

SPECIAL CLASSES OF MERIDIAN SURFACES IN THE FOUR-DIMENSIONAL EUCLIDEAN SPACE

GEORGI GANCHEV AND VELICHKA MILOUSHEVA

ABSTRACT. Meridian surfaces in the Euclidean 4-space are two-dimensional surfaces which are one-parameter systems of meridians of a standard rotational hypersurface. On the base of our invariant theory of surfaces we study meridian surfaces with special invariants. In the present paper we give the complete classification of Chen meridian surfaces and meridian surfaces with parallel normal bundle.

1. Introduction

A fundamental problem of the contemporary differential geometry of surfaces and hypersurfaces in the Euclidean space \mathbb{R}^n is the investigation of the basic invariants characterizing the surfaces. Our aim is to study and classify various important classes of surfaces in the four-dimensional Euclidean space \mathbb{R}^4 characterized by conditions on their invariants.

An invariant theory of surfaces in the four-dimensional Euclidean space \mathbb{R}^4 was developed by the present authors in [7] and [8]. We introduced an invariant linear map γ of Weingarten-type in the tangent plane at any point of the surface, which generates two invariant functions $k = \det \gamma$ and $\varkappa = -\frac{1}{2} \operatorname{tr} \gamma$. On the base of this map γ we introduced principal lines and a geometrically determined moving frame field. Writing derivative formulas of Frenet-type for this frame field, we obtained eight invariant functions $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ and proved a fundamental theorem of Bonnet-type, stating that these eight invariants under some natural conditions determine the surface up to a motion in \mathbb{R}^4 .

The basic geometric classes of surfaces in \mathbb{R}^4 are characterized by conditions on these invariant functions. For example, surfaces with flat normal connection are characterized by the condition $\nu_1 = \nu_2$, minimal surfaces are described by $\nu_1 + \nu_2 = 0$, Chen surfaces are characterized by $\lambda = 0$, and surfaces with parallel normal bundle are characterized by the condition $\beta_1 = \beta_2 = 0$.

Received November 3, 2014; Revised May 11, 2015.

2010 *Mathematics Subject Classification.* Primary 53A55; Secondary 53A07, 53A10.

Key words and phrases. meridian surfaces, Chen surfaces, surfaces with parallel normal bundle.

In [8] we constructed special two-dimensional surfaces which are one-parameter systems of meridians of the rotational hypersurface in \mathbb{R}^4 and called these surfaces *meridian surfaces*. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two-dimensional axis in \mathbb{R}^4 . Hence, the class of meridian surfaces is a new source of examples of two-dimensional surfaces in \mathbb{R}^4 . We classified the meridian surfaces with constant Gauss curvature, constant mean curvature, and constant invariant k [8].

In the present paper we give the invariants $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ of the meridian surfaces and on the base of these invariants we classify completely the Chen meridian surfaces (Theorem 4.1) and the meridian surfaces with parallel normal bundle (Theorem 5.1).

2. Preliminaries

Let \mathbb{R}^4 be the four-dimensional Euclidean space endowed with the metric $\langle \cdot, \cdot \rangle$ and M^2 be a surface in \mathbb{R}^4 . We denote by ∇' and ∇ the Levi Civita connections on \mathbb{R}^4 and M^2 , respectively. Let x and y be vector fields tangent to M^2 and ξ be a normal vector field. The formulas of Gauss and Weingarten give decompositions of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into tangent and normal components:

$$\begin{aligned}\nabla'_x y &= \nabla_x y + \sigma(x, y); \\ \nabla'_x \xi &= -A_\xi x + D_x \xi,\end{aligned}$$

which define the second fundamental tensor σ , the normal connection D and the shape operator A_ξ with respect to ξ . The mean curvature vector field H of the surface M^2 is defined as $H = \frac{1}{2} \operatorname{tr} \sigma$.

