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ONE-HOMOGENEOUS WEIGHT CODES OVER FINITE CHAIN RINGS

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Abstract. This paper determines the structures of one-homogeneous weight codes over finite chain rings and studies the algebraic properties of these codes. We present explicit constructions of one-homogeneous weight codes over finite chain rings. By taking advantage of the distance-preserving Gray map defined in [7] from the finite chain ring to its residue field, we obtain a family of optimal one-Hamming weight codes over the residue field. Further, we propose a generalized method that also includes the examples of optimal codes obtained by Shi *et al.* in [17].

1. Introduction

Constant-weight codes represent an important class of codes within the family of error-correcting codes [11]. A linear code having constant-weight means that every nonzero codeword has the same weight. In the literature there are many papers on binary constant-weight codes which have several applications such as the design of demultiplexers for nano-scale memories [8] and the construction of frequency hopping lists for use in GSM networks [12]. Especially, considerable research has been done on the central problem regarding constant-weight codes which is the determination of $A(n,d,w)$, the largest possible size of a constant-weight code of length n, Hamming distance at least d, and constant weight w. Due to the difficulty in finding good constant-weight codes, various upper and lower bounds on $A(n,d,w)$ have been developed [1, 3, 15, 18]. Moreover, there are further studies in this direction including nonbinary finite fields in [2, 10]. It has been shown that there exists a unique one-weight binary linear code of dimension k such that any two columns in its generator matrix are linearly independent for every positive integer k. Later, this result has been extended to the ring $\mathbb{Z}_4$(integers modulo 4) and to the ring $\mathbb{Z}_{p^m}$(integers modulo p^m), respectively [4, 16]. In [4], it has been shown that for every ordered pair of nonnegative integers (k_1,k_2), there exists a unique (up to equivalence)

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one-weight $\mathbb{Z}_4$-linear code of type $4^{k_1}2^{k_2}$. Wood [19] classified the structure of linear codes of constant weight over the ring $\mathbb{Z}_N$ and gave a general implicit description of constructing constant weight codes over the ring $\mathbb{Z}_N$. The classification in [19] has reproved the classical result about linear codes of constant Hamming weight over a finite field [2] and a recent theorem of Carlet [4] on linear codes of constant Lee weight over the ring $\mathbb{Z}_N$. In [17], Shi *et al.* characterized the structure and properties of one-homogeneous weight linear codes over $\mathbb{F}_p[u]/(u^m)$ and obtained a class of optimal p-ary one-Hamming weight linear codes from one-homogeneous weight linear codes by using the Gray map given in [17]. In this paper, we present an explicit construction of constant weight codes over a finite chain ring.

The organization of this paper is as follows: In Section 2, we give some basic notions and definitions. We also state a distance-preserving map given in [7] and called the Gray map from R^n to $\mathbb{F}_{p^l}^{p^{l(e-1)n}}$, where R is a finite chain ring, $\mathbb{F}_{p^l}$ is the residue field of R with p^l elements and e is the nilpotency index of R. In Section 3, we determine the structures of one-homogeneous weight codes over finite chain rings and study their properties. Yet in this paper, different from [17], we provide a different approach which leads to richer families of one-homogeneous weight codes from the generator matrix of a one-homogeneous weight code presented in Theorem 3.5. By the Gray map, we obtain a class of optimal one-Hamming weight q-ary linear codes which meet the Griesmer bound over $\mathbb{F}_{p^l}$. In Section 3, we give some examples by taking $k = 1$ to illustrate that we derive more optimal one-Hamming weight p-ary linear codes over $R = \mathbb{F}_{p^k}[u]/(u^s)$ than the study in [17]. Finally, we conclude this paper in Section 4.

2. Preliminaries

Throughout this paper, rings are commutative rings with identity $1 \neq 0$. An ideal I of a ring R is said to be principal if it is generated by a single ring element. We say R is a principal ideal ring if every ideal of R is principal. A ring R with a unique maximal ideal is called a local ring. Moreover, a ring R is called a chain ring if its lattice of ideals forms a chain, i.e., its ideals are linearly ordered with respect to set inclusion.

