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A CHARACTERIZATION OF SOME $PGL(2, q)$ BY MAXIMUM ELEMENT ORDERS

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Abstract. In this paper, we characterize some $PGL(2, q)$ by their orders and maximum element orders. We also prove that $PSL(2, p)$ with $p \geqslant 3$ a prime can be determined by their orders and maximum element orders. Moreover, we show that, in general, if $q = p^n$ with p a prime and $n > 1$, $PGL(2, q)$ can not be uniquely determined by their orders and maximum element orders. Several known results are generalized.

All groups considered in this paper are finite.

For a group G, as in [27], we construct its prime graph $\Gamma(G)$ as follows: the vertices are the primes dividing the order of G and two vertices p and r are joined by an edge if and only if G contains an element of order pr. Denote by $T(G) = \{\pi_i(G) \,|\, 1 \leqslant i \leqslant t(G)\}$ the set of all connected components of the graph $\Gamma(G)$, where $t(G)$ is the number of the connected components of $\Gamma(G)$. If the order of G is even, we always assume that $2 \in \pi_1(G)$. Let

$$\pi_e(G) = \{|g| : g \in G\} \quad \text{and} \quad \text{meo}(G) = \max\{|g| : g \in\};$$

that is, $\pi_e(G)$ is the set of orders of elements of G and $\text{meo}(G)$ is the maximum element order of G (see [11]). A group G is called a 2-*Frobenius group* if G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K and G/H are Frobenius groups with kernels H and K/H respectively. A group G is almost simple if there exists a non-abelian simple group S such that $S \leqslant G \leqslant \text{Aut}(S)$. A group G is called a K_n-group if $\pi(G)$ consists of exactly n distinct primes.

Recall that a Singer cycle of $GL(n, q)$ is an element of order $q^n - 1$. Let y be a Singer cycle of $GL(n, q)$ and $Y = \langle y \rangle$. Then the normalizer of Y in $GL(n, q)$ is a split extension of Y by a cyclic group of order n (see, for example, [15, 16]). This fact plays an important role in our discussions.

The other notation and terminologies in this paper are standard and the reader is referred to ATLAS [8] and [12] if necessary.

The connection between arithmetical properties and structural properties of groups have been researched widely. For example, it is proved recently by Shi and Mazurov et al that every finite simple group S can be determined

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by $|S|$ and $\pi_e(S)$ (see [2, 10, 18, 19, 21, 22, 23, 24, 25, 26, 28]). This means that every simple group can be characterized by pure quantitative properties. It is also proved by Chen, Mazurov and Shi et al. in [7] that if q is not a prime, and also $q \neq 9$, and G is a group with $\pi_e(G) = \pi_e(PGL(2,q))$, then $G \simeq PGL(2,q)$. In the exceptional cases, there are infinitely many pairwise non-isomorphic groups G such that $\pi_e(G) = \pi_e(PGL(2,q))$. In this study, all elements of $\pi_e(G)$ of a group G are involved. Hence, it is natural to seek weaker quantitative conditions under which G can be reconstructed. Especially, we want to know: *Can a simple* (*or almost simple or quasisimple*) *group S be determined by a subset of $\pi_e(S)$, or equivalently by fewer element orders of S?* And *what information can we obtain about S from fewer element orders of S?*

In recent years, many authors are interested in the study of the special element orders of a group as well as their connection with the structure of non-abelian simple groups. In [17], Kantor and Seress investigated the influence of three largest element orders on the characteristic of Lie type simple groups of odd characteristic. Let

$$m_1 = m_1(G),\ m_2 = m_2(G),\ m_3 = m_3(G),\ m_1 > m_2 > m_3$$

denote the three largest element orders of a group G. Note that $m_1 = m_1(G) = \mathrm{meo}(G)$. A remarkable result of Kantor and Seress shows that if G and H are simple groups of Lie type of odd characteristic and $m_i(G) = m_i(H)$ for $1 \leqslant i \leqslant 3$, then $\mathrm{ch}(G) = \mathrm{ch}(H)$. This means that the three largest element orders determine the characteristic of simple groups of Lie type of odd characteristic. In addition, the two largest element orders of simple groups of Lie type of odd characteristic are listed in [17]. In [11], Guest, Morris, Praeger and Spiga determine the upper bounds of $\mathrm{meo}(S)$ for S a finite almost simple group. Moreover, these results are applied to determine the primitive permutation groups on a set of size n that contain permutations of order greater than or equal to $n/4$. On the other hand, many authors characterized non-abelian simple groups using their orders and some special element orders. In [13, 14, 29], it is proved that the simple K_3-groups and some $PSL(2,p)$ are determined by their orders and maximal element orders.