Let $M^2 : z = z(u, v), (u, v) \in \mathcal{D} (\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization of M^2 . The tangent space at an arbitrary point $p = z(u, v)$ of M^2 is $T_p M^2 = \operatorname{span}\{z_u, z_v\}$. We use the standard denotations $E(u, v) = \langle z_u, z_u \rangle$, $F(u, v) = \langle z_u, z_v \rangle$, $G(u, v) = \langle z_v, z_v \rangle$ for the coefficients of the first fundamental form. Let $\{n_1, n_2\}$ be an orthonormal normal frame field of M^2 such that the quadruple $\{z_u, z_v, n_1, n_2\}$ is positively oriented in \mathbb{R}^4 . The coefficients of the second fundamental form II of M^2 are introduced by the following functions

$$L = \frac{2}{W} \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad M = \frac{1}{W} \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad N = \frac{2}{W} \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix},$$

where

$$\begin{aligned}c_{11}^1 &= \langle z_{uu}, n_1 \rangle; & c_{12}^1 &= \langle z_{uv}, n_1 \rangle; & c_{22}^1 &= \langle z_{vv}, n_1 \rangle; \\ c_{11}^2 &= \langle z_{uu}, n_2 \rangle; & c_{12}^2 &= \langle z_{uv}, n_2 \rangle; & c_{22}^2 &= \langle z_{vv}, n_2 \rangle.\end{aligned}$$

The second fundamental form II is invariant up to the orientation of the tangent space or the normal space of the surface.

The condition $L = M = N = 0$ characterizes points at which the space $\{\sigma(x, y) : x, y \in T_p M^2\}$ is one-dimensional. We call such points *flat points* of the surface. The surfaces consisting of flat points either lie in \mathbb{R}^3 or are developable ruled surfaces in \mathbb{R}^4 . So, further we consider surfaces free of flat points, i.e., $(L, M, N) \neq (0, 0, 0)$.

Using the functions L, M, N and E, F, G in [7] we introduced a linear map γ of Weingarten type in the tangent space at any point of M^2 similarly to the theory of surfaces in \mathbb{R}^3 . The map γ is invariant with respect to changes of parameters on M^2 as well as to motions in \mathbb{R}^4 . It generates two invariant functions

$$k = \frac{LN - M^2}{EG - F^2}, \quad \varkappa = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

It turns out that the invariant \varkappa is the curvature of the normal connection of the surface (see [7]). As in the theory of surfaces in \mathbb{R}^3 the invariant k divides the points of M^2 into the following types: *elliptic* ($k > 0$), *parabolic* ($k = 0$), and *hyperbolic* ($k < 0$).

The second fundamental form II determines conjugate, asymptotic, and principal tangents at a point p of M^2 in the standard way. A line $c : u = u(q), v = v(q); q \in J \subset \mathbb{R}$ on M^2 is said to be an *asymptotic line*, respectively a *principal line*, if its tangent at any point is asymptotic, respectively principal. The surface M^2 is parameterized by principal lines if and only if $F = 0, M = 0$.

Considering surfaces in \mathbb{R}^4 whose mean curvature vector at any point is non-zero (surfaces free of minimal points), on the base of the principal lines we introduced a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point of such a surface [7]. The tangent vector fields x and y are collinear with the principal directions, the normal vector field b is collinear with the mean curvature vector field H . Writing derivative formulas of Frenet-type for this frame field, we obtained eight invariant functions $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$, which determine the surface up to a rigid motion in \mathbb{R}^4 .

The invariants $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1$, and β_2 are determined by the geometric frame field $\{x, y, b, l\}$ as follows

$$(1) \quad \begin{aligned} \nu_1 &= \langle \nabla'_x x, b \rangle, & \nu_2 &= \langle \nabla'_y y, b \rangle, & \lambda &= \langle \nabla'_x y, b \rangle, & \mu &= \langle \nabla'_x y, l \rangle, \\ \gamma_1 &= \langle \nabla'_x x, y \rangle, & \gamma_2 &= \langle \nabla'_y y, x \rangle, & \beta_1 &= \langle \nabla'_x b, l \rangle, & \beta_2 &= \langle \nabla'_y b, l \rangle. \end{aligned}$$