Let R be a finite chain ring and let γ be a generator of the maximal ideal of R. It is well known that the characteristic of a finite chain ring is a positive power of the characteristic of its residue field. So, $R/(\gamma)$ is called the residue field of R having p^l elements, where p is a prime and $l \geq 1$. The ideals of R are as follows:

$$(0) = (\gamma^e) \subset \left(\gamma^{e-1}\right) \subset \cdots \subset (\gamma) \subset \left(\gamma^0\right) = R,$$

where e is the nilpotency index of R. Let $\mathbb{F}_{p^l}$ be a field of p^l elements. We state the Lemma 2.4 in [13].

Proposition 2.1 ([13]). *Let V be a maximal subset of R and γ be a generator of maximal ideal of R with the property that $\overline{x}_1 \neq \overline{x}_2 \bmod (\gamma)$ for all x_1, $x_2 \in R$ such that $x_1 \neq x_2$. Then,*

(1) *for all $x \in R$, there are unique $x_0, x_1, \ldots, x_{e-1} \in V$ such that $x = x_0 + x_1\gamma + \cdots + x_{e-1}\gamma^{e-1}$;*

(2) $\left|\gamma^j R\right| = \left|\mathbb{F}_{p^l}\right|^{e-j}$ *for $0 \leq j \leq e$.*

By Proposition 2.1, it is clear that if $j = 0$, then $|R| = \left|\gamma^0 R\right| = \left|\ \mathbb{F}_{p^l}\right|^{e-0} = p^{le}$. Also, any element $\mathbf{x} \in R^n$ can be written uniquely as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1\gamma + \cdots + \mathbf{x}_{e-1}\gamma^{e-1}, \tag{1}$$

where $\mathbf{x}_i = (\mathbf{x}_{i,0}, \mathbf{x}_{i,1}, \ldots, \mathbf{x}_{i,n-1}) \in V^n$ for all $i \in \{0, 1, \ldots, e-1\}$. Let Γ denote the natural map from R to $\mathbb{F}_{p^l}^n$ such that

$$\Gamma(\mathbf{x}_i) = (\Gamma(\mathbf{x}_{i,0}), \Gamma(\mathbf{x}_{i,1}), \ldots, \Gamma(\mathbf{x}_{i,n-1})).$$

A code C of length n is a nonempty subset of R^n. A linear code C of length n over R is a R-submodule of R^n. It is given in [13] that any code C of length of n over R is permutation-equivalent to a code with the following generator matrix:

$$G = \begin{pmatrix} I_{k_1} & A_{11} & A_{12} & A_{13} & \cdots & A_{1,e-1} & A_{1,e} \\ 0 & \gamma I_{k_2} & \gamma A_{22} & \gamma A_{23} & \cdots & \gamma A_{2,e-1} & \gamma A_{2,e} \\ 0 & 0 & \gamma^2 I_{k_3} & \gamma^2 A_{33} & \cdots & \gamma^2 A_{3,e-1} & \gamma^2 A_{3,e} \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & \gamma^{e-1} I_{k_e} & \gamma^{e-1} A_{e,e} \end{pmatrix}, \tag{2}$$

where I_{k_i} is $k_i \times k_i$ identity matrix and $A_{i,j}$'s are matrices over R for all $i, j \in \{1, 2, \ldots, e\}$. A code having a generator matrix in this form has $\left(p^l\right)^{\sum_{i=0}^{e-1}(e-i)k_{i+1}}$ elements and C is said to be of type $1^{k_1}\left(p^l\right)^{k_2}\cdots\left(p^{l(e-1)}\right)^{k_e}$.

The Hamming weight $w_H(x)$ of a codeword $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ is the number of nonzero components and the Hamming distance between the codewords $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is defined as $d_H(x, y) = w_H(x - y)$.

In [6], the homogeneous weight of an element x of R in the sense of [5] is defined as follows:

$$w_{\text{hom}}(x) = \begin{cases} p^{l(e-1)}, & x \in \left(\gamma^{e-1}\right) \setminus \{0\}, \\ p^{l(e-2)}\left(p^l - 1\right), & x \in R \setminus \left(\gamma^{e-1}\right), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The homogeneous weight can be extended to R^n componentwisely. Then, the homogeneous weight of $x = (x_0, x_1, \ldots, x_{n-1}) \in R^n$ becomes

$$w_{\text{hom}}(x) = \sum_{i=0}^{n-1} w_{\text{hom}}(x_i). \tag{4}$$

Also, the homogeneous distance between $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ in R^n is defined in [5] as follows:

$$d_{\text{hom}}(x, y) = w_{\text{hom}}(x - y).$$

By the following definition in [7], we present the Gray map from R^n to $\mathbb{F}_{p^l}^{p^{l(e-1)}}$:

