\right) + \frac{2}{k+2}\frac{\theta_2^{k+1}}{\theta_1} \\ &+ \alpha\left[\frac{2}{k+2}\frac{\theta_2^{k+1}}{\theta_1} - \frac{3}{k+3}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\frac{\theta_2^{k+2}}{\theta_1} + \frac{4}{k+4}\frac{\theta_2^{k+2}}{\theta_1^2}\right] \end{aligned} \tag{2.5}$$

and

$$E(m^k) = \frac{1}{k+1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\theta_2^{k+1} + \frac{1}{k+1}\left(\theta_2^k - \frac{\theta_1^{k+1}}{\theta_2}\right) - E(M^k).$$

(b) If $\theta_1 < \theta_2$, then

$$\begin{aligned} E(M^k) =& \frac{1}{k+1}\left(\theta_2^k - \frac{\theta_1^{k+1}}{\theta_2}\right) + \frac{2}{k} + 2\frac{\theta_1^{k+1}}{\theta_2} \\ &+ \alpha\left[\frac{2}{k+2}\frac{\theta_1^{k+1}}{\theta_2} - \frac{3}{k+3}\left(\frac{1}{\theta_1\theta_2}+\frac{1}{\theta_2^2}\right)\theta_1^{k+2} + \frac{4}{k+4}\frac{\theta_1^{k+2}}{\theta_2^2}\right] \end{aligned} \tag{2.6}$$

and

$$E(m^k) = \frac{1}{k+1}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\theta_1^{k+1} + \frac{1}{k+1}\left(\theta_1^k - \frac{\theta_2^{k+1}}{\theta_1}\right) - E(M^k).$$

Especially, if $\theta_1 = \theta_2 = \theta$ in the density function (2.1), then

$$E(M^k) = 2\left[\frac{1}{k+2} + \alpha\left(\frac{1}{k+2} - \frac{3}{k+3} + \frac{2}{k+4}\right)\right]\theta^k \text{ and } E(m^k) = \frac{2}{k+1}\theta^k - E(M^k).$$

From now on it is enough for us to consider the following properties when $\theta_1 >= \theta_2$ in the density function (2.1), because the results of (a) and (b) in the results (2.5) and (2.6) can hold as θ_1 and θ_2 interchange.

Let $E(X) = \theta_1/2 \equiv \mu_1$, $E(Y) = \theta_2/2 \equiv \mu_2$, and $\alpha/3 = \rho$ from (2.2). Then means and variances of $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$ are represented by μ_1, μ_2 and ρ as follows ;

$$\begin{aligned} E(M) &= \mu_1 + \frac{1}{3}\frac{\mu_2^2}{\mu_1} - 3\rho\left(\frac{1}{6}\frac{\mu_2^2}{\mu_1} - \frac{1}{10}\frac{\mu_2^3}{\mu_1^2}\right), \\ E(m) &= 2\mu_2 - \mu_1 - \frac{1}{3}\frac{\mu_2^2}{\mu_1} + 3\rho\left(\frac{1}{6}\frac{\mu_2^2}{\mu_1} - \frac{1}{10}\frac{\mu_2^3}{\mu_1^2}\right), \end{aligned}$$

$$Var(M) = \frac{4}{3}\mu_1^2 + \frac{2}{3}\frac{\mu_2^3}{\mu_1} - 3\rho\left[\frac{2}{5}\frac{\mu_2^3}{\mu_1} - \frac{4}{15}\frac{\mu_2^4}{\mu_1^2}\right] - \left[\mu_1 + \frac{1}{3}\frac{\mu_2^2}{\mu_1} - 3\rho\left(\frac{1}{6}\frac{\mu_2^2}{\mu_1} - \frac{1}{10}\frac{\mu_2^3}{\mu_1^2}\right)\right]^2 \tag{2.7}$$

and

$$Var(m) = \frac{4}{3}\left(\frac{\mu_2^3}{\mu_1} - \frac{\mu_1^3}{\mu_2}\right) + \frac{8}{3}\mu_2^2 - E(M^2) - E^2(m),$$

where $E(M^2) = \frac{4}{3}\mu_1^2 + \frac{2}{3}\frac{\mu_2^3}{\mu_1} - 3\rho[\frac{2}{5}\frac{\mu_2^3}{\mu_1} - \frac{4}{15}\frac{\mu_2^4}{\mu_1^2}]$.