The invariants k, \varkappa , and the Gauss curvature K of M^2 are expressed by the functions $\nu_1, \nu_2, \lambda, \mu$ as follows:

$$k = -4\nu_1 \nu_2 \mu^2, \quad \varkappa = (\nu_1 - \nu_2)\mu, \quad K = \nu_1 \nu_2 - (\lambda^2 + \mu^2).$$

The normal mean curvature vector field of M^2 is $H = \frac{\nu_1 + \nu_2}{2} b$. The norm $\|H\|$ of the mean curvature vector is expressed as

$$\|H\| = \frac{|\nu_1 + \nu_2|}{2} = \frac{\sqrt{\varkappa^2 - k}}{2|\mu|}.$$

The geometric meaning of the invariant λ is connected with the notion of Chen submanifolds. Let M be an n -dimensional submanifold of $(n + m)$ -dimensional Riemannian manifold \widetilde{M} and ξ be a normal vector field of M . B.-Y. Chen [3] defined the *allied vector field* $a(\xi)$ of ξ by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^m \{\text{tr}(A_1 A_k)\} \xi_k,$$

where $\{\xi_1 = \frac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is an orthonormal base of the normal space of M , and $A_i = A_{\xi_i}$, $i = 1, \dots, m$ is the shape operator with respect to ξ_i . The allied vector field $a(H)$ of the mean curvature vector field H is called the *allied mean curvature vector field* of M in \widetilde{M} . B.-Y. Chen defined the \mathcal{A} -submanifolds to be those submanifolds of \widetilde{M} for which $a(H)$ vanishes identically [3]. Ruled \mathcal{A} -surfaces in Euclidean space \mathbb{R}^n were treated in [16]. \mathcal{A} -surfaces with flat normal connection were studied in [19]. Spherical \mathcal{A} -surfaces were considered in [9, 12]. Several examples of \mathcal{A} -surfaces in Euclidean space can be found in [17].

In [10], [11] the \mathcal{A} -submanifolds are called *Chen submanifolds*. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial. A nice survey of results on Chen submanifolds in Euclidean space \mathbb{R}^n , especially on Chen surfaces in \mathbb{R}^4 , can be seen in [18]. Further developments of the theory of Chen submanifolds and non-trivial examples are given also in [1, 2, 5, 6]. Chen surfaces in \mathbb{R}^4 are characterized in terms of the normal curvature ellipse in [13].

In [7] we showed that the allied mean curvature vector field of M^2 is expressed as follows

$$a(H) = \frac{\sqrt{\kappa^2 - k}}{2} \lambda l.$$

Hence, if M^2 is free of minimal points, then $a(H) = 0$ if and only if $\lambda = 0$. This gives the geometric meaning of the invariant λ : M^2 is a non-trivial Chen surface if and only if the invariant λ is zero.

Now we shall discuss the geometric meaning of the invariants β_1 and β_2 . It follows from (1) that

$$\begin{aligned} \nabla'_x b &= -\nu_1 x - \lambda y + \beta_1 l; & \nabla'_x l &= -\mu y - \beta_1 b; \\ \nabla'_y b &= -\lambda x - \nu_2 y + \beta_2 l; & \nabla'_y l &= -\mu x - \beta_2 b. \end{aligned}$$

Hence, $\beta_1 = \beta_2 = 0$ if and only if $D_x b = D_y b = 0$ (or equivalently, $D_x l = D_y l = 0$).

A normal vector field ξ is said to be *parallel in the normal bundle* (or simply *parallel*) [4], if $D_x \xi = 0$ holds identically for any tangent vector field x . Hence, $\beta_1 = \beta_2 = 0$ if and only if the geometric normal vector fields b and l are parallel in the normal bundle.

Surfaces admitting a geometric normal frame field $\{b, l\}$ of parallel normal vector fields, we shall call *surfaces with parallel normal bundle*. They are characterized by the condition $\beta_1 = \beta_2 = 0$. Note that if M^2 is a surface free of minimal points with parallel mean curvature vector field (i.e., $DH = 0$), then M^2 is a surface with parallel normal bundle, but the converse is not true in general. It is true only in the case $\|H\| = \text{const}$.