Every element $\epsilon \in \mathbb{Z}_{p^l}$ can be viewed as $\epsilon = \nu_0(\epsilon) + \nu_1(\epsilon)p + \cdots + \nu_{l-1}(\epsilon)p^{l-1}$, where $\nu_i(\gamma) \in \{0, 1, \ldots, p-1\}$ for all $0 \leq i \leq l-1$. Let α be a primitive element of $\mathbb{F}_{p^l}$. Then, the corresponding element to every $\epsilon \in \mathbb{Z}_{p^l}$ is given by $\alpha_\epsilon := \nu_0(\epsilon) + \nu_1(\epsilon)\alpha + \cdots + \nu_{l-1}(\epsilon)\alpha^{l-1}$. Also, an element $\omega \in \mathbb{Z}_{p^{l(e-1)}}$ can be written as the p^l-adic representation $w = \overline{\nu}_0(w) + \overline{\nu}_1(w)p^l + \cdots + \overline{\nu}_{e-2}(w)\ p^{l(e-2)}$, where $\overline{\nu}_i(w) \in 0, 1, \ldots, p^l-1$ for every $0 \leq i \leq e-2$. The Gray map $\varphi : R^n \to \mathbb{F}_{p^l}^{p^{l(e-1)}n}$ is defined by $\varphi(x) = \left(a_0, a_1, \ldots, a_{p^{l(e-1)}n-1}\right)$, for all $x_i = x_{0,i} + x_{1,i}\gamma + \cdots + x_{e-1}\gamma^{e-1}$, $i \in \left\{0, 1, \ldots, p^{l(e-1)}n - 1\right\}$, where

$$a_{(wp^l+\varepsilon)n+j} = \Gamma(x_{e-1,j}) + \left(\sum_{l=1}^{e-1} \alpha_{\bar{\nu}_{l-1}(w)}\Gamma(x_{l,j})\right) + \alpha_\varepsilon \Gamma(x_{0,j}) \tag{5}$$

for all $0 \leq w \leq p^{l(e-2)} - 1$, $0 \leq \epsilon \leq\ p^l - 1$ and $0 \leq j \leq n-1$.

Theorem 2.2 ([7]). *The Gray map φ is an isometry from (R^n, d_{hom}) to $\left(\mathbb{F}_{p^l}^{p^{l(e-1)}n},\ d_H\right)$, where d_H denotes the Hamming distance on $\mathbb{F}_{p^l}^{p^{l(e-1)}n}$.*

It is well known that $[n, k, d]_q$ refers to a linear code of length n and minimum distance d over $\mathbb{F}_q$, where $q = p^l$, p is a prime and $l \geq 1$. Recall that $A_q(n, d)$ is the maximum size of a code C having length n and minimum distance d. The number $A_q(n, d)$ is very important in coding theory. We state the well known Griesmer bound which applies specifically to linear codes.

Lemma 2.3 ([9]). *Let C be a q-ary code of parameters $[n, k, d_H]_q$, where $k \geq 1$. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{q^i} \right\rceil. \tag{6}$$

Note that if a linear code C over a finite field $\mathbb{F}_q$ meets the Griesmer bound, then C is called optimal.

3. One-homogeneous weight codes over finite chain rings

Throughout rest of the this paper, we denote $C_{k_1,\ldots,k_e}$ as a code with type $1^{k_1}\left(p^l\right)^{k_2} \cdots \left(p^{l(e-1)}\right)^{k_e}$ and we take R as a finite chain ring with residue field $\mathbb{F}_{p^l}$ and nilpotency index e. The characterization of one-Hamming weight linear codes is studied in [3, 11] and [14]. According to [14], we can give the following proposition.