By the above research, we see that the three largest element orders of non-abelian simple groups have significant impact on their structure and provide much structural information of simple groups. Hence, one may wonder *whether the three largest element orders of a simple* (*or almost simple or quasisimple*) *group S as well as the order of S are enough to determine S*. The purpose of the present paper is to give such a characterization for some $PGL(2,q)$ as well as some $PSL(2,q)$, where q is a power of a prime. We focus our attention on $PSL(2,p)$ and its automorphism group $PGL(2,p)$, where p is a prime. Our study is motivated by [11] and [17], which contain the maximum element orders of simple groups of Lie type. It is well known that

$$\mathrm{meo}(PGL(2,q)) = q+1,$$

where $q = p^n$ is a prime power, and

$$\text{meo}(PSL(2, p)) = p$$

with $p > 2$ a prime (see [11] and [17] respectively or see Dickson's book [9]).

Our first result shows that the maximum element orders of $PGL(2, p)$ together with their orders in turn determine $PGL(2, p)$. Along the way, we also obtain the same characterization for $PGL(2, 2^n)$ with $2^n \pm 1$ a prime and $PSL(2, p)$. However, we observe that if $q = p^n$ where p is a prime and $n > 1$, then $PGL(2, q)$ can not be uniquely determined by these two quantitative properties.

Theorem 1. *Let G be a group. Then $G \simeq PGL(2, p)$ if and only if $|G| = |PGL(2, p)|$ and $\text{meo}(G) = \text{meo}(PGL(2, p)) = p + 1$, where p is a prime.*

In order to prove Theorem 1, we need the following lemmas.

Lemma 2. *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$.*

Proof. See Lemma 1.6 in [5]. □

Lemma 3. *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, the order of G/K divides the order of the automorphism group of K/H, and both G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$ and G is soluble.*

Proof. This is Lemma 1.7 in [5]. □

Lemma 4. *Let G be a group with more than one prime graph component. Then G is one of the following:*

(i) *a Frobenius or 2-Frobenius group;*

(ii) *G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Besides, $\pi_i(G) \in T(K/H)$ for $i \geqslant 2$.*

Proof. It follows straight forward from Lemmas 1-3 in [27], Lemma 1.5 in [3] and Lemma 7 in [6]. □

The next lemma due to Brauer and Reynolds is crucial in our proofs so that we can avoid the application of the classification of non-abelian simple groups whose prime graphs are not connected.

Lemma 5. *Let G be a non-abelian simple group whose order g is divisible by a prime $p > g^{1/3}$. Then G is isomorphic either to $PSL(2, p)$ where $p > 3$ is a prime, or to $PSL(2, p - 1)$ where $p > 3$ is a Fermat prime, $p = 2^n + 1$ with integral n.*

Proof. See [1] or [20]. □

Now, we are ready to prove Theorem 1. We divide the proof into several lemmas.

Lemma 6. *Let G be a group such that $|G| = |PGL(2,2)|$ and* $\mathrm{meo}(G) = \mathrm{meo}(PGL(2,2))$*. Then $G \simeq PGL(2,2)$.*

Proof. By the hypothesis, $|G| = 6$ and $\mathrm{meo}(G) = 3$. It follows that $G \simeq PGL(2,2)$. □

Lemma 7. *Let G be a group such that $|G| = |PGL(2,3)|$ and* $\mathrm{meo}(G) = \mathrm{meo}(PGL(2,3))$*. Then $G \simeq PGL(2,3)$.*