3. Meridian surfaces in \mathbb{R}^4 and their invariants

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in \mathbb{R}^4 , and $S^2(1)$ be a 2-dimensional sphere in $\mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$, centered at the origin O . Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) + \dot{g}^2(u) = 1$, $u \in I$. The standard rotational hypersurface M^3 in \mathbb{R}^4 , obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ around the Oe_4 -axis, is parameterized as follows:

$$M^3 : Z(u, w^1, w^2) = f(u)l(w^1, w^2) + g(u)e_4,$$

where $l(w^1, w^2)$ is the unit radius-vector of $S^2(1)$ in \mathbb{R}^3 :

$$l(w^1, w^2) = \cos w^1 \cos w^2 e_1 + \cos w^1 \sin w^2 e_2 + \sin w^1 e_3.$$

The rotational hypersurface M^3 is a two-parameter system of meridians.

In [8] we constructed a family of surfaces lying on the rotational hypersurface M^3 which are one-parameter systems of meridians of the rotational hypersurface. The construction is as follows. We consider a smooth curve $c : l = l(v) = l(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length, i.e., $l'^2(v) = 1$. Denote $t(v) = l'(v)$ and consider the moving frame field $\text{span}\{t(v), n(v), l(v)\}$ of the curve c on $S^2(1)$. With respect to this orthonormal frame field we have the following Frenet formulas:

$$\begin{aligned} l' &= t; \\ t' &= \kappa n - l; \\ n' &= -\kappa t, \end{aligned}$$

where $\kappa = \kappa(v)$ is the spherical curvature of c .

We construct a surface \mathcal{M} lying on M^3 in the following way:

$$(2) \quad \mathcal{M} : z(u, v) = f(u)l(v) + g(u)e_4, \quad u \in I, v \in J.$$

Since \mathcal{M} is a one-parameter system of meridians of M^3 , we call \mathcal{M} a *meridian surface*.

The tangent space of \mathcal{M} is spanned by the vector fields:

$$z_u = \dot{f}l + \dot{g}e_4; \quad z_v = ft,$$

and hence, the coefficients of the first fundamental form of \mathcal{M} are $E = 1$; $F = 0$; $G = f^2(u)$. Denote $X = z_u$, $Y = \frac{z_v}{f} = t$ and consider the orthonormal

normal frame field of \mathcal{M} defined by:

$$n_1 = n(v); \quad n_2 = -\dot{g}(u)l(v) + \dot{f}(u)e_4.$$

Thus we obtain a positive orthonormal frame field $\{X, Y, n_1, n_2\}$ of \mathcal{M} . With respect to this frame field we get the following derivative formulas:

$$(3) \quad \begin{aligned} \nabla'_X X &= \kappa_m n_2; & \nabla'_X n_1 &= 0; \\ \nabla'_X Y &= 0; & \nabla'_Y n_1 &= -\frac{\kappa}{f} Y; \\ \nabla'_Y X &= \frac{\dot{f}}{f} Y; & \nabla'_X n_2 &= -\kappa_m X; \\ \nabla'_Y Y &= -\frac{\dot{f}}{f} X + \frac{\kappa}{f} n_1 + \frac{\dot{g}}{f} n_2; & \nabla'_Y n_2 &= -\frac{\dot{g}}{f} Y, \end{aligned}$$

where κ_m denotes the curvature of the meridian curve m , i.e.,

$$\kappa_m(u) = \dot{f}(u)\ddot{g}(u) - \dot{g}(u)\ddot{f}(u) = \frac{-\ddot{f}(u)}{\sqrt{1 - \dot{f}^2(u)}}.$$

The coefficients of the second fundamental form of \mathcal{M} are $L = N = 0$, $M = -\kappa_m(u)\kappa(v)$. The invariants k , \varkappa , and the Gauss curvature K are given by the following formulas [8]:

$$k = -\frac{\kappa_m^2(u)\kappa^2(v)}{f^2(u)}; \quad \varkappa = 0; \quad K = -\frac{\ddot{f}(u)}{f(u)}.$$

