Bull. Korean Math. Soc. **52** (2015), No. 6, pp. 2047–2069
http://dx.doi.org/10.4134/BKMS.2015.52.6.2047

JOINING OF CIRCUITS IN $PSL(2,\mathbb{Z})$-SPACE

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Abstract. The coset diagrams are composed of fragments, and the fragments are further composed of circuits at a certain common point. A condition for the existence of a certain fragment γ of a coset diagram in a coset diagram is a polynomial f in $\mathbb{Z}[z]$. In this paper, we answer the question: how many polynomials are obtained from the fragments, evolved by joining the circuits (n, n) and (m, m), where $n < m$, at all points.

1. Introduction

It is well known that the modular group $PSL(2,\mathbb{Z})$ ([1], [3], [4] and [5]) is generated by the linear fractional transformations $x : z \to \frac{-1}{z}$ and $y : z \to \frac{z-1}{z}$ which satisfy the relations

$$x^2 = y^3 = 1. \tag{1.1}$$

The extended modular group $PGL(2,\mathbb{Z})$, is the group of linear fractional transformations $z \to \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$. If t is $z \to \frac{1}{z}$ which belongs to $PGL(2,\mathbb{Z})$ but not to $PSL(2,\mathbb{Z})$, then x, y, t satisfy the relations

$$x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1. \tag{1.2}$$

Let q be a power of a prime p, and $PL(F_q)$ denote the projective line over the finite field F_q, that is $PL(F_q) = F_q \cup \{\infty\}$. The group $PGL(2, q)$ is the group of all linear fractional transformations $z \to \frac{az+b}{cz+d}$ such that a, b, c, d are in F_q and $ad - bc$ is non-zero, while $PSL(2, q)$ is its subgroup consisting of transformations (1.1) such that a, b, c, d are in F_q and $ad - bc$ is a quadratic residue in F_q.

Professor Graham Higman introduced a new type of graph called coset diagram for $PGL(2,\mathbb{Z})$. The three-cycles of y are denoted by small triangles whose vertices are permuted counter-clockwise by y and any two vertices which are interchanged by x are joined by an edge. The fixed points of x and y are denoted by heavy dots. Since $(yt)^2 = 1$ is equivalent to $tyt = y^{-1}$, therefore t reverses the orientation of the triangles representing the three-cycles of y. Thus, there is

Received November 4, 2014; Revised May 4, 2015.
2010 *Mathematics Subject Classification.* Primary 05C25; Secondary 20G40.
Key words and phrases. modular group, coset diagrams, projective line over finite field.

no need to make the diagram complicated by introducing t-edges. Consider the action of $PGL(2,\mathbb{Z})$ on $PL(F_{19})$. We calculate the permutation representations x, y and t by $(z)\,x = \frac{-1}{z}$, $(z)\,y = \frac{z-1}{z}$ and $(z)\,t = \frac{1}{z}$ respectively. So

$$x : (0\ \infty)(1\ 18)(2\ 9)(3\ 6)(4\ 14)(5\ 15)(7\ 8)(10\ 17)(11\ 12)(13\ 16),$$
$$y : (0\ \infty\ 1)(2\ 10\ 18)(3\ 7\ 9)(4\ 15\ 6)(5\ 16\ 14)(13\ 17\ 11)(8)(12),$$
$$t : (0\ \infty)(2\ 10)(3\ 13)(4\ 5)(6\ 16)(7\ 11)(8\ 12)(9\ 17)(14\ 15)(1)(18).$$

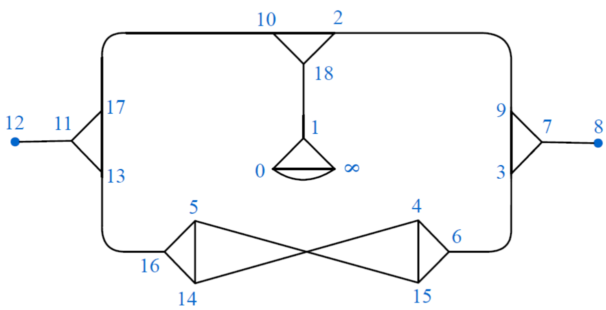

FIGURE 1

For more on coset diagrams, we suggest reading of [2], [8], [9] and [11].

Two homomorphisms α and β from $PGL(2,\mathbb{Z})$ to $PGL(2,q)$ are called conjugate if $\beta = \alpha\rho$ for some inner automorphism ρ on $PGL(2,q)$. We call α to be non-degenerate if neither of x, y lies in the kernel of α. In [7] it has been shown that there is a one to one correspondence between the conjugacy classes of non-degenerate homomorphisms from $PGL(2,\mathbb{Z})$ to $PGL(2,q)$ and the elements $\theta \neq 0, 3$ of F_q under the correspondence which maps each class to its parameter θ. As in [7], the coset diagram corresponding to the action of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ via a homomorphism α with parameter θ is denoted by $D\,(\theta, q)$.

2. Occurrence of fragments in $D\,(\theta, q)$

By a circuit in a coset diagram for an action of $PGL(2,\mathbb{Z})$ on $PL(F_q)$, we shall mean a closed path of triangles and edges. Let $n_1, n_2, n_3, \ldots, n_{2k}$ be a sequence of positive integers. The circuit which contains a vertex, fixed by $w = (xy)^{n_1}(xy^{-1})^{n_2}\cdots(xy^{-1})^{n_{2k}} \in PSL(2,\mathbb{Z})$ for some $k \geq 1$, we shall mean the circuit in which n_1 triangles have one vertex inside the circuit and n_2 triangles have one vertex outside the circuit and so on. Since it is a cycle $(n_1, n_2, n_3, \ldots, n_{2k})$, so it does not make any difference if n_1 triangles have one vertex outside the circuit and n_2 triangles have one vertex inside the circuit and so on. The circuit of the type

$$(n_1, n_2, \ldots, n_{2k'}, n_1, n_2, \ldots, n_{2k'}, \ldots, n_1, n_2, \ldots, n_{2k'})$$

is called a periodic circuit and the length of its period is $2k'$. A circuit that is not of this type is non-periodic.

Consider two simple circuits $(n_1, n_2, n_3, \ldots, n_{2k})$ and $(m_1, m_2, m_3, \ldots, m_{2l})$. Let v_1 be any vertex in $(n_1, n_2, n_3, \ldots, n_{2k})$ fixed by a word w_1 and v_2 be any vertex in $(m_1, m_2, m_3, \ldots, m_{2l})$ fixed by a word w_2. In order to connect these two circuits at v_1 and v_2, we choose, without loss of generality $(n_1, n_2, n_3, \ldots, n_{2k})$ and apply w_2 on v_1 in such a way that w_2 ends at v_1. Consequently, we get a fragment say γ, containing a vertex $v = v_1 = v_2$ fixed by the pair w_1, w_2.

Example 1. We connect the vertex v, fixed by $(xy)\left(xy^{-1}\right)^3(xy)^3$, in $(4,3)$ with the vertex u, fixed by $(xy)^3\left(xy^{-1}\right)^2$ of $(3,2)$, and compose a fragment γ as follows

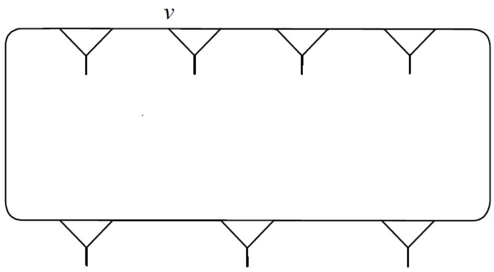

FIGURE 2

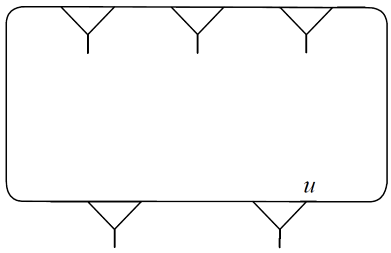

FIGURE 3

One can see that the vertex $v = u$ in γ (Figure 4) is fixed by a pair of words $(xy)\left(xy^{-1}\right)^3(xy)^3, (xy)^3\left(xy^{-1}\right)^2$.

