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TWISTED QUADRATIC MOMENTS FOR DIRICHLET L-FUNCTIONS

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Abstract. Given c, a positive integer, we set

$$M(f,c) := \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \chi(c)|L(1,\chi)|^2,$$

where X_f^- is the set of the $\phi(f)/2$ odd Dirichlet characters mod $f > 2$, with $\gcd(f,c) = 1$. We point out several mistakes in recently published papers and we give explicit closed formulas for the f's such that their prime divisors are all equal to ± 1 modulo c. As a Corollary, we obtain closed formulas for $M(f,c)$ for $c \in \{1,2,3,4,5,6,8,10\}$. We also discuss the case of twisted quadratic moments for primitive characters.

1. Introduction: a general formula

Throughout this paper, $c \geq 1$ is a positive integer and $f > 2$ is an integer coprime with c. Let X_f^- be the set of the $\phi(f)/2$ odd Dirichlet characters modulo f. Set

$$\begin{aligned} M(f,c) &:= \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \chi(c)|L(1,\chi)|^2 \\ &= \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \overline{\chi(c)}|L(1,\chi)|^2 \quad (\gcd(f,c)=1). \end{aligned}$$

Our aim is to point out mistakes in the literature for explicit formulas for these mean values and to replace them by correct statements (see Theorems 4 and 10). Our starting point is the formula (see [6, Proposition 1]):

$$L(1,\chi) = \frac{\pi}{2f} \sum_{a=1}^{f-1} \chi(a) \cot(\pi a/f) \quad (\chi \in X_f^-). \tag{1}$$

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Setting

$$\delta_f^-(a,b) := \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \chi(a)\overline{\chi(b)} \tag{2}$$

and using the orthogonality relations

$$\delta_f^-(a,b) = \begin{cases} +1 & \text{if } b \equiv +a \pmod f \text{ and } \gcd(a,f)=1 \\ -1 & \text{if } b \equiv -a \pmod f \text{ and } \gcd(a,f)=1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

we readily obtain

$$M(f,c) = \frac{\pi^2}{2f^2}\tilde{S}(f,c), \tag{4}$$

where

$$\tilde{S}(c,d) := \sum_{\substack{a=1 \\ \gcd(a,c)=1}}^{c-1} \cot\left(\frac{\pi a}{c}\right)\cot\left(\frac{\pi a d}{c}\right) \quad (\gcd(c,d)=1)$$

depends only on d modulo c. This formula has two drawbacks. First, we sum over indices a coprime with c. This drawback disappears when dealing with prime moduli:

$$M(p,c) = \frac{\pi^2}{2p^2}S(p,c), \tag{5}$$

where

$$S(c,d) := \sum_{a=1}^{c-1} \cot\left(\frac{\pi a}{c}\right)\cot\left(\frac{\pi a d}{c}\right) = -S(c,-d) \quad (\gcd(c,d)=1). \tag{6}$$

The second drawback is that we would rather have a result that would depend on f modulo c (the present one depends on c modulo f). To this end, we use the so-called reciprocity law (e.g. see [10, Lemma 5]):

$$dS(c,d) + cS(d,c) = (c^2+d^2-3cd+1)/3 \qquad (c>1,\ d>1). \tag{7}$$

In particular, using (5) and (7), for $p \neq q$ two odd prime numbers we have

$$pM(p,q) + qM(q,p) = \frac{\pi^2}{6}\frac{p^2+q^2-3pq+1}{pq}. \tag{8}$$

Moreover, $S(c,1)$ is easy to compute by determining the polynomials whose roots are the $\cot(\pi a/c)$'s (see [6, Lemma (a)]):

$$S(c,1) = (c-1)(c-2)/3. \tag{9}$$

Now, $S(1,d)$ is not defined, but the sum in (6) being empty for $c=1$, we set

$$S(1,d) = 0 \qquad (d \in \mathbb{Z}). \tag{10}$$

With this convention the reciprocity law (7) is now valid for $c \geq 1$, $d \geq 1$. Using (5) and (7) we obtain

$$M(p,c) = \frac{\pi^2}{6c}\frac{p^2 - b_c(p)p + c^2 + 1}{p^2}, \tag{11}$$

where $b_c(p) := 3(c + S(c,p))$ depends only on p modulo c. For $c = 1$, using (9), we recover H. Walum's formula (see [14]) for $M(p,1)$, $p \geq 3$ a prime. Using it with $c = 2$, 4 and 8 we recover [15, Corollary 1.2]:

Proposition 1. *Let $p \geq 3$ be an odd prime number. Then*

$$M(p,8) = \frac{\pi^2}{48p^2} \times \begin{cases} p^2 - 66p + 5 & \text{if } p \equiv 1 \pmod 8, \\ p^2 - 30p + 65 & \text{if } p \equiv 3 \pmod 8, \\ p^2 - 18p + 65 & \text{if } p \equiv 5 \pmod 8, \\ p^2 + 18p + 65 & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Proof. For example, using (7) we obtain for $p \equiv 7 \pmod 8$: $S(8,p) = S(8,7) = -\frac{18}{7} - \frac{8}{7}S(7,8) = -\frac{18}{7} - \frac{8}{7}S(7,1) = -\frac{18}{7} - \frac{8}{7}10 = -14$. □

To conclude this introduction, we come back to not necessarily prime moduli. Using Möbius inversion formula, we have

$$\tilde{S}(f,c) = \sum_{d|f} \mu(f/d)S(d,c), \tag{12}$$

where $S(d,c)$ is defined in (6). Using (12) and (7) we end up with the following general formula for $M(f,c)$ proved in [10]:

Theorem 2. *Let $c \geq 1$ be a positive integer. Set $\phi(f) := f\prod_{p|f}(1-1/p)$ and $\Psi(f) := f\prod_{p|f}(1+1/p)$. For $\gcd(f,c) = 1$, we have*:

$$M(f,c) = \frac{\pi^2}{6c}\frac{\phi(f)}{f}\left(\frac{\Psi(f)}{f} - \frac{3c}{f} - \frac{3}{\phi(f)}\sum_{d|f}\frac{\mu(d)}{d}S(c,f/d)\right).$$

2. The cases $c \in \{1,2\}$

Since $S(1,d) = 0$ for $d \geq 1$, by (10), and $S(2,d) = 0$ for $d \geq 1$ odd, by (9), we obtain (see also [6], [13], and see [9] for applications):

$$M(f,c) = \frac{\pi^2}{6c}\frac{\phi(f)}{f}\left(\frac{\Psi(f)}{f} - \frac{3c}{f}\right) \quad (c \in \{1,2\}, \gcd(f,c) = 1). \tag{13}$$

3. The cases $c \in \{3,4,6\}$

In this section we show that [11, Theorems 1.3, 1.4 and 1.5] are false and give correct statements replacing them. In particular, in Theorem 4 we give closed formulas for $M(f,c)$, when $c \in \{3,4,6\}$.

Lemma 3. *Assume that* $c > 2$ *and* $\gcd(c, f) = 1$. *Set*

$$\Psi_\chi(f) = \prod_{p|f} \left(1 - \frac{\chi(p)}{p}\right) \text{ and } S_c(\chi) = \sum_{\substack{1 \le a < c/2 \\ \gcd(c,a)=1}} \chi(a) S(c, a).$$

Then

$$\sum_{d|f} \frac{\mu(d)}{d} S(c, f/d) = \frac{2}{\phi(c)} \sum_{\chi \in X_c^-} \overline{\chi(f)} \Psi_\chi(f) S_c(\chi). \tag{14}$$

Proof. Set

$$\tilde{S}_c(\chi) = \sum_{a=1}^{c-1} \chi(a) S(c, a).$$

Let X_c be the group of the $\phi(c)$ Dirichlet characters modulo c. Then

$$\delta_c(a, x) = \frac{1}{\phi(c)} \sum_{\chi \in X_c} \chi(a) \overline{\chi(x)} = \begin{cases} 1 & \text{if } x \equiv a \pmod{c} \text{ and } \gcd(c, a) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$S(c, x) = \sum_{\substack{a=1 \\ \gcd(c;a)=1}}^{c-1} S(c, a) \delta_c(a, x) \qquad (\gcd(c, x) = 1)$$