Proof. By the hypothesis, $|G| = 24$ and $\mathrm{meo}(G) = 4$. It is obvious that $t(G) > 1$ and $\{3\} \in T(G)$. By Lemma 2, we know that G is not a Frobenius group. Then G is a 2-Frobenius group. By Lemma 3, G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K/H is cyclic of order 3 and G/K is of order 2. Then $|H| = 4$. Clearly, H is non-cyclic and therefore H is elementary abelian. Let P be a Sylow 3-group of G. Then P is not normal in G; if not, G has an element of order 6. Therefore, G has 4 Sylow 3-subgroups and $H = N_G(P)$ is of order 6. Moreover H is non-cyclic. Consider the action of G on the right cosets of H. Since this action is faithful, we see that G is isomorphic to S_4, which implies that $G \simeq PGL(2,3)$. □

Lemma 8. *Let G be a group and $p > 3$ a prime. If $|G| = |PGL(2,p)|$ and* $\mathrm{meo}(G) = \mathrm{meo}(PGL(2,p))$*, then $G \simeq PGL(2,p)$.*

Proof. By the hypothesis, $|G| = (p-1)p(p+1)$ and $\mathrm{meo}(G) = p+1$. Therefore, $t(G) > 1$ and $\{p\}$ is a component of $\Gamma(G)$.

Suppose that G is a Frobenius group with Frobenius kernel H and Frobenius complement K. Then $|K| = p$ by Lemma 2. Let T be a Sylow t-subgroup of H with t odd and $t \in \pi(p+1)$ or $\pi(p-1)$. If $t \in \pi(p+1)$, then T is cyclic because $\mathrm{meo}(G) = p+1$. Since p does not divide $|\mathrm{Aut}(T)|$, G has an element of order pt, a contradiction. Assume that $t \in \pi(p-1)$. Since TK is also a Frobenius group, we have that p divides $|T| - 1$, which is impossible. Hence, G is not a Frobenius group.

If G is a 2-Frobenius group, then by Lemma 3, G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K/H is cyclic of order p. If $\pi(H)$ contains an odd prime t, then, as above, we derive a contradiction. Suppose that $\pi(H) = \{2\}$. Then $p = 2^r - 1$ is a Mersenne prime with r a prime. Since $2^r - 1$ does not divide $2^{r+1} - 1$, we obtain that $|H| = p + 1 = 2^r$ and therefore G/K is cyclic of order $p - 1$. Let $\Omega = \Omega_1(Z(H))$ and $|\Omega| = 2^s$. If $s < r$, then 2 and p are joined as p does not divide $|GL(s,2)|$, a contradiction. Hence $s = r$ and so H is an elementary abelian group. Thus, $H \leq C_G(H)$. If $H < C_G(H)$, then $C_G(H)/H$ is a normal subgroup of G/H, from which we conclude that G has an element of order $2p$, a contradiction. Consequently, $C_G(H) = H$ and so $\overline{G} = G/H$ can be viewed as a subgroup of $GL(r, 2)$. Let $y \in \overline{G}$ such that $|y| = p$. Then y is a Singer cycle of $GL(r, 2)$ and $\overline{K} = K/H = \langle y \rangle$. Since $N_{GL(r,2)}(\langle y \rangle)$ is a split extension of $\langle y \rangle$ by a cyclic group of order r, we have that $2^r - 2$ divides r. It follows that $r = 2$, which is impossible because $p \geqslant 5$ by the hypothesis. Thus, G is not a 2-Frobenius groups either.

Now, by Lemma 4, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Besides, $\{p\}$ is a prime graph component of K/H. By the hypothesis, $|K/H| \leqslant |G| = p^3 - p < p^3$. In view of the result of Brauer and Reynolds (Lemma 5), K/H is isomorphic to $PSL(2, p')$ where $p' > 3$ is a prime, or to $PSL(2, p' - 1)$ where $p' > 3$ is a Fermat prime, $p' = 2^n + 1$ with integral n.

Suppose first that K/H is isomorphic to $PSL(2, p' - 1)$ where $p' > 3$ is a Fermat prime, $p' = 2^n + 1$ with integral n. Since p is the largest prime in $\pi(K/H)$, $p = p' = 2^n + 1$. By comparing the orders of G and K/H, we obtain that $2^n + 2$ is divisible by $2^n - 1$, from which we conclude that $n = 2$. Then $p = 5$ and so $K/H \simeq PSL(2, 4) \simeq PSL(2, 5)$. Since $\mathrm{meo}(G) = 6$, $H = 1$ and so G is isomorphic to $PGL(2, 5)$.