The action of $PGL(2,\mathbb{Z})$ on $PL(F_{q^2})$ yields two components, namely $PL(F_q)$ and $PL(F_{q^2}) \diagdown PL(F_q)$. For sake of simplicity, let $\overline{PL(F_q)}$ denote the complement $PL(F_{q^2}) \diagdown PL(F_q)$. In what follows, by γ, we shall mean a non-simple fragment consisting of two connected, non-trivial circuits such that neither of

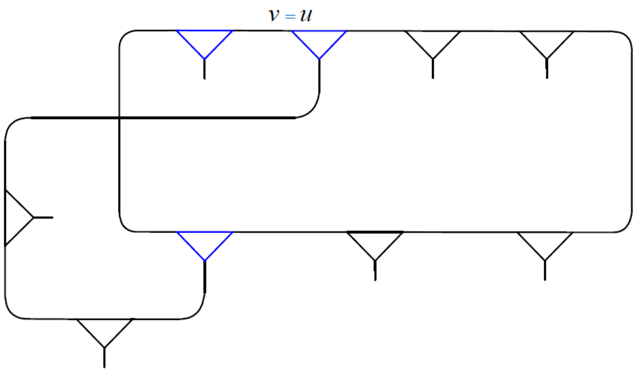

FIGURE 4

them is periodic. The coset diagram $D(\theta, q)$ is made of fragments. It is therefore necessary to ask, when a fragment exists in $D(\theta, q)$. In [6] this question is answered in the following way.

Theorem 1. *Given a fragment, there is a polynomial f in $\mathbb{Z}[z]$ such that*

(i) *if the fragment occurs in $D(\theta, q)$, then $f(\theta) = 0$,*

(ii) *if $f(\theta) = 0$ then the fragment, or a homomorphic image of it occurs in $D(\theta, q)$ or in $\overline{PL(F_q)}$.*

In [6], the method of calculating a polynomial from a fragment is given. In [10], it has been proven that, the polynomial obtained from a fragment is unique.

Let the fragment δ, evolved by joining two periodic circuits, and $f(\theta)$ be the polynomial obtained from δ. Then corresponding to each root of $f(\theta)$, the homomorphic image of δ (instead of δ) exists in the respective coset diagrams [6]. Therefore, we are dealing with only those fragments, which are composed by joining a pair of non-periodic circuits.

Let γ be formed by joining the vertex v_1 of one circuit with the vertex v_2 of the other circuit, then we denote this point of connection by $v_1 \leftrightarrow v_2$. Note that $v_1 \leftrightarrow v_2$ is not a unique point of connection for γ. The following theorems proved in [10], are useful for finding all the points of connection for γ.

Theorem 2. *Let the fragment γ be constructed by joining a vertex v_1 of $(n_1, n_2, n_3, \ldots, n_{2k})$ with the vertex v_2 of $(m_1, m_2, \ldots, m_{2l})$. Then γ is obtainable also, if the vertex $(v_1)\,w$ of $(n_1, n_2, n_3, \ldots, n_{2k})$ is joined with the vertex $(v_2)\,w$ of $(m_1, m_2, \ldots, m_{2l})$.*

Theorem 3. *Let P be the set of words such that for any $w \in P$, both vertices $(v_1)\,w$ and $(v_2)\,w$ lie on the circuits $(n_1, n_2, n_3, \ldots, n_{2k})$ and $(m_1, m_2, \ldots, m_{2l})$. Let s be the number of points of connection of the circuits to compose γ. Then $s = |P|$.*

Example 2. As in Example 1, the vertex v in $(4,3)$ is connected with the vertex u in $(3,2)$, and a fragment γ is evolved. Then one can see that

$$P = \{y, y^{-1}, e, x, xy, xy^{-1}, xyx, xyxy, xyxy^{-1}\}$$

is the set of words such that for any $w \in P$, both vertices $(v)\,w$ and $(u)\,w$ lie on $(4,3)$ and $(3,2)$. So by Theorem 2, the same fragment γ is formed if we join

$$(v)\,y \text{ with } (u)\,y, (v)\,y^{-1} \text{ with } (u)\,y^{-1}, (v)\,e \text{ with } (u)\,e, (v)\,x \text{ with } (u)\,x,$$
$$(v)\,xy \text{ with } (u)\,xy, (v)\,xy^{-1} \text{ with } (u)\,xy^{-1}, (v)\,xyx \text{ with } (u)\,xyx,$$
$$(v)\,xyxy \text{ with } (u)\,xyxy \text{ or } (v)\,xyxy^{-1} \text{ with } (u)\,xyxy^{-1}.$$

Since $|P| = 9$, there are nine points of connection

$$(v)\,y \leftrightarrow (u)\,y, (v)\,y^{-1} \leftrightarrow (u)\,y^{-1}, (v)\,e \leftrightarrow (u)\,e, (v)\,x \leftrightarrow (u)\,x, (v)\,xy \leftrightarrow (u)\,xy,$$
$$(v)\,xy^{-1} \leftrightarrow (u)\,xy^{-1}, (v)\,xyx \leftrightarrow (u)\,xyx, (v)\,xyxy \leftrightarrow (u)\,xyxy$$
$$\text{and } (v)\,xyxy^{-1} \leftrightarrow (u)\,xyxy^{-1}$$

of $(4,3)$ and $(3,2)$ to compose γ.

Let a fragment γ (Figure 5) occur in a coset diagram. Since a coset diagram

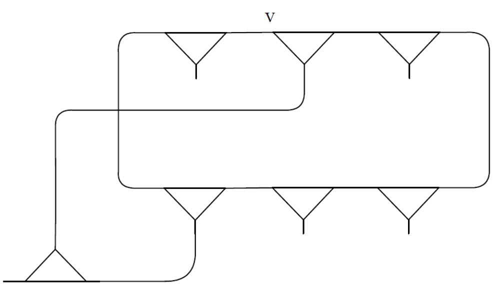

FIGURE 5

admits a vertical axis of symmetry, the mirror image of γ under the permutation t will also occur. Through-out this paper, we denote the mirror image of a

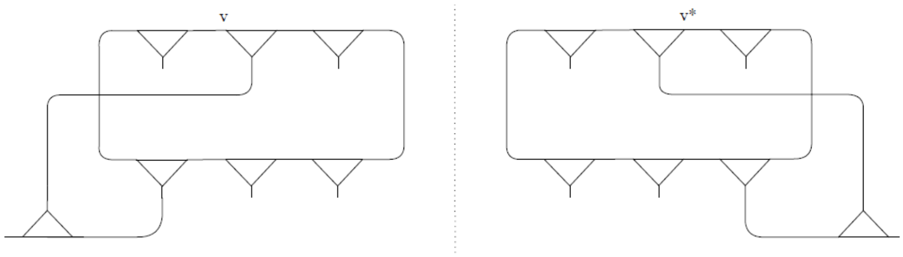

FIGURE 6

fragment γ by γ^*. If $w = xy^{\eta_1}xy^{\eta_2}\cdots xy^{\eta_n}$ ($\eta = 1$ or -1) is a word, then let

$$w^* = xy^{-\eta_1}xy^{-\eta_2}\cdots xy^{-\eta_n}.$$

If a vertex v is fixed by w, then the vertex fixed by w^* is denoted by v^*.

Remark 1. Since t reverses the orientation of the triangles representing the three-cycles of y (as reflection does), so if γ have a vertex v fixed by the pair w_1, w_2, then obviously its mirror image γ^* contains a vertex v^* fixed by the pair w_1^*, w_2^*. Since $D(\theta, q)$ has a vertical axis of symmetry, therefore if γ exists in $D(\theta, q)$, then its mirror image γ^* also exists in $D(\theta, q)$. So condition for the existence of γ and γ^* in $D(\theta, q)$ is the same, implying that, a unique polynomial is obtained from γ and γ^*. There are certain fragments which have the same orientations as those of their mirror images. These kinds of fragments admits a vertical axis of symmetry and may have fixed points of t. A fragment γ containing a vertex v fixed by the pair w_1, w_2, has the same orientations as that of its mirror image if and only if it contains a vertex v^* fixed by the pair w_1^*, w_2^*. For example, the fragment formed by joining a vertex v_1, fixed by $(xy)^{n_1}(xy^{-1})^{n_2}$ in (n_1, n_2) with the vertex v_2, fixed by $(xy)^{\frac{m_1+n_1}{2}}\left(xy^{-1}\right)^{m_2}(xy)^{\frac{m_1-n_1}{2}}$ in (m_1, m_2) has the same orientation as that of its mirror image. Diagrammatically, it means: Consider the circuit (n, n) and