(notice that $a \mapsto S(c, a)$ is c-periodic). Hence, using $\sum_{d|f} \frac{\mu(d)}{d} \chi(d) = \Psi_\chi(f)$ and noticing that $\chi(a) = 0$ for $\gcd(c, a) > 1$, we have

$$\sum_{d|f} \frac{\mu(d)}{d} S(c, f/d) = \frac{1}{\phi(c)} \sum_{\chi \in X_c} \overline{\chi(f)} \Psi_\chi(f) \tilde{S}_c(\chi).$$

Changing a to $c - a$ and using $S(c, c-a) = -S(c, a)$ and $\chi(c-a) = \chi(-1)\chi(a)$, we obtain $\tilde{S}_c(\chi) = 0$ if $\chi(-1) = +1$ and $\tilde{S}_c(\chi) = 2S_c(\chi)$ if $\chi(-1) = -1$. $\square$

Theorem 4. *Assume that* $c > 2$. *Set* $\epsilon_c(n) = +1$ *if* $n \equiv +1 \pmod{c}$ *and* $\epsilon_c(n) = -1$ *if* $n \equiv -1 \pmod{c}$. *If* p *divides* f *implies* $p \equiv \pm 1 \pmod{c}$, *which is always the case for* $c \in \{3, 4, 6\}$ *and* $\gcd(c, f) = 1$, *then*

$$\sum_{d|f} \frac{\mu(d)}{d} S(c, f/d) = S(c, 1) \epsilon_c(f) \prod_{p|f} \left(1 - \frac{\epsilon_c(p)}{p}\right)$$

and

$$M(f, c) = \frac{\pi^2}{6c} \frac{\phi(f)}{f} \left(\frac{\Psi(f)}{f} - \frac{3c}{f} - (c-1)(c-2) \frac{\epsilon_c(f)}{f} \prod_{p|f} \frac{p - \epsilon_c(p)}{p-1} \right).$$

Proof. If $x \equiv \epsilon_c(x) \pmod c$, with $\epsilon_c(x) \in \{\pm 1\}$, and if $\chi \in X_c^-$, then $\chi(x) = \epsilon_c(x)$ does not depend on $\chi \in X_c^-$. Hence, $\overline{\chi(f)}\Psi_\chi(f) = \epsilon_c(f)\prod_{p|f}\left(1 - \frac{\epsilon_c(p)}{p}\right)$ does not depend on $\chi \in X_c^-$ and

$$\sum_{d|f}\frac{\mu(d)}{d}S(c,f/d) = \left(\sum_{\substack{1\leq a<c/2\\ \gcd(c,a)=1}} S(c,a)\delta_c^-(a,f)\right)\epsilon_c(f)\prod_{p|f}\left(1-\frac{\epsilon_c(p)}{p}\right),$$

where $\delta_c^-(a,x)$ is defined in (2). Since $f \equiv \epsilon_c(f) \equiv \pm 1 \pmod c$, the only index a for which $\delta_c^-(a,f)$ is not equal to 0 is $a = 1$ and $\delta_c^-(1,f) = \epsilon_c(f)$, by (3). The result follows by using (9). □

4. The cases $c \in \{5, 8, 10\}$

Corollary 5. *Let χ_5 be the odd Dirichlet character modulo 5 defined by $\chi_5(1) = 1$, $\chi_5(2) = \zeta_4$, $\chi_5(3) = -\zeta_4$ and $\chi_5(4) = -1$. Then*

$$\sum_{d|f}\frac{\mu(d)}{d}S(5,f/d) = 4\Re\left(\overline{\chi_5(f)}\prod_{p|f}\left(1-\frac{\chi_5(p)}{p}\right)\right).$$

Proof. Clearly $X_5^- = \{\chi_5, \overline{\chi_5}\}$. Moreover, $S(5,1) = 4$ and $S(5,2) = 0$. Hence, $S_5(\chi) = \chi(1)S(5,1) + \chi(2)S(5,2) = 4$, $\chi \in X_5^-$, and the desired result follows, by (14). □

Corollary 6. *We have*

$$\sum_{d|f}\frac{\mu(d)}{d}S(8,f/d) = 8\left(\frac{-8}{f}\right)\prod_{p|f}\left(1-\frac{\left(\frac{-8}{p}\right)}{p}\right) + 3\left(\frac{-4}{f}\right)\prod_{p|f}\left(1-\frac{\left(\frac{-4}{p}\right)}{p}\right).$$