The equality $\varkappa = 0$ implies that \mathcal{M} is a surface with flat normal connection. The mean curvature vector field H is given by

$$H = \frac{\kappa}{2f} n_1 + \frac{\dot{g} + f\kappa_m}{2f} n_2.$$

We distinguish the following three cases:

I. $\kappa(v) = 0$, i.e., the curve c is a great circle on $S^2(1)$. In this case $n_1 = \text{const}$, and \mathcal{M} is a planar surface lying in the constant 3-dimensional space spanned by $\{X, Y, n_2\}$. Particularly, if in addition $\kappa_m(u) = 0$, i.e., the meridian curve m lies on a straight line, then \mathcal{M} is a developable surface in the 3-dimensional space $\text{span}\{X, Y, n_2\}$.

II. $\kappa_m(u) = 0$, i.e., the meridian curve m is part of a straight line. In such case $k = \varkappa = K = 0$, and \mathcal{M} is a developable ruled surface. If in addition $\kappa(v) = \text{const}$, i.e., c is a circle on $S^2(1)$, then \mathcal{M} is a developable ruled surface in a 3-dimensional space. If $\kappa(v) \neq \text{const}$, i.e., c is not a circle on $S^2(1)$, then \mathcal{M} is a developable ruled surface in \mathbb{R}^4 .

III. $\kappa_m(u)\kappa(v) \neq 0$, i.e., c is not a great circle on $S^2(1)$, and m is not a straight line. In this case the invariant function $k < 0$, which implies that

there exist two systems of asymptotic lines on \mathcal{M} . The parametric lines of the surface \mathcal{M} , defined by (2) are asymptotic.

In the first two cases the surface \mathcal{M} consists of flat points. So, we consider meridian surfaces of the third (general) case, i.e., we assume that $\kappa_m \neq 0$ and $\kappa \neq 0$. Note that the orthonormal frame field $\{X, Y, n_1, n_2\}$ of \mathcal{M} is not the geometric frame field defined in Section 2. The principal tangents of \mathcal{M} are

$$x = \frac{X + Y}{\sqrt{2}}; \quad y = \frac{-X + Y}{\sqrt{2}}.$$

The geometric normal frame field $\{b, l\}$ is given by

$$b = \frac{\kappa n_1 + (\dot{g} + f\kappa_m) n_2}{\sqrt{\kappa^2 + (\dot{g} + f\kappa_m)^2}}; \quad l = \frac{-(\dot{g} + f\kappa_m) n_1 + \kappa n_2}{\sqrt{\kappa^2 + (\dot{g} + f\kappa_m)^2}}.$$

Applying formulas (1) for the geometric frame field $\{x, y, b, l\}$ of \mathcal{M} and derivative formulas (3), we obtain the following invariants of \mathcal{M} :

$$(4) \quad \begin{aligned} \gamma_1 = -\gamma_2 &= \frac{\dot{f}}{\sqrt{2}f}; \\ \nu_1 = \nu_2 &= \frac{\sqrt{\kappa^2 + (\dot{g} + f\kappa_m)^2}}{2f}; \\ \lambda &= \frac{\kappa^2 + \dot{g}^2 - f^2\kappa_m^2}{2f\sqrt{\kappa^2 + (\dot{g} + f\kappa_m)^2}}; \\ \mu &= \frac{-\kappa\kappa_m}{\sqrt{\kappa^2 + (\dot{g} + f\kappa_m)^2}}; \\ \beta_1 &= \frac{1}{\sqrt{2}(\kappa^2 + (\dot{g} + f\kappa_m)^2)} \left(\kappa \frac{d}{du} (\dot{g} + f\kappa_m) - \frac{d}{dv} (\kappa) \frac{\dot{g} + f\kappa_m}{f} \right); \\ \beta_2 &= -\frac{1}{\sqrt{2}(\kappa^2 + (\dot{g} + f\kappa_m)^2)} \left(\kappa \frac{d}{du} (\dot{g} + f\kappa_m) + \frac{d}{dv} (\kappa) \frac{\dot{g} + f\kappa_m}{f} \right). \end{aligned}$$

In [8] we described and classified the meridian surfaces with constant Gauss curvature K , constant mean curvature $\|H\|$, and constant invariant k . Meridian surfaces of Weingarten type were studied in [15]. A special class of meridian surfaces in Heisenberg spacetime is considered in [14].