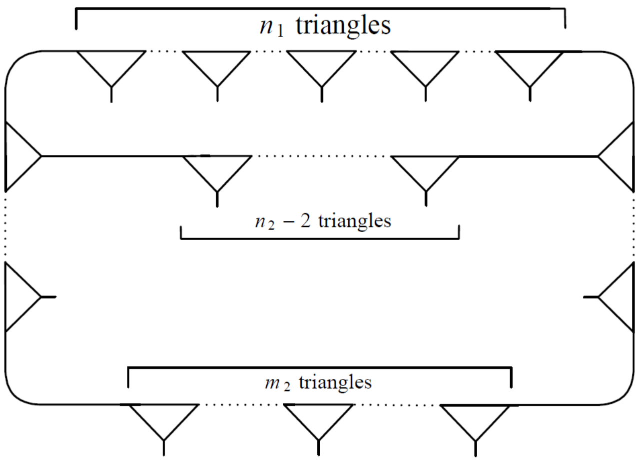

FIGURE 7

(m, m), where $n < m$. Let us join (n, n) and (m, m) at a certain point, and obtain a fragment γ. As, a fragment has many points of connection in (n, n) and (m, m). So if we change the point of connection in (n, n) and (m, m), it is not necessary that we get a fragment different from γ. It is therefore necessary to ask, how many distinct fragments (polynomials), we obtain, if we join the circuits (n, n) and (m, m) at all points of connection? In this paper, we not only

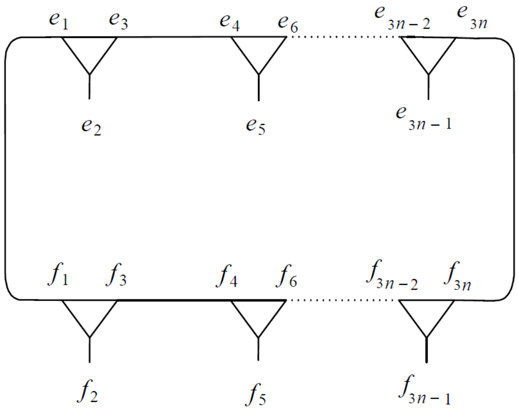

FIGURE 8

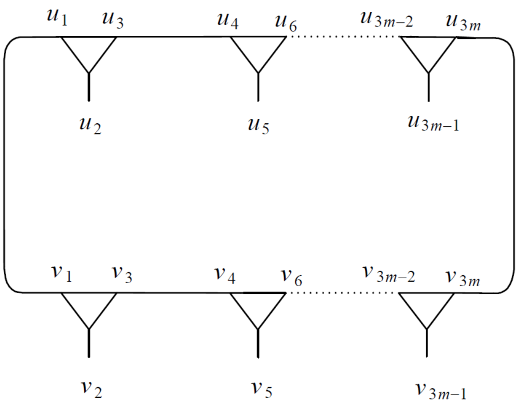

FIGURE 9

give the answer of this question, but also mention those points of connection in (n,n) and (m,m) , which are important. There is no need to join (n,n) and (m,m) at the points, which are not mentioned as important. Because if we join (n,n) and (m,m) at such a point, we obtain a fragment, which we have already obtained by joining at important points.

Remark 2. Recall, if $w = xy^{\eta_1}xy^{\eta_2}\cdots xy^{\eta_n}$ $(\eta = 1 \text{ or } -1)$ is a word, then $w^* = xy^{-\eta_1}xy^{-\eta_2}\cdots xy^{-\eta_n}$. If a vertex v is fixed by w, then the vertex fixed by w^* is denoted by v^*. In Figures 8 and 9, one can see that, for each $i = 1,2,3,\ldots,n$ and $j = 1,2,3,\ldots,m$ we have

$$e_i^* = e_{3n-(i-1)}, f_i^* = f_{3n-(i-1)}, u_j^* = u_{3m-(j-1)} \text{ and } v_j^* = v_{3m-(j-1)}.$$

Since the number of vertices in each circuit (n,n) and (m,m) is $6n$ and 6m respectively. So there are $36nm$ points of connection in (n,n) and (m,m). We connect a vertex in (n,n) with the vertex in (m,m) and compose a fragment. By using Theorem 3, we count all the points of connection in (n,n) and (m,m) for this fragment. Then we check whether the fragment has the same orientation as that of its mirror image or not. If the fragment has different orientation as that of its mirror image, then we double its points of connection, as there are same number of points of connection for the mirror image of the fragment. After that, we connect (n,n) and (m,m) at one of the remaining points. This process continues until all the points of connection, $36nm$ of these circuits are exhausted. Hence we get all fragments composed by joining (n,n) and (m,m) at all points.

We first prove some theorems, which are used in our main result.

Theorem 4. *There are n distinct fragments evolved, as a result of joining the vertex f_{3n} in (n,n) with the vertices u_{3l+1}, where $l=0,1,2,\ldots,n-1$, in (m,m). Moreover total number of points of connection of these fragments, and their mirror images, are $6\sum_{l=0}^{n-1}(l+2)$.*

Proof. Let us join the vertex f_{3n}, fixed by $(xy)^n\left(xy^{-1}\right)^n$ with the vertices u_{3l+1}, fixed by $(xy)^l\left(xy^{-1}\right)^m(xy)^{m-l}$ and obtain a class of fragments γ_l. Then

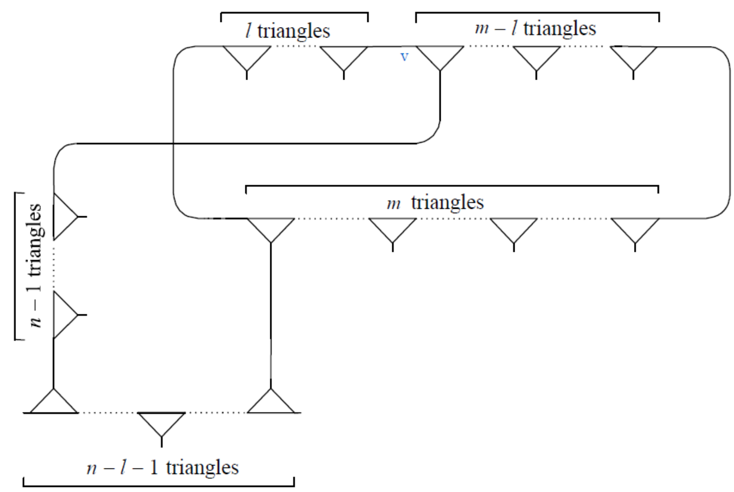

FIGURE 10

$P=\{y,y^{-1},e,x,xy^{-1},xy,xyx,xyxy^{-1},(xy)^2,\ldots,(xy)^l x,(xy)^l xy^{-1},(xy)^{l+1}\}$ is the set of words such that for any $w\in P$, both the vertices $(f_{3n})w$ and $(u_{3l+1})w$ lie on (n,n) and (m,m) respectively. Since $|P|=3(l+2)$, therefore by Theorem 3, each fragment γ_l has $3(l+2)$ points of connection. From Figure 10, it is clear that, all fragments in $\{\gamma_l : l = 0, 1, 2, \dots, n-1\}$ have different number of triangles. Therefore all these fragments are different and none of them is a mirror image of the other. Also Figure 10 shows that, no fragment has a vertical axis of symmetry, implying that none of them has the same orientation as that of its mirror image.

Hence $|\gamma_l| = n$, so there are $3\sum_{l=0}^{n-1}(l+2)$ points of connection for the fragments in $\{\gamma_l : l = 0, 1, 2, \dots, n-1\}$. Since the same number of points of connection for the mirror images of the fragments in $\{\gamma_l : l = 0, 1, 2, \dots, n-1\}$. Hence there are $6\sum_{l=0}^{n-1}(l+2)$ points of connection for the fragments in $\{\gamma_l : l = 0, 1, 2, \dots, n-1\}$ and their mirror images. □

Theorem 5. *If the vertex f_{3n} in (n,n) is connected with the vertices v_{3l+1} in (m,m), then there are n distinct fragments, and there are $6\sum_{l=0}^{n-1}(l+2)$ points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 4, with the only difference that the set of fragments obtained, is denoted by $\{\gamma'_l : l = 0, 1, 2, \dots, n-1\}$.

Theorem 6. *If the vertex e_{3n} in (n,n) is connected with the vertices $u_{3l'+1}$, where $l' = 1, 2, \dots, n-1$, in (m,m), then there are $n-1$ distinct fragments, and there are $6\sum_{l'=1}^{n-1}(l'+2)$ points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 4, with the only difference that the set of fragments obtained, is denoted by $\{\lambda_{l'} : l' = 1, 2, \dots, n-1\}$.

Theorem 7. *If the vertex e_{3n} in (n,n) is connected with the vertices $v_{3l'+1}$ in (m,m), then there are $n-1$ distinct fragments, and there are $6\sum_{l'=1}^{n-1}(l'+2)$ points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 4, with the only difference that the set of fragments obtained, is denoted by $\{\lambda'_{l'} : l' = 1, 2, 3, \dots, n-1\}$.