Proof. Here $1 \leq a < 8/2$ and $\gcd(8,a) = 1$ implies $a = 1$ or $a = 3$. Now, $S(8,1) = 14$, $S(8,3) = 2$ and $X_{10}^- = \{\left(\frac{-4}{\bullet}\right), \left(\frac{-8}{\bullet}\right)\}$. □

Corollary 7. *Let χ_{10} be the odd Dirichlet character modulo 10 defined by $\chi_{10}(1) = 1$, $\chi_{10}(3) = \zeta_4$, $\chi_{10}(7) = -\zeta_4$ and $\chi_{10}(9) = -1$. Then*

$$\sum_{d|f}\frac{\mu(d)}{d}S(10,f/d) = 24\Re\left(\overline{\chi_{10}(f)}\prod_{p|f}\left(1-\frac{\chi_{10}(p)}{p}\right)\right).$$

Proof. Here $1 \leq a < 10/2$ and $\gcd(10,a) = 1$ implies $a = 1$ or $a = 3$. Now, $S(10,1) = 24$, $S(10,3) = 0$ and $X_{10}^- = \{\chi_{10}, \overline{\chi_{10}}\}$. □

For $c = 2$, formula (13) agrees with [11, Theorem 1.1]. However, Theorem 4 contradicts [11, Theorems 1.3 and 1.4]. Using numerical examples we could not find any disagreement with Theorem 4 whereas [11, Theorems 1.3 and 1.4] were indeed contradicted. In the same way, numerical examples readily show that [11, Theorem 1.5] is false. In fact, you cannot expect a closed formula valid for all $f \equiv 1 \pmod c$:

Corollary 8. *Let $c > 2$ be given. Let $f = p^m$ be a power of a prime $p \geq 3$.*
If $p \equiv 1 \pmod{c}$, then $f \equiv 1 \pmod{c}$ and

$$M(f,c) = \frac{\pi^2}{6c}\left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p} - \frac{c^2+2}{f}\right)$$

(*in agreement with* [11, Theorem 1.5]).
If $p \equiv -1 \pmod{c}$ and m is even then $f \equiv 1 \pmod{c}$ and

$$M(f,c) = \frac{\pi^2}{6c}\left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p} - \frac{c^2+2}{f} - 2\frac{c^2-3c+2}{(p-1)f}\right)$$

(*in disagreement with* [11, Theorem 1.5]).

Let us now identify the mistake in [11]'s proofs. For example, let us have a look at [11, Proof of Theorem 1.3, pages 751–752]. They use their identity (1.4):

$$S(3,q) = \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\chi\in X_d^-}\chi(3)|L(1,\chi)|^2$$

and their Proposition 2.1 (Möbius inversion formula) to obtain

$$\sum_{\chi\in X_q^-}\chi(3)|L(1,\chi)|^2 = \sum_{d|q}\mu(d)S(3,q/d). \tag{15}$$

The mistake is now that they compute a formula for $S(3,q)$ according as $q \equiv 1 \pmod 3$ or $q \equiv 2 \pmod 3$, and plug their formula inside (15) without realizing that knowing q modulo 3 does not give them the values for q/d modulo 3 as d ranges over the divisors of q.

5. The case of primitive characters

In this section we give a simpler and clearer exposition of the results obtained in [7], namely that is hopeless to find closed formulas for twisted quadratic moments of L-functions over odd primitive characters. Let $J(f)$ denote the number of primitive Dirichlet characters modulo $f > 2$. The number $J(f)$ of primitive character modulo f is a multiplicative function of f such that $J(p^m) = \phi(p^m) - \phi(p^{m-1}) = \phi(p^m)^2/p^m$ for p a prime and $m \geq 2$. Hence, $J(f) = \phi(f)^2/f$ for f square-full (to be used in the next section). Notice also that $J(f) > 0$ if and only if $2 < f \not\equiv 2 \pmod 4$. Let P_f^- be the set of the $J(f)/2$ odd primitive Dirichlet characters modulo f. If ψ is a character modulo a divisor of f, we let $\tilde{\psi}$ denote the character modulo f induced by ψ. Using the inclusion-exclusion principle we have

$$\sum_{\chi\in P_f^-}\chi(c)|L(1,\chi)|^2 = \sum_{\substack{d|f\\ d\neq f}}\mu(d)\sum_{\psi\in X_{f/d}^-}\chi(c)|L(1,\tilde{\psi})|^2. \tag{16}$$