In the following sections we shall classify completely the Chen meridian surfaces and the meridian surfaces with parallel normal bundle.

4. Chen meridian surfaces

Let \mathcal{M} be a meridian surface of the general class, i.e., $\kappa_m \neq 0$ and $\kappa \neq 0$. The invariants of \mathcal{M} are given by (4). Recall that \mathcal{M} is a non-trivial Chen surface if and only if $\lambda = 0$. The Chen meridian surfaces of the general class are described in the following theorem.

Theorem 4.1. *Let \mathcal{M} be a meridian surface in \mathbb{R}^4 of the general class. Then \mathcal{M} is a Chen surface if and only if the curve c on $S^2(1)$ is a circle with constant spherical curvature $\kappa = \text{const} = b$, $b > 0$, and the meridian m is determined by $\dot{f} = y(f)$ where*

$$y(t) = \frac{\pm 1}{2t^{\pm 1}} \sqrt{4t^{\pm 2} - a \left(t^{\pm 2} - \frac{b^2}{a} \right)^2}, \quad a = \text{const} \neq 0,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

Proof. It follows from (4) that $\lambda = 0$ if and only if

$$\kappa^2(v) = f^2(u) \kappa_m^2(u) - \dot{g}^2(u),$$

which implies

$$(5) \quad \begin{aligned} \kappa &= \text{const} = b, \quad b > 0; \\ f^2(u) \kappa_m^2(u) - \dot{g}^2(u) &= b^2. \end{aligned}$$

The first equality of (5) implies that the spherical curve c has constant spherical curvature $\kappa = b$, i.e., c is a circle on $S^2(1)$.

Using that $\dot{f}^2 + \dot{g}^2 = 1$, and $\kappa_m = \dot{f}\ddot{g} - \dot{g}\ddot{f}$, from the second equality of (5) we obtain that the function $f(u)$ is a solution of the following differential equation:

$$(6) \quad f^2(\ddot{f})^2 = b^2(1 - \dot{f}^2) + (1 - \dot{f}^2)^2.$$

The function $g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

The solutions of differential equation (6) can be found as follows. Setting $\dot{f} = y(f)$ in equation (6), we obtain that the function $y = y(t)$ is a solution of the equation:

$$(7) \quad \frac{t^2}{4} ((y^2)')^2 = b^2(1 - y^2) + (1 - y^2)^2.$$

We set $z(t) = 1 - y^2(t)$ and obtain

$$\frac{t}{2} z' = \pm \sqrt{b^2 z + z^2}.$$

The last equation is equivalent to

$$(8) \quad \frac{z'}{\sqrt{b^2 z + z^2}} = \pm \frac{2}{t}.$$

Integrating both sides of (8), we get

$$(9) \quad z + \frac{b^2}{z} + \sqrt{b^2 z + z^2} = c t^{\pm 2}, \quad c = \text{const}.$$

It follows from (9) that

$$z(t) = \frac{(a t^{\pm 2} - b^2)^2}{4a t^{\pm 2}}, \quad a = 2c.$$

Hence, the general solution of differential equation (7) is given by

$$(10) \quad y(t) = \frac{\pm 1}{2t^{\pm 1}} \sqrt{4t^{\pm 2} - a \left(t^{\pm 2} - \frac{b^2}{a} \right)^2}, \quad a = \text{const} \neq 0.$$

The function $f(u)$ is determined by $\dot{f} = y(f)$, where y satisfies (10). □

5. Meridian surfaces with parallel normal bundle

In the present section we shall describe the meridian surfaces with parallel normal bundle. Recall that a surface in \mathbb{R}^4 has parallel normal bundle if and only if $\beta_1 = \beta_2 = 0$.