Let $r = \begin{cases} 0 & \text{if } m+n \text{ is even integer} \\ 1 & \text{if } m+n \text{ is odd integer.} \end{cases}$

Theorem 8. *If the vertex f_{3n} in (n,n) is connected with the vertices u_{3p+1}, where $p = \frac{m+n+r}{2}, \frac{m+n+2+r}{2}, \dots, m-1$, in (m,m), then there are $\frac{1}{2}(m-n-r)$ distinct fragments, and there are $3(n+2)(m-n-1)$ points of connection of these fragments and their mirror images.*

Proof. Let us join the vertex f_{3n}, fixed by $(xy)^n(xy^{-1})^n$ with the vertices u_{3p+1}, fixed by $(xy)^p(xy^{-1})^m(xy)^{m-p}$ and obtain a class of fragments μ_p. Then

$$P = \left\{ \begin{array}{c} x, xy^{-1}, xy, xyx, xyxy^{-1}, (xy)^2, \dots, (xy)^{n-1}x, (xy)^{n-1}xy^{-1}, (xy)^n, \\ (xy)^n x, (xy)^n xy^{-1}, (xy)^{n+1}, e, y^{-1}, y \end{array} \right\}$$

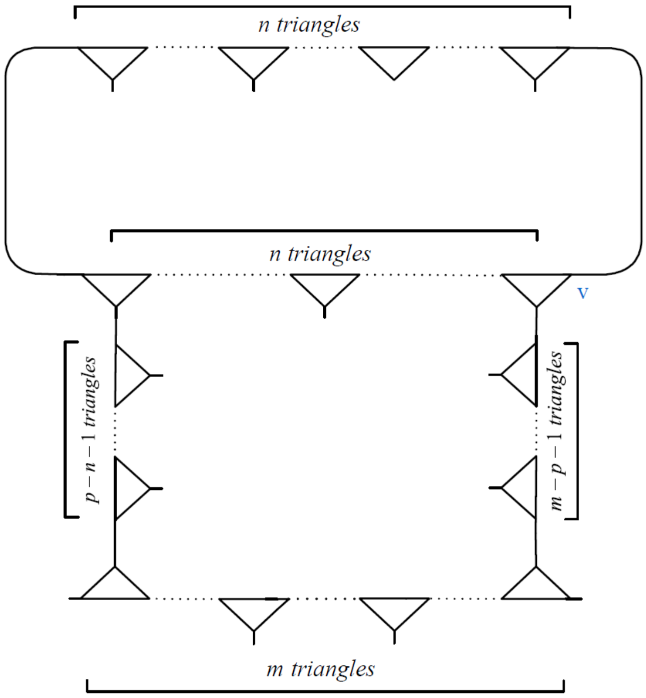

Figure 11

is the set of words such that for any $w \in P$, both the vertices $(f_{3n})\,w$ and $(u_{3p+1})\,w$ lie on (n,n) and (m,m) respectively. By Theorem 3, each fragment μ_p has $|P| = 3\,(n+2)$ points of connection.

Let $\mu_h, \mu_k \in \{\mu_p\}$, then μ_k is obtained by joining f_{3n} with u_{3k+1} and μ_h is obtained by joining f_{3n} with u_{3l+1}.

Let μ_h and μ_k be the same fragments. This means that μ_h is obtainable also, if we join f_{3n} with u_{3k+1}, implying that $f_{3n} \leftrightarrow u_{3k+1}$ is one of the points of connection for μ_h. So by Theorem 2, there exists a word $w \in P$ such that $(f_{3n})\,w = f_{3n}$ and $(u_{3k+1})\,w = u_{3h+1}$. There is only word $e \in P$ for which $(f_{3n})\,w = f_{3n}$, but $(u_{3l+1})\,e \neq u_{3k+1}$. Hence μ_h and μ_k are distinct fragments. Since $p = \frac{m+n+r}{2}, \frac{m+n+2+r}{2}, \ldots, m-1$, therefore $|\mu_p| = \frac{1}{2}\,(m-n-r)$.

Let μ_h and μ_k be the mirror images of each other, that is $\mu_h = \mu_k^*$. Then by Remark 1, μ_h is obtainable also, if we join f_{3n}^*, with u_{3k+1}^*, implying that $f_{3n}^* \leftrightarrow u_{3k+1}^*$ is one of the points of connection for γ_h. So by Theorem 2, there exists a word $w \in P$ such that $(f_{3n})\,w = f_{3n}^*$ and $(u_{3h+1})\,w = u_{3k+1}^*$. There is only one word $(xy)^n\,x \in P$ for which $(f_{3n})\,w = f_{3n}^*$. Now $(u_{3h+1})\,(xy)^n\,x = u_{3(m+n-h)+1}^*$, this implies that for $k = m+n-h$, the fragments μ_k and μ_h are mirror images of each other. Now for all $h \in \left\{\frac{m+n+r}{2}, \frac{m+n+2+r}{2}, \frac{m+n+4+r}{2}, \ldots,\right.$

$m-1\} \setminus \frac{m+n}{2}$, we have $k = m+n-h < \frac{m+n+r}{2}$, implying that $\mu_{m+n-h} \notin \{\mu_p\}$. But for $h = \frac{m+n}{2}$, we get $m+n-h = \frac{m+n}{2}$, therefore $\mu_{\frac{m+n}{2}}$ has the same orientation as that of its mirror image.

Let $m+n$ be an even integer, then there is only one fragment $\mu_{\frac{m+n}{2}} \in \{\mu_p\}$ having the same orientation as that of its mirror image, and all other $\frac{1}{2}(m-n-2)$ fragments have different orientations from their mirror images. Hence there are

$$\begin{aligned} 2|P|\left(\frac{m-n-2}{2}\right) + |P| &= 6(n+2)\left(\frac{m-n-2}{2}\right) + 3(n+2) \\ &= 3(n+2)(m-n-1) \end{aligned}$$

points of connection for the fragments in $\{\mu_p\}$ and their mirror images.

Let $m+n$ be an odd integer, then all fragments in $\{\mu_p\}$ have different orientations from their mirror images. Hence there are

$$2|P|\left(\frac{m-n-1}{2}\right) = 6(n+2)\left(\frac{m-n-1}{2}\right) = 3(n+2)(m-n-1)$$

points of connection for the fragments in $\{\mu_p\}$ and their mirror images. □

Theorem 9. *If the vertex f_{3n} in (n,n) is connected with the vertices v_{3p+1}, where $p = \frac{m+n+r}{2}, \frac{m+n+2+r}{2}, \ldots, m-1$, in (m,m), then there are $\frac{1}{2}(m-n-r)$ distinct fragments, and there are $3(n+2)(m-n-1)$ points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 8, with the only difference that the set of fragments obtained, is denoted by $\{\mu'_p\}$.

Theorem 10. *If the vertex e_{3n} in (n,n) is connected with the vertices u_{3p+1}, where $p = \frac{m+n+r}{2}, \frac{m+n+2+r}{2}, \ldots, m-1$, in (m,m), then there are $\frac{1}{2}(m-n-r)$ distinct fragments, and there are $3(n+2)(m-n-1)$ points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 8, with the only difference that the set of fragments obtained, is denoted by $\{\nu_p\}$.

Theorem 11. *If the vertex e_{3n} in (n,n) is connected with the vertices v_{3p+1}, where $p = \frac{m+n+r}{2}, \frac{m+n+2+r}{2}, \ldots, m-1$, in (m,m), then there are $\frac{1}{2}(m-n-r)$ distinct fragments and there are $3(n+2)(m-n-1)$ points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 8, with the only difference that the set of fragments obtained, is denoted by $\{\nu'_p\}$.

Let $\sigma = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

Theorem 12. *If the vertices e_{3i}, where $i = 1, 2, 3, \ldots, \frac{n-(\sigma+2)}{2}$, in (n, n) are connected with the vertices u_{3j+1}, where $j = 1, 2, 3, \ldots, m-1$, in (m, m), then there are $\frac{1}{2}(n-(\sigma+2))(m-1)$ different fragments, and there are*

$$6(n-(\sigma+2))(m-1)$$

points of connection of these fragments and their mirror images.