Theorem 9. *If $q > p \geq 3$ are prime numbers and if $q \equiv 1 \pmod p$, then*

$$\sum_{\chi \in P_{pq}^-} |L(1,\chi)|^2 = \frac{\pi^2}{12} \frac{(p-2)(p-1)(p+1)(q-1)^2}{p^2 q}.$$

Proof. By (16), we have

$$\sum_{\chi \in P_{pq}^-} |L(1,\chi)|^2 = \sum_{\chi \in X_{pq}^-} |L(1,\chi)|^2 - \sum_{\psi \in X_p^-} |L(1,\tilde{\psi})|^2 - \sum_{\psi \in X_q^-} |L(1,\tilde{\psi})|^2.$$

If $\psi \in X_p^-$, then $L(1,\tilde{\psi}) = (1-\psi(q)/q)L(1,\psi)$ and

$$\frac{2}{\phi(p)} \sum_{\psi \in X_p^-} |L(1,\tilde{\psi})|^2 = \left(1 + \frac{1}{q^2}\right) M(p,1) - \frac{2}{q} M(p,q).$$

Using the same identity where p and q are exchanged, we obtain:

$$\sum_{\chi \in P_{pq}^-} |L(1,\chi)|^2 = \frac{\phi(pq)}{2} M(pq,1) - \frac{\phi(p)}{2}\left(1 + \frac{1}{q^2}\right) M(p,1) + \frac{\phi(p)}{q} M(p,q)$$
$$- \frac{\phi(q)}{2}\left(1 + \frac{1}{p^2}\right) M(q,1) + \frac{\phi(q)}{p} M(q,p).$$

By its definition $M(f,c)$ depends on c modulo f only. Hence, $M(p,q) = M(p,1)$. Using (8) to express $M(q,p)$ in terms of $M(p,q) = M(p,1)$ and using (13) to compute $M(pq,1)$, $M(p,1)$ and $M(q,1)$, we obtain the desired result. □

As explained in [7], Theorem 9 which gives a result with is not symmetrical in p and q dampens hopes of ever finding a simple formula for the mean value of $|L(1,\chi)|^2$ for primitive odd Dirichlet characters modulo $f > 2$; in particular, the formula conjectured in [17] (see also [MR1077163 (91j:11068)]) is wrong.

6. Twisted moments for primitive characters of square-full conductors

The only situation where we can readily use (16) is when $L(1,\tilde{\psi}) = L(1,\psi)$ for any square-free d dividing f and any $\psi \in X_{f/d}^-$. Since

$$L(1,\psi) = L(1,\tilde{\psi}) \prod_{p \mid f} (1 - \tilde{\psi}(p)/p), \psi \in X_{f/d}^-,$$

we want to have $\tilde{\psi}(p) = 0$ for any prime p dividing f and any $\psi \in X_{f/d}^-$, i.e., we want to have $p \mid f/d$ for any prime p dividing f and any square-free d dividing f, i.e., we want f to be square-full. So, let us assume that $f > 2$ is square-full, i.e., such that p divides f implies p^2 divides f. Then for any square-free divisor d of f, we have $\tilde{\psi} = \psi$, $\psi \in X^-_{f/d}$, and (16) yields (after changing d into f/d)

$$\tilde{M}(f,c) := \frac{2}{J(f)} \sum_{\chi \in P_f^-} \chi(c)|L(1,\chi)|^2 = \frac{f}{\phi(f)^2} \sum_{d|f} \mu(f/d)\phi(d)M(d,c). \tag{17}$$

Theorem 10. *Let $c > 2$ be a given integer. Let f be a square-full integer such that p divides f implies $p \equiv \pm 1 \pmod{c}$, which is always the case for $c \in \{3,4,6\}$. If there exists a prime $p \equiv 1 \pmod{c}$ dividing f, then*

$$\tilde{M}(f,c) = \frac{\pi^2}{6c} \prod_{p|f} (1 - \frac{1}{p^2}).$$

If all the prime p dividing f satisfy $p \equiv -1 \pmod{c}$, then

$$\tilde{M}(f,c) = \frac{\pi^2}{6c} \left(\prod_{p|f} (1 - \frac{1}{p^2}) - \epsilon_c(f) \frac{(c-1)(c-2)}{f} \prod_{p|f} 2\frac{p+1}{p-1} \right).$$