Theorem 5.1. *Let \mathcal{M} be a meridian surface in \mathbb{R}^4 of the general class. Then \mathcal{M} has parallel normal bundle if and only if one of the following cases holds:*

(i) *the meridian m is defined by*

$$\begin{aligned} f(u) &= \pm \sqrt{u^2 + 2cu + d}; \\ g(u) &= \pm \sqrt{d - c^2} \ln |u + c + \sqrt{u^2 + 2cu + d}| + a, \end{aligned}$$

where $a, c,$ and d are constants, $d > c^2$;

(ii) *the curve c is a circle on $S^2(1)$ with constant spherical curvature $\kappa = \text{const} = b, b > 0,$ and the meridian m is determined by $\dot{f} = y(f)$ where*

$$y(t) = \pm \frac{\sqrt{(1 - a^2)t^2 - 2act - c^2}}{t}, \quad a = \text{const} \neq 0, \quad c = \text{const},$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

Proof. Using formulas (4) we get that $\beta_1 = \beta_2 = 0$ if and only if

$$(11) \quad \begin{aligned} \kappa f \frac{d}{du} (\dot{g} + f\kappa_m) - \frac{d}{dv} (\kappa) (\dot{g} + f\kappa_m) &= 0; \\ \kappa f \frac{d}{du} (\dot{g} + f\kappa_m) + \frac{d}{dv} (\kappa) (\dot{g} + f\kappa_m) &= 0. \end{aligned}$$

It follows from (11) that there are two possible cases:

Case 1. $\dot{g} + f\kappa_m = 0$. Using that $\dot{g} + f\kappa_m = \frac{1 - \dot{f}^2 - f\ddot{f}}{\sqrt{1 - \dot{f}^2}}$, we get the differential equation

$$1 - \dot{f}^2 - f\ddot{f} = 0,$$

whose general solution is $f(u) = \pm \sqrt{u^2 + 2cu + d}, c = \text{const}, d = \text{const}$. Since $\dot{g}^2 = 1 - \dot{f}^2$, we get $\dot{g}^2 = \frac{d - c^2}{u^2 + 2cu + d}$, and hence $d > c^2$. Integrating both sides of the equation

$$\dot{g} = \pm \frac{\sqrt{d - c^2}}{\sqrt{u^2 + 2cu + d}},$$

we obtain $g(u) = \pm\sqrt{d-c^2} \ln|u+c+\sqrt{u^2+2cu+d}| + a$, $a = \text{const}$. Hence, in this case the meridian m is defined as described in (i).

Case 2. $\dot{g} + f\kappa_m = a = \text{const}$, $a \neq 0$ and $\kappa = b = \text{const}$, $b \neq 0$. In this case we obtain that the meridian m is determined by the following differential equation:

$$(12) \quad 1 - \dot{f}^2 - f\ddot{f} = a\sqrt{1 - \dot{f}^2}, \quad a = \text{const} \neq 0.$$

The solutions of differential equation (12) can be found in the following way. Setting $\dot{f} = y(f)$ in equation (12), we obtain that the function $y = y(t)$ is a solution of the equation:

$$(13) \quad 1 - y^2 - \frac{t}{2}(y^2)' = a\sqrt{1 - y^2}.$$

If we set $z(t) = \sqrt{1 - y^2(t)}$ we get

$$z' + \frac{1}{t}z = \frac{a}{t}.$$

The general solution of the above equation is given by the formula $z(t) = \frac{c+at}{t}$, $c = \text{const}$. Hence, the general solution of (13) is

$$(14) \quad y(t) = \pm \frac{\sqrt{(1-a^2)t^2 - 2act - c^2}}{t}, \quad c = \text{const}.$$

The function $f(u)$ is determined by $\dot{f} = y(f)$, where y is defined by (14). \square