Proof. Let us join the vertices e_{3i}, fixed by $(xy^{-1})^{n-i}(xy)^n(xy^{-1})^i$ with the vertices u_{3j+1}, fixed by $(xy)^j(xy^{-1})^m(xy)^{m-j}$ and obtain a class of fragments $\alpha_{(i,j)}$. Then $P = \{y, y^{-1}, e, x, xy^{-1}, xy\}$ is the set of words such that for any $w \in$

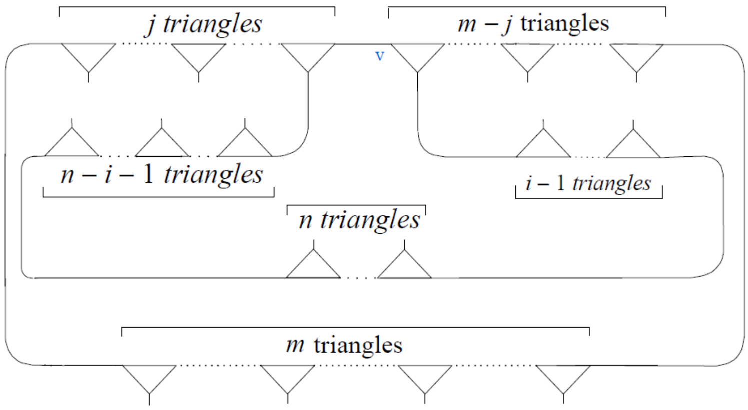

FIGURE 12

P, both the vertices $(e_{3i})w$ and$(u_{3j+1})w$ lie on (n, n) and (m, m) respectively. Since $|P| = 6$, therefore by Theorem 3, each fragment $\alpha_{(i,j)}$ has 6 points of connection.

Let $\alpha_{(h_1,k_1)}, \alpha_{(h_2,k_2)} \in \{\alpha_{(i,j)}\}$. Then $\alpha_{(h_1,k_1)}$ is obtained by joining e_{3h_1} with u_{3k_1+1} and $\alpha_{(h_2,k_2)}$ is obtained by joining e_{3h_2} with u_{3k_2+1}.

Let $\alpha_{(h_1,k_1)}$ and $\alpha_{(h_2,k_2)}$ be the same fragments. This means that $\alpha_{(h_1,k_1)}$ is obtainable also, if we join e_{3h_2} with u_{3k_2+1}, implying that $e_{3h_2} \leftrightarrow u_{3k_2+1}$ is one of the 6 points of connection for $\alpha_{(h_1,k_1)}$. So by Theorem 2, there exists a word $w \in P$ such that $(e_{3h_1})w = e_{3h_2}$ and $(u_{3k_1+1})w = u_{3k_2+1}$. There is only word $xy^{-1} \in P$ for which $(e_{3h_1})w = e_{3h_2}$ where $h_2 = h_1 + 1$, but $(u_{3k_1+1})xy^{-1} \neq u_{3k_2+1}$. Hence $\alpha_{(h_1,k_1)}$ and $\alpha_{(h_2,k_2)}$ are distinct fragments.

Let $\alpha_{(h_1,k_1)}$ and $\alpha_{(h_2,k_2)}$ be the mirror images of each other, that is $\alpha_{(h_1,k_1)} = \alpha^*_{(h_2,k_2)}$. Then by Remark 1, $\alpha_{(h_1,k_1)}$ is obtainable also, if we join $e^*_{3h_2}$ with $u^*_{3k_2+1}$, implying that $e^*_{3h_2} \leftrightarrow u^*_{3k_2+1}$ is one of the 6 points of connection for $\alpha_{(h_1,k_1)}$. So there exists a word $w \in P$ such that $(e_{3h_1})w = e^*_{3h_2}$ and $(u_{3k_1+1})w = u^*_{3k_2+1}$. There is only word $x \in P$ for which $(e_{3h_1})w = e_{3h_1+1} = e^*_{3(n-h_1)}$. Now $(u_{3k_1+1})x = u_{3k_1} = u^*_{3(m-k_1)+1}$, this implies that for $h_2 = n-h_1$

and $k_2 = m - k_1$, $\alpha_{(h_1,k_1)}$ and $\alpha_{(h_2,k_2)}$ are the mirror images of each other. But for all $h_1 \in \left\{1, 2, 3, \ldots, \frac{n-(\sigma+2)}{2}\right\}$, we have $h_2 = n - h_1 > 1, 2, \ldots, \frac{n-(\sigma+2)}{2}$, implying that $\alpha_{(h_2,k_2)} = \alpha_{(n-h_1,m-k_1)} \notin \left\{\alpha_{(i,j)}\right\}$. Therefore $\alpha_{(h_1,k_1)}, \alpha_{(h_2,k_2)} \in \left\{\alpha_{(i,j)}\right\}$ are not the mirror images of each other.

Hence $|\alpha_{(i,j)}| = \frac{1}{2}(n - (\sigma + 2))(m - 1)$, so there are $3(n - (\sigma + 2))(n - 1)$ points of connection for the fragments in $\left\{\alpha_{(i,j)}\right\}$. Since the same number of points of connection for the mirror images of the fragments in $\left\{\alpha_{(i,j)}\right\}$. Therefore the total number of points of connection for the fragments in $\left\{\alpha_{(i,j)}\right\}$ and their mirror images are $6(n - (\sigma + 2))(m - 1)$. □

Theorem 13. *If the vertices e_{3i}, where $i = 1, 2, 3, \ldots, \frac{n-(\sigma+2)}{2}$, in (n, n) are connected with the vertices v_{3j+1}, where $j = 1, 2, 3, \ldots, n - 1$, in (m, m), then there are $\frac{1}{2}(n - (\sigma + 2))(m - 1)$ different fragments, and there are*

$$6(n - (\sigma + 2))(m - 1)$$

points of connection of these fragments and their mirror images.

The proof is similar to that of Theorem 12, with the only difference that the set of fragments obtained, is denoted by $\left\{\alpha'_{(i,j)}\right\}$.

Theorem 14. *If the vertices f_{3i}, where $i = 1, 2, 3, \ldots, \frac{n-(\sigma+2)}{2}$, in (n, n) are connected with the vertices u_{3j+1}, where $j = 1, 2, 3, \ldots, n - 1$, in (m, m), then there are $\frac{1}{2}(n - (\sigma + 2))(m - 1)$ different fragments, and there are*

$$6(n - (\sigma + 2))(m - 1)$$

points of connection of these fragments and their mirror images.

The proof is similar to that of Theorem 12, with the only difference that the set of fragments obtained, is denoted by $\left\{\beta_{(i,j)}\right\}$.

Theorem 15. *If the vertices f_{3i}, where $i = 1, 2, 3, \ldots, \frac{n-(\sigma+2)}{2}$, in (n, n) are connected with the vertices v_{3j+1}, where $j = 1, 2, 3, \ldots, n - 1$, in (m, m), then there are $\frac{1}{2}(n - (\sigma + 2))(m - 1)$ different fragments, and there are*

$$6(n - (\sigma + 2))(m - 1)$$

points of connection of these fragments and their mirror images.

The proof is similar to that of Theorem 12, with the only difference that the set of fragments obtained, is denoted by $\left\{\beta'_{(i,j)}\right\}$.

Recall $\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ and let $j' = \begin{cases} 1, 2, 3, \ldots, m - 1 & \text{if } n \text{ is odd} \\ 1, 2, 3, \ldots, \frac{m-\sigma}{2} & \text{if } n \text{ is even,} \end{cases}$ $\sigma' = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even.} \end{cases}$

Theorem 16. *If the vertex* $e_{\frac{3(n-\sigma)}{2}}$ *in* (n,n) *is connected with the vertices* $u_{3j'+1}$ *in* (m,m), *then there are* $\begin{cases} m-1 & \text{if } n \text{ is odd} \\ \frac{m-\sigma'}{2} & \text{if } n \text{ is even} \end{cases}$, *different fragments, and there are* $\begin{cases} 12(m-1) & \text{if } n \text{ is odd} \\ 6(m-1) & \text{if } n \text{ is even} \end{cases}$, *points of connection of these fragments and their mirror images.*

Proof. Let us join the vertex $e_{\frac{3(n-\sigma)}{2}}$, fixed by $\left(xy^{-1}\right)^{\frac{n+\sigma}{2}}(xy)^n\left(xy^{-1}\right)^{\frac{n-\sigma}{2}}$ with the vertices $u_{3j'+1}$, fixed by $(xy)^{j'}\left(xy^{-1}\right)^m(xy)^{m-j'}$ and obtain a class of fragments $\alpha_{\left(\frac{n-\sigma}{2},j'\right)}$. Then $P=\{y,y^{-1},e,x,xy^{-1},xy\}$ is the set of words such that