In particular, if $c \in \{3,4,6\}$, then this formulas holds true for any square-full $f > 2$ coprime with c. Moreover,

$$\tilde{M}(f,1) = \frac{\pi^2}{6} \prod_{p|f} (1 - \frac{1}{p^2}) \ (f \text{ square-full}),$$

as in [17], *and*

$$\tilde{M}(f,2) = \frac{\pi^2}{12} \prod_{p|f} (1 - \frac{1}{p^2}) \ (f \text{ odd and square-full}).$$

Proof. Assume that $c > 2$. We start from (17) and apply Theorem 4 to each $M(d,c)$ (notice that if f is in E_c then any divisor d of f is also in E_c):

$$\phi(d)M(d,c) = \frac{\pi^2}{6c} X(d) \Big(Y(d)\phi(d) - 3cX(d) - (c-1)(c-2)\epsilon_c(d)X(d)Z_c(d) \Big),$$

where $X(d) = \phi(d)/d = \prod_{p|d}(1 - 1/p)$, $Y(d) = \Psi(d)/d = \prod_{p|d}(1 + 1/p)$ and $Z_c(d) = \prod_{p|d}(p - \epsilon_c(p))/(p-1)$. The key point is that if d is a divisor of a square-full integer $f > 2$ such that $\mu(f/d) \neq 0$, then d and f have the same prime divisors and $X(d) = X(f)$, $Y(d) = Y(f)$ and $Z_c(d) = Z_c(f)$ do not depend on d. Hence, we obtain

$$\tilde{M}(f,c) = \frac{f}{\phi(f)^2} \frac{\pi^2}{6c} X(f) \left(Y(f) \sum_{d|f} \mu(f/d)\phi(d) - 3cX(f) \sum_{d|f} \mu(f/d) \right.$$
$$\left. -(c-1)(c-2)X(f)Z_c(f) \sum_{d|f} \mu(f/d)\epsilon_c(d) \right),$$

and using $\sum_{d|f}\mu(f/d)\phi(d)=\phi(f)^2/f$ (for $f>1$ square-full), $\sum_{d|f}\mu(f/d)=0$ (for $f>1$) and $\sum_{d|f}\mu(f/d)\epsilon_c(d)=\epsilon_c(f)\prod_{p|f}(1-\epsilon_c(p))$ the desired result follows. For $c=1$ and $c=2$ the proof is even simpler using (17) and (13). □

7. The case $c = 2^k$

Let $p \geq 3$ be an odd prime number. Z. Wu and W. Zhang gave in [15] formulas for $M(p,2)$, $M(p,4)$ and $M(p,8)$ (they readily follow from (11)), and conjectured a formula for the $b_{2^k}(p)$'s, $k \geq 2$ (see (11)). The truth of this Conjecture for a given $k \geq 2$ is equivalent to the truth of the following one:

Conjecture 11 (See [15, Conjecture 2.1])**.** *For $m>1$, let $R_m(n)$ denote the unique integer in $\{0,1,\dots,m-1\}$ equal to n modulo m. For $m>1$ even, let $L_m(n)$ denote the unique integer in $\{-m/2+1,\dots,m/2\}$ equal to n modulo m. Then for $p>2^k$ be a prime number it holds that*

$$f_p(2^{k-1}) := \#\{a;\ 1 \leq a \leq p-1 \text{ and } R_p(2^{k-1}a) \not\equiv a \pmod 2\} = \frac{p-L_{2^k}(p)}{2}.$$