If $K/H \simeq PSL(2, p')$, then $p = p'$ as above. Also, H is trivial as $\mathrm{meo}(G)$ is $p + 1$. It follows that $K \simeq PSL(2, p)$ and consequently $G \simeq PGL(2, p)$, which concludes the proof. □

Proof of Theorem 1. The necessity is clear and the sufficiency follows from Lemmas 6-8. □

The proof of our next result is similar to Theorem 1.

Theorem 9. *Let G be a group and $2^n + 1$ be a Fermat prime. Suppose that $|G| = |PGL(2, 2^n)|$ and $\mathrm{meo}(G) = \mathrm{meo}(PGL(2, 2^n))$. Then $G \simeq PGL(2, 2^n)$.*

Theorem 10. *Let G be a group such that $|G| = |PGL(2, 2^n)|$ and $\mathrm{meo}(G) = \mathrm{meo}(PGL(2, 2^n))$, where $2^n - 1$ is a Mersenne prime. Then $G \simeq PGL(2, 2^n)$.*

Proof. Write $p = 2^n - 1$. Then, by the hypothesis, $|G| = p(p + 1)(p + 2)$ and $\mathrm{meo}(G) = p + 2$. It follows that the prime graph $\Gamma(G)$ of G is not connected and $\{p\}$ is a component of $\Gamma(G)$. Similar to Lemma 8, one can show that G is neither a Frobenius group nor a 2-Frobenius groups. Hence, by Lemma 4, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group such that

(1) $\{p\}$ is an isolated point of $\Gamma(K/H)$, where p is a Mersenne prime and maximal in $\pi(K/H)$, and

(2) $|K/H| \leqslant |G| = p(p + 1)(p + 2)$.

By Tables 1-4 in [4], we know that K/H is isomorphic to one of the following:

(i) $PSL(3, 2)$;

(ii) $PSL(2, q)$ with $4 \mid q + 1$;

(iii) $PSL(2, q)$ with $4 \mid q - 1$;

(iv) $PSL(2,q)$ with $2 \mid q$.

If $K/H \simeq PSL(3,2)$, then $|H| = 3$ and so $21 \in \pi_e(G)$, a contradiction since $\mathrm{meo}(G) = 9$.

If $K/H \simeq PSL(2,q)$ with $4 \mid q+1$, then $p = q$ or $p = (q-1)/2$ (see [4, Table 2]). If $p = q$, then we have that $(p-1)/2$ divides $p+2$, that is

$$(2^{n-1} - 1) \mid (2^n + 1).$$

This holds only when $n = 2$ or 3 and therefore $p = q = 3$ or 7. Since K/H is nonabelian simple, $q = 7$ and so $K/H \simeq PSL(2,7)$, which implies that $21 \in \pi_e(G)$ as $PSL(2,7) \simeq PSL(3,2)$, a contradiction. If $p = (q-1)/2$, then $|PSL(2,q)| = p(2p+1)(2p+2) \geqslant |G| = p(p+1)(p+2)$, which is impossible.

Analogous to the foregoing argument, we have that K/H can not be isomorphic to $PSL(2,q)$ with $4 \mid q-1$.

Hence, K/H must be isomorphic to $PSL(2,q)$ with $q = 2^m$. In this case, by [4, Table 2], $p = q+1$ or $q-1$. If $p = q+1$, we also have that $(p-1)/2$ divides $p+2$, a contradiction as above. Thus,

$$p = q - 1 = 2^m - 1 = 2^n - 1$$

and so $m = n$. It follows that $K/H \simeq PSL(2,2^n)$ with $2^n - 1$ a Mersenne prime and so G is isomorphic to $PSL(2,2^n) = PGL(2,2^n)$, as wanted. □

It should be noted that, in general, $PGL(2,q)$ with q composite can not be uniquely determined by their orders and maximum element orders, which is indicated by the following proposition.

Proposition 11. *Let G be a group and $q =$ 9, 25, 27, 49, 64, 81, 125. Suppose that $|G| = |PGL(2,q)|$ and $\mathrm{meo}(G) = q+1$. Then G has a chief factor isomorphic to $PSL(2,q)$.*

Proof. We proceed the proof case by case.

(1) $q = 9$.