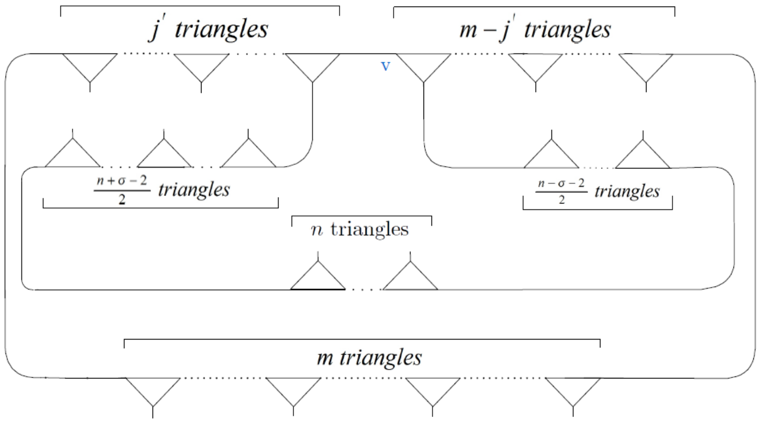

FIGURE 13

for any $w\in P$, both the vertices $\left(e_{\frac{3(n-\sigma)}{2}}\right)w$ and$(u_{3j'+1})\,w$ lie on (n,n) and (m,m) respectively. Since $|P|=6$, therefore by Theorem 3, each fragment $\alpha_{\left(\frac{n-\sigma}{2},j'\right)}$ has 6 points of connection.

We prove in Theorem 12 that all fragments in $\left\{\alpha_{(i,j)}\right\}$ are distinct, and the mirror image of $\alpha_{(h_1,k_1)}\in\left\{\alpha_{(i,j)}\right\}$ is $\alpha_{(n-h_1,m-k_1)}$. Similarly we have, all the fragments in $\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ are different, and the mirror image of $\alpha_{\left(\frac{n-\sigma}{2},k_1'\right)}\in\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ is $\alpha_{\left(\frac{n+\sigma}{2},m-k_1'\right)}$. Now

(i) If n is odd, then $\alpha_{\left(\frac{n+1}{2},m-k_1'\right)}\notin\left\{\alpha_{\left(\frac{n-1}{2},j'\right)}\right\}$, this shows that none of the fragments in $\left\{\alpha_{\left(\frac{n-1}{2},j'\right)}\right\}$ is the mirror image of the other. Hence $|\alpha_{\left(\frac{n-1}{2},j'\right)}|=m-1$, and so there are $12(m-1)$ points of connection for the fragments of $\left\{\alpha_{\left(\frac{n-1}{2},j'\right)}\right\}$ and their mirror images.

(ii) If n is even, and m is an odd integer, then for all $k_1' \in \left\{1, 2, 3, \ldots, \frac{m-1}{2}\right\}$, we have $k_1' > \frac{m-1}{2}$ implying that $\alpha_{(\frac{n}{2}, m-k_1')} \notin \left\{\alpha_{(\frac{n}{2}, j')}\right\}$. So none of the fragments in $\left\{\alpha_{(\frac{n}{2}, j')}\right\}$ is the mirror image of the other, implies that $|\alpha_{(\frac{n}{2}, j')}| = \frac{m-1}{2}$. Hence total number of points of connection for the fragments in $\left\{\alpha_{(\frac{n}{2}, j')}\right\}$ and their mirror images are $2|P|\left(\frac{m-1}{2}\right) = 6(m-1)$.

(iii) If n and m are both even, then for all $k_1' \in \left\{1, 2, 3, \ldots, \frac{m}{2}\right\} \setminus \frac{m}{2}$, we have $k_1' > \frac{m}{2}$ implying that $\alpha_{(\frac{n}{2}, m-k_1')} \notin \left\{\alpha_{(\frac{n}{2}, j')}\right\}$, and for $k_1' = \frac{m}{2}$, we have $\alpha_{(\frac{n}{2}, m-k_1')} = \alpha_{(\frac{n}{2}, \frac{m}{2})} \in \left\{\alpha_{(\frac{n}{2}, j')}\right\}$. This shows that, for $j' < \frac{m}{2}$, none of the fragments in $\left\{\alpha_{(\frac{n}{2}, j')}\right\}$ is the mirror image of the other and $\alpha_{(\frac{n}{2}, \frac{m}{2})}$ is the mirror image of itself, implies that $|\alpha_{(\frac{n}{2}, j')}| = \frac{m}{2}$. Hence there are $2|P|\left(\frac{m-2}{2}\right) + |P| = 6(m-1)$ points of connection for the fragments in $\left\{\alpha_{(\frac{n}{2}, j')}\right\}$ and their mirror images. □

Theorem 17. *If the vertex $e_{\frac{3(n-\sigma)}{2}}$ in (n, n) is connected with the vertices $v_{3j'+1}$ in (m, m), then there are $\begin{cases} m-1 & \text{if } n \text{ is odd} \\ \frac{m-\sigma'}{2} & \text{if } n \text{ is even} \end{cases}$, different fragments, and there are $\begin{cases} 12(m-1) & \text{if } n \text{ is odd} \\ 6(m-1) & \text{if } n \text{ is even} \end{cases}$, points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 16, with the only difference that the set of fragments obtained, is denoted by $\left\{\alpha'_{(\frac{n-\sigma}{2}, j')}\right\}$.

Theorem 18. *If the vertex $f_{\frac{3(n-\sigma)}{2}}$ in (n, n) is connected with the vertices $u_{3j'+1}$ in (m, m), then there are $\begin{cases} m-1 & \text{if } n \text{ is odd} \\ \frac{m-\sigma'}{2} & \text{if } n \text{ is even} \end{cases}$, different fragments, and there are $\begin{cases} 12(m-1) & \text{if } n \text{ is odd} \\ 6(m-1) & \text{if } n \text{ is even} \end{cases}$, points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 16, with the only difference that the set of fragments obtained, is denoted by $\left\{\beta_{(\frac{n-\sigma}{2}, j')}\right\}$.

Theorem 19. *If the vertex $f_{\frac{3(n-\sigma)}{2}}$ in (n, n) is connected with the vertices $v_{3j'+1}$ in (m, m), then there are $\begin{cases} m-1 & \text{if } n \text{ is odd} \\ \frac{m-\sigma'}{2} & \text{if } n \text{ is even} \end{cases}$, different fragments, and there are $\begin{cases} 12(m-1) & \text{if } n \text{ is odd} \\ 6(m-1) & \text{if } n \text{ is even} \end{cases}$, points of connection of these fragments and their mirror images.*

The proof is similar to that of Theorem 16, with the only difference that the set of fragments obtained, is denoted by $\left\{\beta_{\left(\frac{n-\sigma'}{2},j'\right)}\right\}$.

Theorem 20. *Let η be the fragment obtained by joining the vertex f_{3n} in (n,n) with the vertex v_{3m} in (m,m). Then there are $12(n+1)$ points of connection for η and its mirror image.*

Proof. Let us join the vertex f_{3n}, fixed by $(xy)^n(xy^{-1})^n$ in (n,n) with the vertex v_{3m}, fixed by $(xy)^m(xy^{-1})^m$ in (m,m), and obtain a fragment η.

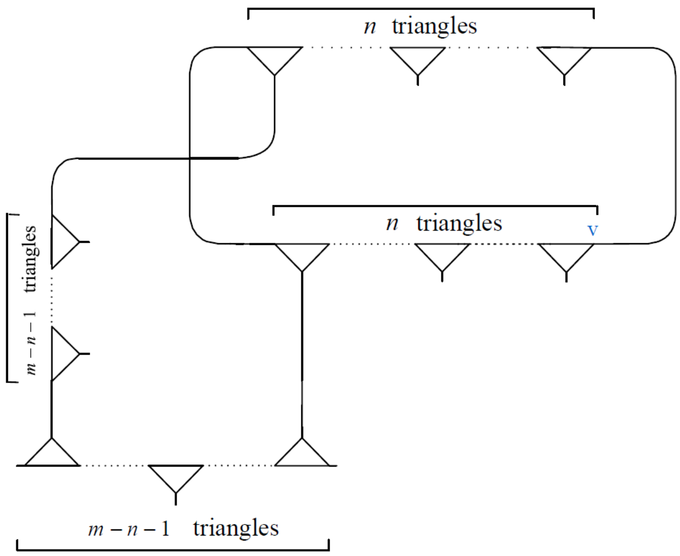

FIGURE 14

Then

$$P=\left\{\begin{array}{c} x,xy^{-1},xy,xyx,xyxy^{-1},(xy)^2,\ldots,(xy)^{n-1}x,(xy)^{n-1}xy^{-1},(xy)^n, \\ (xy)^n x,(xy)^n xy^{-1},(xy)^{n+1},y^{-1},y,yx,yxy^{-1},yxy,yxyx,yxyxy^{-1}, \\ yxyxy,yxyxyx,\ldots,(yx)^{n-1}y^{-1},(yx)^{n-1}y,(yx)^n,(yx)^n y^{-1},(yx)^n y,e \end{array}\right\}$$

is the set of words such that for any $w \in P$, both the vertices $(f_{3n})\,w$ and $(v_{3m})\,w$ lie on (n,n) and (m,m) respectively. By Theorem 3, η has $|P| = 6(n+1)$ points of connection.