Whereas their Conjecture 2.1 is true for $2^k=2$ and $2^k=4$ (see [15, Lemma 2.2] or Corollary 13):

$$f_p(2)=\begin{cases}\frac{p-1}{2} & \text{if } p\equiv 1 \pmod 4\\ \frac{p+1}{2} & \text{if } p\equiv -1 \pmod 4\end{cases} \quad\text{and}\quad f_p(4)=\begin{cases}\frac{p-1}{2} & \text{if } p\equiv 1 \pmod 8\\ \frac{p-3}{2} & \text{if } p\equiv 3 \pmod 8\\ \frac{p+1}{2} & \text{if } p\equiv -1 \pmod 8\\ \frac{p+3}{2} & \text{if } p\equiv -3 \pmod 8\end{cases}$$

we will prove in Corollary 14 that it is not true for $2^k=8$: if $p\equiv 3 \pmod{16}$ then $f_p(8)=(p+1)/2=(p+4-L_{16}(p))/2\neq(p-L_{16}(p))/2$.

Proposition 12. *For $p>2$, $c>1$ and $\gcd(p,c)=1$, we have*

$$\begin{aligned} f_p(c) &:= \#\{a;\ 1\leq a\leq p-1 \text{ and } R_p(ac)\not\equiv a \pmod 2\}\\ &= \frac{p-1}{2}-\frac{5S(p,c)-2S(p,2c)-2S(p,2^*c)}{2p}, \end{aligned}$$

where $2^*\cdot 2\equiv 1 \pmod p$.

Proof. By [15, Lemma 2.1] we have

$$f_p(c)=\frac{p-1}{2}-\frac{2p}{\pi^2(p-1)}\sum_{\chi\in X_P^-}\chi(c)(5-2\chi(2)-2\chi(2^*))|L(1,\chi)|^2,$$

and the desired result follows by using (5) and the definition of $M(p,c)$. □

Corollary 13. *Let $c>2$ be even and p coprime with c. We have*

$$f_p(c)=\frac{p}{2}+\frac{5S(c,p)-S(2c,p)-4S(c/2,p)}{2c}, \tag{18}$$

which depends on p mod $2c$ only. For example, we have

$$f_p(c) = \begin{cases} \frac{p-1}{2} & \text{if } p \equiv 1 \pmod{2c} \\ \frac{p-(c-1)}{2} & \text{if } p \equiv c-1 \pmod{2c} \\ \frac{p+(c-1)}{2} & \text{if } p \equiv c+1 \pmod{2c} \\ \frac{p+1}{2} & \text{if } p \equiv 2c-1 \pmod{2c}. \end{cases} \tag{19}$$

If $p \equiv r \pmod{2c}$ with $r > 1$, then

$$f_p(c) = \frac{p-1}{2} - \frac{5S(r,c) - 2S(r,2c) - 2S(r,c/2)}{2r}. \tag{20}$$

Proof. To obtain (18), use (7): $S(p,d) = F(p,d) - pS(d,p)/d$, with $F(p,d) := \frac{p^2-3dp+d^2+1}{3d}$, and notice that $5F(p,c) - 2F(p,2c) - 2F(p,c/2) = -p$. Then (19) follows using (7). To obtain (20) apply (7) once again and notice that $5F(c,p) - F(2c,p) - 4F(c/2,p) = -c$. □

Corollary 14. *For $k \geq 1$ and $p \equiv 3 \pmod{2^{k+1}}$ we have $L_{2^{k+1}}(p) = 3$ and*

$$f_p(2^k) = \left(p - 1 - 2(-1)^k\right)/2.$$

Proof. Use $S(3,2^l) = S(3,(-1)^l) = (-1)^l S(3,1) = \frac{2}{3}(-1)^l$. □

Since $(p - L_{2^{k+1}}(p))/2 = (p-3)/2$ for $k \geq 2$ and $p \equiv 3 \pmod{2^{k+1}}$, [15, Conjecture 2.1] is false for $k \geq 2$ and k odd, hence false for $k = 3$.

8. Conclusion

We refer the reader to the bibliography for the recent literature regarding more complicated mean values of the type

$$M(f_1,f_2,c) = \frac{4}{\phi(f_1)\phi(f_2)} \sum_{\chi_1 \in X_{f_1}^-} \sum_{\chi_1 \in X_{f_1}^-} \chi(c) L(m,\chi_1) \overline{L(n,\chi_2)},$$

where $m \geq 1$ and $n \geq 1$ are both odd (e.g., see [1], [3], [4], [5], [8], [12]).

See also [2] and [16] for recent papers dealing with L-functions.

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