References

- [1] S. Carter and U. Dursun, *On generalized Chen and k-minimal immersions*, Beiträge Algebra Geom. **38** (1997), no. 1, 125–134.
- [2] ———, *Partial tubes and Chen submanifolds*, J. Geom. **63** (1998), no. 1-2, 30–38.
- [3] B.-Y. Chen, *Geometry of Submanifolds*. Marcel Dekker, Inc., New York, 1973.
- [4] ———, *Pseudo-Riemannian Geometry, δ -Invariants and Applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [5] U. Dursun, *On product k-Chen submanifolds*, Glasgow Math. J. **39** (1997), no. 3, 243–249.
- [6] ———, *On minimal and Chen immersions in space forms*, J. Geom. **66** (1999), no. 1-2, 104–115.
- [7] G. Ganchev and V. Milousheva, *On the theory of surfaces in the four-dimensional Euclidean space*, Kodai Math. J. **31** (2008), no. 2, 183–198.
- [8] ———, *Invariants and Bonnet-type theorem for surfaces in \mathbb{R}^4* , Cent. Eur. J. Math. **8** (2010), no. 6, 993–1008.
- [9] O. Garay, *Spherical Chen surfaces which are mass-symmetric and of 2-type*, J. Geom. **33** (1988), no. 1-2, 39–52.
- [10] L. Gheysens, P. Verheyen, and L. Verstraelen, *Sur les surfaces \mathcal{A} ou les surfaces de Chen*, C. R. Acad. Sci. Paris, Sér. I Math. **292** (1981), no. 19, 913–916.
- [11] ———, *Characterization and examples of Chen submanifolds*, J. Geom. **20** (1983), no. 1, 47–62.
- [12] C. Houh, *On spherical \mathcal{A} -submanifolds*, Chinese J. Math. **2** (1974), no. 1, 128–134.
- [13] E. İyigün, K. Arslan, and G. Öztürk, *A characterization of Chen surfaces in \mathbb{E}^4* , Bull. Malays. Math. Sci. Soc. **31** (2008), no. 2 209–215.

- [14] T. Körpınar and E. Turhan, *A new class of time-meridian surfaces of biharmonic S -particles and its Lorentz transformation in Heisenberg spacetime*, Int. J. Theor. Phys. **2015** (2015); DOI 10.1007/s10773-015-2621-3.
- [15] G. Öztürk, B. Bulca, B. Bayram, and K. Arslan, *Meridian surfaces of Weingarten type in 4-dimensional Euclidean space \mathbb{E}^4* , preprint available at ArXiv:1305.3155.
- [16] B. Rouxel, *Ruled \mathcal{A} -surfaces in Euclidean space \mathbb{E}^n* , Soochow J. Math. **6** (1980), 117–121.
- [17] ———, *\mathcal{A} -submanifolds in Euclidean space*, Kodai Math. J. **4** (1981), no. 1, 181–188.
- [18] ———, *Chen submanifolds*, Geometry and Topology of Submanifolds, VI, World Sci. Publ., River Edge, NJ, 1994.
- [19] L. Verstraelen, *\mathcal{A} -surfaces with flat normal connection*, J. Korean Math. Soc. **15** (1978), no. 1, 1–7.

GEORGI GANCHEV
INSTITUTE OF MATHEMATICS AND INFORMATICS
BULGARIAN ACADEMY OF SCIENCES
ACAD. G. BONCHEV STR. BL. 8, 1113 SOFIA, BULGARIA
E-mail address: `ganchev@math.bas.bg`

VELICHKA MILOUSHEVA
INSTITUTE OF MATHEMATICS AND INFORMATICS
BULGARIAN ACADEMY OF SCIENCES
ACAD. G. BONCHEV STR. BL. 8, 1113, SOFIA, BULGARIA
AND
“L. KARAVELOV” CIVIL ENGINEERING HIGHER SCHOOL
175 SUHODOLSKA STR., 1373 SOFIA, BULGARIA
E-mail address: `vmil@math.bas.bg`