Let η has the same orientation from its mirror image η^*. Then by Remark 1, η is obtainable also, if we join f_{3n}^* with v_{3m}^*, implying that $f_{3n}^* \leftrightarrow v_{3m}^*$ is one of the points of connection for η. So by Theorem 2, there exists a word $w \in P$ such that $(f_{3n})\,w = f_{3n}^*$ and $(v_{3m})\,w = v_{3m}^*$. But P does not contain such a word. Therefore η has different orientation from its mirror image η^*. Hence there are $12\,(n-1)$ points of connection for the η and its mirror image η^*. $\square$

Theorem 21. *Let η' be the fragment obtained by joining the vertex e_{3n} in (n,n) with the vertex v_{3m} in (m,m). Then there are $12(n+1)$ points of connection for η' and its mirror image.*

The proof is similar to that of Theorem 20.

Remark 3. In Figures 8 and 9, one can see that the vertices f_{3n} and e_{3n} are fixed by the same word $(xy)^n\left(xy^{-1}\right)^n$. Moreover the vertices u_{3k+1} and v_{3k+1}, for some $k \in \{1,2,3,\ldots,n-1\}$, are fixed by the same word

$$(xy)^{3k}\left(xy^{-1}\right)^m(xy)^{m-3k}.$$

Since $\gamma_k, \gamma'_k, \lambda_k$ and λ'_k are composed by joining f_{3n} with u_{3k+1}, f_{3n} with v_{3k+1}, e_{3n} with u_{3k+1} and e_{3n} with v_{3k+1} respectively. Therefore $\gamma_k, \gamma'_k, \lambda_k$ and λ'_k contain a vertex fixed by a pair of words

$$(xy)^n\left(xy^{-1}\right)^n, (xy)^k\left(xy^{-1}\right)^m(xy)^{m-k}.$$

Hence $\{\gamma_l\}\diagdown\gamma_0, \{\gamma'_l\}\diagdown\gamma'_0, \{\lambda_{l'}\}$ and $\left\{\lambda'_{l'}\right\}$ are the same sets of fragments. Similarly it is easy to show that

(i) γ_0 and γ'_0 are the same fragments.

(ii) $\left\{\mu_p\right\}, \left\{\mu'_p\right\}, \{\nu_p\}$ and $\left\{\nu'_p\right\}$ are the same sets of fragments.

(iii) $\left\{\alpha_{(i,j)}\right\}, \left\{\alpha'_{(i,j)}\right\}, \left\{\beta_{(i,j)}\right\}$ and $\left\{\beta'_{(i,j)}\right\}$ are the same sets of fragments.

(iv) $\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}, \left\{\alpha'_{\left(\frac{n-\sigma}{2},j'\right)}\right\}, \left\{\beta_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ and $\left\{\beta'_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ are the same sets of fragments.

(v) η and η' are the same fragments.

Thus, we are left with five, $\{\gamma_l\}, \{\mu_p\}, \left\{\alpha_{(i,j)}\right\}, \left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ and $\{\eta\}$, sets of fragments. Now we show that these sets are mutually disjoint.

It is clear from figures 11 to 14 that none of the fragments in $\left\{\mu_p\right\}$, $\left\{\alpha_{(i,j)}\right\}$, $\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ and $\{\eta\}$ contains a vertex fixed by

$$(xy)^n\left(xy^{-1}\right)^n, (xy)^l\left(xy^{-1}\right)^m(xy)^{m-l}, \text{ where } l < n.$$

This implies that

$$\{\gamma_l\} \cap \left\{\mu_p\right\} = \{\gamma_l\} \cap \left\{\alpha_{(i,j)}\right\} = \{\gamma_l\} \cap \left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\} = \{\gamma_l\} \cap \{\eta\} = \phi. \tag{2.1}$$

From Figures 12 to 14, one can see that, none of the fragments in $\left\{\alpha_{(i,j)}\right\}$, $\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ and $\{\eta\}$ contains a vertex fixed by

$$(xy)^n\left(xy^{-1}\right)^n, (xy)^p\left(xy^{-1}\right)^m(xy)^{m-p}, \text{ where } p > n.$$

So

$$\left\{\mu_p\right\} \cap \left\{\alpha_{(i,j)}\right\} = \left\{\mu_p\right\} \cap \left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\} = \left\{\mu_p\right\} \cap \{\eta\} = \phi. \tag{2.2}$$

From Figures 13 to 14, it is quite obvious that none of the fragments in $\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ and $\{\eta\}$ contains a vertex fixed by

$$\left(xy^{-1}\right)^{n-i}(xy)^{n}\left(xy^{-1}\right)^{i},(xy)^{j}\left(xy^{-1}\right)^{m}(xy)^{m-j},$$

where $i<\frac{n-\sigma}{2}$. Therefore

$$\left\{\alpha_{(i,j)}\right\}\cap\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}=\left\{\alpha_{(i,j)}\right\}\cap\{\eta\}=\phi. \tag{2.3}$$

From Figure 13, it is easy to verify that none of the fragments in $\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ contains a vertex fixed by $(xy)^{n}\left(xy^{-1}\right)^{n}$,$(xy)^{m}\left(xy^{-1}\right)^{m}$. This implies that

$$\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}\cap\{\eta\}=\phi. \tag{2.4}$$

From equations 2.1 to 2.4, we have $\left\{\mu_p\right\},\left\{\alpha_{(i,j)}\right\},\left\{\alpha_{\left(\frac{n-\sigma}{2},j'\right)}\right\}$ and $\{\eta\}$ are mutually disjoint.

Let

$$\rho=\begin{cases}0 & \text{if } n \text{ is even and } m \text{ is odd}\\ 0 & \text{if } n \text{ is odd and } m \text{ is even}\\ 1 & \text{if both } n \text{ and } m \text{ are odd}\\ 2 & \text{if both } n \text{ and } m \text{ are even.}\end{cases}$$

Now we are in a position, to prove our main result.

Theorem 22. *There are $\frac{1}{2}(nm+2+\rho)$ polynomials obtained by joining the circuit (n,n) with (m,m) at all points.*

Proof. Let us connect the following vertices
(i) f_{3n} with u_{3l+1},u_{3p+1}.
(ii) e_{3i} with u_{3j+1}.
(iii) $e_{\frac{3(n-\sigma)}{2}}$ with $u_{3j'+1}$.
(iv) f_{3n} with v_{3m}.
Then by Theorems 4, 8, 12, 16, 20 and Remark 3, we obtain the set of fragments $F=\left\{\gamma_l,\mu_p,\alpha_{(i,j)},\alpha_{\left(\frac{n-\sigma}{2},j'\right)},\eta\right\}$, and there are

$$\begin{aligned}S=12\sum_{l=0}^{n-1}(l+2)+12\sum_{l'=1}^{n-1}(l'+2)+12(n+2)(m-n-1)\\ +24(n-1)(m-1)+24(n+1)\end{aligned}$$

points of connection of these fragments and their mirror images. Since $S=36nm$, so S is the total points of connection in (n,n) and (m,m). Also $|F|=\frac{1}{2}(nm+2+\rho)$, hence there are $\frac{1}{2}(nm+2+\rho)$ different fragments, composed, by joining (n,n) and (m,m) at all points of connection. Since a unique polynomial is obtained from a fragment, so there are $\frac{1}{2}(nm+2+\rho)$ polynomials obtained by joining the circuit (n,n) with (m,m) at all points. □

Conclusion

Total number of points of connection in (n,n) and (m,m) are $36nm$, out of which only $\frac{1}{2}(nm+2+\rho)$ points of connection are important. There is no need to join (n,n) and (m,m) at the points, which are not mentioned as important. Because if we join (n,n) and (m,m) at such a point v, we obtain a fragment, which we have already obtained by joining at important points.