In this case, $|G| = 720$ and $\mathrm{meo}(G) = 10$. We claim that G is insoluble. If not, the Hall $\{3,5\}$-subgroup of G contains an element of order 15, a contradiction. Let K denote the largest soluble normal subgroup of G. Then $\overline{G} = G/K$ has a unique minimal normal subgroup $\overline{L} = L/K$, whence $G/K \leq \mathrm{Aut}(L/K)$. By [8], $\overline{L} \simeq A_5$ or $PSL(2,9)$. If $\overline{L} \simeq A_5$, then $15 \in \pi_e(G)$, a contradiction. Thus, $\overline{L} \simeq PSL(2,9)$.

(2) $q = 25$.

By the hypothesis, $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 13$ and $\mathrm{meo}(G) = 26$. We show that G has a chief factor M/N such that $\{5,13\} \subseteq \pi(M/N)$. We first choose a chief factor M/N of G with $\pi(N) \cap \{5,13\} = \emptyset$ and $\pi(M) \cap \{5,13\} \neq \emptyset$. If $13 \in \pi(M)$ and $5 \notin \pi(M)$, then $G = MN_G(G_{13})$, where G_{13} is a Sylow 13-subgroup of G contained in M. This implies that $65 \in \pi_e(G)$, a contradiction. If $5 \in \pi(M)$ but $13 \notin \pi(M)$, then $G = MN_G(M_5)$, where M_5 is a Sylow 5-subgroup of M with order 5 or 25. Hence, $13 \in \pi(N_G(M_5))$. Since $(13, |\mathrm{Aut}(M_5)|) = 1$, G has an element of order 65 as above, a contradiction. The foregoing argument shows that both 5 and 13 must be contained in $\pi(M)$ and therefore M/N is a non-abelian simple group. By [8], M/N is isomorphic to $PSL(2,25)$, as wanted.

(3) $q = 49$.

If $q = 49$, then $|G| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^2$ and $\mathrm{meo}(G) = 50$. Thus, the Sylow 5-subgroups of G are cyclic. We assert that G is insoluble. Otherwise, G contains a Hall $\{5,7\}$-subgroup PQ with $|P| = 5^2$ and $|Q| = 7^2$. Since P is cyclic, P normalizes Q. Since $(5, |\mathrm{Aut}(Q)|) = 1$, we see that G has an element of order 175, contrary to the fact $\mathrm{meo}(G) = 50$. Hence G is insoluble, as desired. Let K be the largest soluble normal subgroup of G and $\overline{G} = G/K$. Then $\overline{G}$ has a unique minimal normal subgroup $\overline{L} = L/K$. By the order of G and the cyclicity of P, we have that $\overline{L}$ is a non-abelian simple group. By [8], $\overline{L}$ is isomorphic to A_5, $PSL(2,7)$ or $PSL(2,49)$. If $\overline{L} \simeq A_5$ or $PSL(2,7)$, then $175 \in \pi_e(G)$, a contradiction. Thus, $\overline{L}$ is isomorphic to $PSL(2,49)$, as desired.

(4) $q = 64$.

In this case, $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and $\mathrm{meo}(G) = 65$. As in (2), G has a non-abelian simple chief factor M/N with $\{7,13\} \subseteq \pi(M/N)$. Set $\overline{G} = G/N$. By [8], $\overline{M} = M/N$ is isomorphic to $PSL(2,13)$, $Sz(8)$ or $PSL(2,64)$. Suppose that $\overline{M} \simeq PSL(2,13)$. Then, by the order of G, there exist at most two minimal normal subgroup in $\overline{G}$. If $\overline{M}$ is the only minimal subgroup of $\overline{G}$, then G has an element whose order is greater than 65, a contradiction. Assume that $\overline{G}$ has another minimal normal subgroup, say $\overline{H} = H/N$. Then, by [8], $\overline{H} \simeq A_5$. If $|N| < 4$, then $\mathrm{meo}(G) > 65$, a contradiction. If $|N| = 4$, then $N \leq Z(G)$, which implies that $130 \in \pi_e(G)$, a contradiction again. Hence, it is impossible that $\overline{M} \simeq PSL(2,13)$. Suppose that $\overline{M} \simeq Sz(8)$. Then $|N| = 3^2$ and it follows that $195 \in \pi_e(G)$, a contradiction. Thus, M/N must be isomorphic to $PSL(2,64)$ and so $G \simeq PSL(2,64) = PGL(2,64)$.