Especially, if $\theta_1 = \theta_2 = \theta$ in the density function (2.1), then

$$\begin{aligned} E(M) =& 4\mu\left(\frac{1}{3} - \frac{1}{20}\rho\right), \\ Var(M) =& 2\mu^2\left(1 - \frac{1}{5}\rho\right) - 16\mu^2\left(\frac{1}{3} - \frac{1}{20}\rho\right)^2, \\ E(m) =& 2\mu\left(\frac{1}{3} + \frac{1}{10}\rho\right), \end{aligned} \tag{2.8}$$

and

$$Var(m) = \mu^2\left(\frac{2}{3} + \frac{2}{5}\rho\right) - 4\mu^2\left(\frac{1}{3} + \frac{1}{10}\rho\right)^2,$$

where $\mu = \theta/2$.

The following proposition shows how means and variances of $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$ in (2.7) vary with respect to the means μ_1, μ_2 and the correlation ρ.

Proposition 2.1 Let (X, Y) have the Morgenstern type bivariate uniform density function having $\theta_1 \geq \theta_2$ in (2.1). Then

(a) $E(M)$ is a monotone decreasing function of ρ, but $E(m)$ is a monotone increasing function of ρ.

(b) $E(M)$ is a monotone increasing function of μ_1 if $\rho > 0$ and $\frac{5}{6} \leq \frac{\mu_2}{\mu_1} < 1$ or $\rho < 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{5}{6}$, but $E(m)$ is a monotone decreasing function of μ_1 if $\rho > 0$ and $\frac{5}{6} \leq \frac{\mu_2}{\mu_1} < 1$ or $\rho < 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{5}{6}$.

(c) $E(M)$ and $E(m)$ are monotone increasing functions of μ_2.

(d) $Var(M)$ is a monotone increasing function of ρ if $\rho < 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{5}{9}$, and $Var(m)$ is a monotone decreasing function of ρ if $\rho > 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{5}{9}$.

(e) $Var(M)$ is a monotone increasing function of μ_1 if $\rho > 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{3}{4}$, but $Var(m)$ is a monotone decreasing function of μ_1 if $\rho > 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{3}{4}$.

(f) $Var(M)$ is a monotone decreasing function of μ_2 if $\rho < 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{10}{27}$, and $Var(m)$ is a monotone increasing function of μ_2 if $\rho < 0$ and $0 < \frac{\mu_2}{\mu_1} < \frac{10}{27}$.

Corollary 2.2 For $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$ in the underlying Morgenstern type bivariate uniform density having $\theta_1 ge \theta_2$ in (2.1), if $E(X) = \mu_1$, $Var(X) = \sigma_1^2$, $E(Y) = \mu_2$ and $Var(Y) = \sigma_2^2$, then monotone properties of $E(M)$, $Var(M)$, $E(m)$ and $Var(m)$ with respective to σ_i^2 are the same trends as that of those with respective to μ_i for i =1 and 2, respectively.

Proof It follows since $\mu_i = \sqrt{3}(\sigma_i^2)^{1/2}$ in (2.1) for i =1 and 2, if $H \equiv H(\mu_1, \mu_2, \rho) = H(\sigma_1^2, \sigma_2^2, \rho)$ is a representation function of σ_i^2 and ρ instead of $E(M)$, $Var(M)$, $E(m)$ and $Var(m)$ which they are functions of μ_i and ρ, then for each i =1 and 2.

$$\frac{\partial H}{\partial \sigma_i^2} = \frac{\partial H}{\partial \mu_i}\frac{\partial \mu_i}{\partial \sigma_i^2} = \frac{\sqrt{3}}{2}\frac{\partial H}{\partial \mu_i} \cdot (\sigma_i^2)^{-1/2}.$$

Since signs of $\frac{\partial H}{\partial \sigma_i^2}$ and $\frac{\partial H}{\partial \mu_i}$ are the same, we have done.

3. Concluding remarks

In this paper, we obtained the conditional expectations in the underlying Morgenstern type bivariate uniform variables. And we obtained k-moments for $\max\{X, Y\}$ and $\min\{X, Y\}$ in the underlying Morgenstern type bivariate uniform variables X and Y with same scale parameters and different scale parameters, respectively. From above results, We have considered the conditional expectations to know the dependence of one variable on the other variable and we have considered the behaviors of means and variances of $\max\{X, Y\}$ and $\min\{X, Y\}$ with respect to changes in means, variances, and the correlation coefficient of the underlying Morgenstern type bivariate uniform variables with same scale parameters and different scale parameters, respectively

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