Example 3. Consider two circuits $(2,2)$ and $(3,3)$. By Theorem 22 the set of

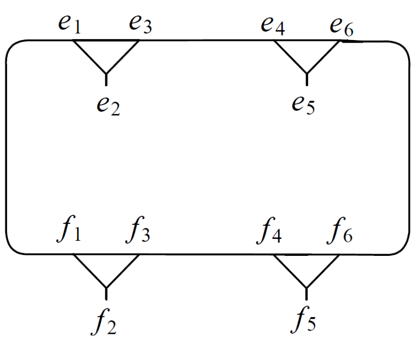

FIGURE 15

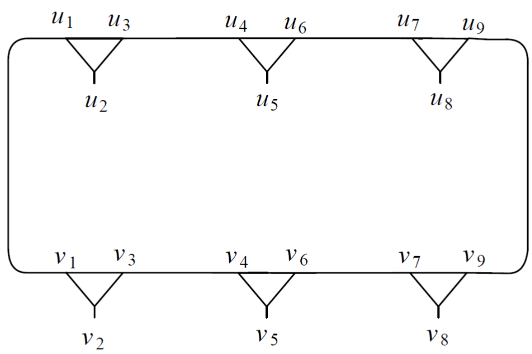

FIGURE 16

all fragments evolved by joining the circuits (n,n) and (m,m) at all points, is $\left\{\gamma_l, \mu_p, \alpha_{(i,j)}, \alpha_{\left(\frac{n-\sigma}{2}, j'\right)}, \eta\right\}$. Therefore $\{\gamma_0, \gamma_1, \alpha_{(1,1)}, \eta\}$ is set of all fragments evolved by joining the circuits $(2,2)$ and $(3,3)$.

1. The fragment γ_0 is obtained by joining the vertex f_6, fixed by

$$(xy)^2(xy^{-1})^2$$

in $(2,2)$ with the vertex u_1, fixed by $(xy^{-1})^3(xy)^3$ in $(3,3)$. By using the method

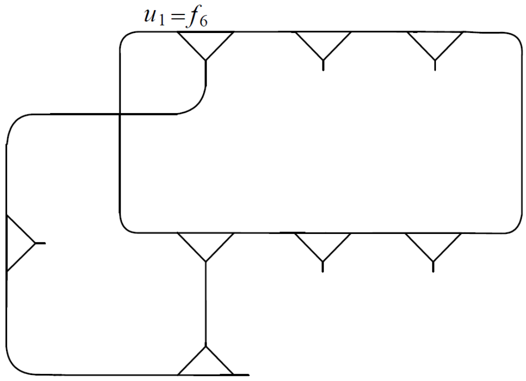

FIGURE 17

of calculating a polynomial from a fragment given in [6], the polynomial evolved from γ_0 is

$$f_1(\theta) = -\theta^8 + 11\theta^7 - 48\theta^6 + 106\theta^5 - 126\theta^4 + 80\theta^3 - 25\theta^2 + 3\theta.$$

2. The fragment γ_1 is obtained by joining the vertex f_6, fixed by

$$(xy)^2(xy^{-1})^2$$

in $(2,2)$ with the vertex u_4, fixed by $(xy)(xy^{-1})^3(xy)^2$ in $(3,3)$. By using the

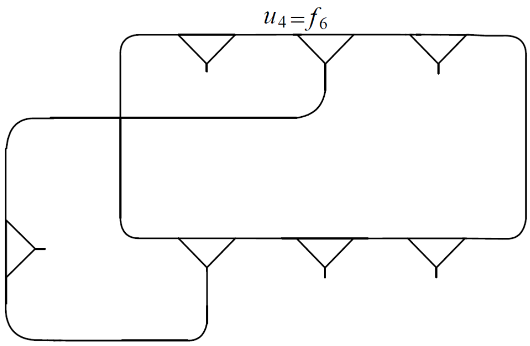

FIGURE 18

method of calculating a polynomial from a fragment, the polynomial evolved from γ_1 is

$$f_2(\theta) = -\theta^7 + 9\theta^6 - 31\theta^5 + 50\theta^4 - 35\theta^3 + 5\theta^2 + 3\theta.$$

3. The fragment $\alpha_{(1,1)}$ is obtained by joining the vertex e_3, fixed by

$$(xy^{-1})(xy)^2(xy^{-1})$$

in $(2,2)$ with the vertex u_4, fixed by $(xy)(xy^{-1})^3(xy)^2$ in $(3,3)$. By using the

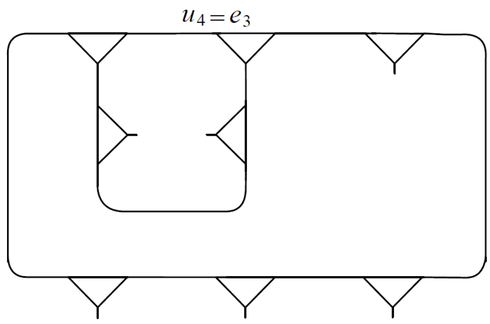

FIGURE 19

method of calculating a polynomial from a fragment, the polynomial evolved from $\alpha_{(1,1)}$ is

$$f_3(\theta) = \theta^8 - 11\theta^7 + 48\theta^6 - 106\theta^5 + 124\theta^4 - 70\theta^3 + 11\theta^2 + 3\theta.$$

4. The fragment η is obtained by joining the vertex f_6, fixed by $(xy)^2(xy^{-1})^2$ in $(2,2)$ with the vertex v_9, fixed by $(xy)^3(xy^{-1})^3$ in $(3,3)$. By using the method

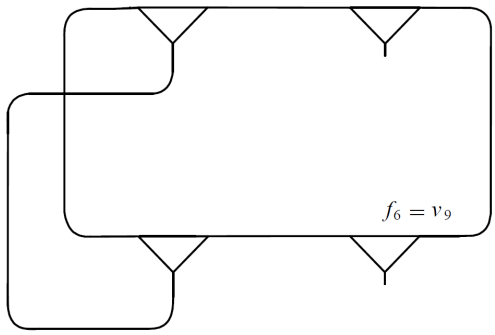

FIGURE 20

of calculating a polynomial from a fragment, the polynomial evolved from η is

$$f_4(\theta) = -\theta^4 + 5\theta^3 - 7\theta^2 + 3\theta.$$

Since the total number of points of connection in (n,n) and (m,m) are $36nm$. So there are 216 points of connection in $(2,2)$ and $(3,3)$. Theorem 22 assures us, in order to create all fragments by joining $(2,2)$ and $(3,3)$, we just have to connect only 4 points (mentioned in Theorem 22). There is no need to connect $(2,2)$ and $(3,3)$ at the remaining 212 points.

Hence there are 216 points of connection in $(2,2)$ and $(3,3)$, which compose only 4 distinct fragments $\gamma_0, \gamma_1, \alpha_{(1,1)}$ and η. The polynomials obtained from these fragments are

$$f_1(\theta) = -\theta^8 + 11\theta^7 - 48\theta^6 + 106\theta^5 - 126\theta^4 + 80\theta^3 - 25\theta^2 + 3\theta,$$
$$f_2(\theta) = -\theta^7 + 9\theta^6 - 31\theta^5 + 50\theta^4 - 35\theta^3 + 5\theta^2 + 3\theta,$$
$$f_3(\theta) = \theta^8 - 11\theta^7 + 48\theta^6 - 106\theta^5 + 124\theta^4 - 70\theta^3 + 11\theta^2 + 3\theta,$$
$$f_4(\theta) = -\theta^4 + 5\theta^3 - 7\theta^2 + 3\theta.$$

These polynomials split linearly in suitable Galois fields [6] and corresponding to each zero θ, we have a coset diagram $D(\theta, q)$ [7], and ultimately we obtain a class of permutation groups.

3. Open problem for future study

Each fragment is related with a polynomial $f(\theta)$, which splits linearly in a suitable Galois field [6] and corresponding to each zero θ, we get a coset diagram $D(\theta, q)$ [7] and hence a permutation group. This shows that each pair of circuits can be related to a class of groups. In a private communication with Q. Mushtaq, Professor Graham Higman claims that there must be a common property in the groups related to a pair of circuits, and he will feel very happy if some one could find the common property in the groups obtained from the same pair of circuits. In order to pursue this problem, we first have to deal with the question: how many distinct fragments (polynomials) are obtained, if we join the circuits (n,n) and (m,m) at all points of connection?

The answer of this question is given in this article. We are working on the classes of groups obtained from the pair of circuits $(2,2)$ and (m,m), where $m \leq 5$. We will share some interesting results regarding this open problem in a separate paper in the future.

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