The discussions of the remaining cases for $q = 27, 81, 125$ are similar to that of Step (2) and so is omitted. □

Remark 12. Let G be a group such that $|G| = 720$ and $\mathrm{meo}(G) = 10$. Then, by Proposition 11, G is isomorphic to one of the following:

$$PGL(2,9),\ 2.PSL(2,9),\ PSL(2,9) \times Z_2.$$

By Theorems 9 and 10 together with Proposition 11, the following question seems reasonable.

Question 13. Let G be a non-soluble group and $q = p^n$ with p a prime and $n > 1$ integral. Assume that G is of order $(q-1)q(q+1)$ and $\mathrm{meo}(G) = q+1$. Is it true that G has a non-abelian simple chief factor isomorphic to $PSL(2,q)$?

Recall that if $n = 1$, the answer is affirmative in view of Theorem 1.

Arguing as in the proof of Theorem 1, we conclude the following result.

Theorem 14. *Let G be a group and $p > 3$ a prime. Suppose that $|G| = |PSL(2,p)| = (p-1)p(p+1)/2$ and $\mathrm{meo}(G) = \mathrm{meo}(PSL(2,p)) = p$. Then G*

is isomorphic to $PSL(2,p)$ *or to*

$$[[Z_2 \times Z_2 \times Z_2]Z_7]Z_3,$$

a 2*-Frobenius group of order* 168*.*

Proof. The hypothesis implies that $t(G) > 1$ and $\{p\} \in T(G)$. As in Theorem 1, one can show that G is not a Frobenius group. If G is a 2-Frobenius group, then, similarly, we have that G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K/H is cyclic of order p, $p = 2^r - 1$ a Mersenne prime with r a prime. Furthermore, H is an elementary abelian group of order 2^r and G/K is cyclic of order $2^{r-1} - 1$ which is an odd number. Analogous to Theorem 1, we deduce that $2^{r-1} - 1$ must divide r. Hence, by the hypothesis, r must be equal to 3. Thus, G is isomorphic to the 2-Frobenius group

$$[[Z_2 \times Z_2 \times Z_2]Z_7]Z_3.$$

Assume that G is neither a Frobenius group nor a 2-Frobenius group. Then, by Lemma 4, G has a chief factor K/H which is a non-abelian simple group. It follows from Lemma 5 that K/H is isomorphic to $PSL(2,p')$ where $p' > 3$ is a prime, or to $PSL(2,p'-1)$ where $p' > 3$ is a Fermat prime. As in Theorem 1, we have that K/H is isomorphic to $PSL(2,p)$ and therefore G is isomorphic to $PSL(2,p)$. Thus, the proof is complete. □

Remark 15. Theorem 14 is also valid for $p = 3$; that is, for a group G, if $|G| = |PSL(2,3)|$ and $\mathrm{meo}(G) = \mathrm{meo}(PSL(2,3)) = 3$, then G is isomorphic to $PSL(2,3)$.

The corollaries below follow directly from Theorems 1 and 14.

Corollary 16. *Let* G *be a group and* $p > 3$ *a prime. Let* $S = PGL(2,p)$ *or* $PSL(2,p)$*. Then* $G \simeq S$ *if and only if* $|G| = |S|$ *and* $\pi_e(G) = \pi_e(S)$*.*

Corollary 17 (Zhang, Shi, [29])**.** *Let* G *be a group and* $p = 8n+3 > 3$ *a prime with* n *a natural number. If* $|G| = |PSL(2,p)|$ *and* $\mathrm{meo}(G) = \mathrm{meo}(PSL(2,p))$*, then* $G \simeq PSL(2,p)$*.*

Corollary 18 (He, Chen, [14])**.** *Let* G *be a group and* S *a simple* K_4*-group of type* $PSL(2,p)$*, where* p *is a prime but not* $2^n - 1$*. Then* $G \simeq S$ *if and only if* $|G| = |PSL(2,p)|$ *and* $\mathrm{meo}(G) = \mathrm{meo}(PSL(2,p))$*.*

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