Proposition 3.1 ([14]). *Let C be a linear code of length n over $\mathbb{F}_q$, where $q = p^l$, p is a prime and $l \geq 1$. If for each $i \in \{1, \dots, n\}$ there exists a codeword $c = (c_1, \dots, c_n) \in C$ such that $c_i \neq 0$, then $\sum\limits_{c \in C} w_H(c) = \left(p^l - 1\right) |C| \, n / p^l$.*

By making use of Proposition 3.1, we can derive the sum of the homogeneous weights of all codewords of a linear code C over R.

Theorem 3.2. *Let C be a linear code of length n over R. If for each $i \in \{1, \dots, n\}$ there exists a codeword $c = (c_1, \dots, c_n) \in C$ such that $c_i \neq 0$, then $\sum\limits_{c \in C} w_{\text{hom}}(c) = p^{l(e-2)} \left(p^l - 1\right) |C| \, n$.*

Proof. Consider the $|C| \times n$ array of all codewords in C. Then, each column corresponds to one of the following cases:

- the column contains $x_1, x_2, \dots, x_{p^{le}}$ equally often, where $x_i \in R$ and $x_i \neq x_j$ if $i \neq j$, $i, j \in \left\{1, \dots, p^{le}\right\}$.
- the column contains $x_1, x_2, \dots, x_{p^{l(e-1)}}$ equally often, where $x_i \in \gamma R$ and $x_i \neq x_j$ if $i \neq j$, $i, j \in \left\{1, \dots, p^{l(e-1)}\right\}$.

  $\vdots$
- the column contains $x_1, x_2, \dots, x_{p^{2l}}$ equally often, where $x_i \in \gamma^{e-2} R$ and $x_i \neq x_j$ if $i \neq j$, $i, j \in \left\{1, \dots, p^{2l}\right\}$.
- the column contains $x_1, x_2, \dots, x_{p^l}$ equally often, where $x_i \in \gamma^{e-1} R$ and $x_i \neq x_j$ if $i \neq j$, $i, j \in \left\{1, \dots, p^l\right\}$.

Let N_1 be the number of columns which corresponds to the first case and let N_2 be the number of columns which corresponds to the second case. Similarly, let N_e be the number of columns which corresponds to the e-th case. Note that $\sum_{i=1}^{e} N_i = n$. Therefore we can conclude that

$$
\begin{aligned}
&\sum_{c \in C} w_{\text{hom}}(c) \\
&= |C| \sum_{i=1}^{e} \frac{N_s}{p^{l(e-i+1)}} \left[p^{l(e-1)} \left(p^l - 1\right) + p^{l(e-2)} \left(p^l - 1\right) \left(p^{l(e-i+1)} - p^l\right)\right] \\
&= p^{l(e-2)} \left(p^l - 1\right) |C| \sum_{i=1}^{e} N_i \\
&= p^{l(e-2)} \left(p^l - 1\right) |C| \, n.
\end{aligned}
$$

□

Proposition 3.3. *Let $C_{k_1, \dots, k_e}$ be a linear code of length n over R. If the columns of the generator matrix $G_{(k_1 + \dots + k_e) \times n}$ are all distinct nonzero vectors*

$$
\left(c_1, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2}, \dots, c_{1+\sum\limits_{i=1}^{e-1} k_i}, \dots, c_{\sum\limits_{i=1}^{e} k_i}\right)^{\top},
$$

where $c_{i_1} \in R$ *for all* $i_1 \in \{1, \ldots, k_1\}$, $c_{i_2} \in (\gamma)$ *for all* $i_2 \in \{k_1+1, \ldots, k_1+k_2\}$, $\ldots$, *and* $c_{i_e} \in \left(\gamma^{e-1}\right)$ *for all* $i_e \in \left\{1+\sum_{i=1}^{e-1} k_i, \ldots, \sum_{i=1}^{e} k_i\right\}$, *then* $C_{k_1,\ldots,k_e}$ *is a one-homogeneous weight code with nonzero weight*

$$w_0 = p^{l(e-2)}\left(p^l-1\right)\left|C_{k_1,\ldots,k_e}\right|$$

and $n = \left|C_{k_1,\ldots,k_e}\right| - 1$.

Proof. Without loss of generality, we let $k_1 \neq 0$. Consider a column of the generator matrix such that first entry differs from zero. Let a be the first entry of the column. Observe that the number of such columns with the first entry a is exactly $\left(p^l\right)^{e(k_1-1)}\left(p^l\right)^{\sum\limits_{i=1}^{e-1}(e-i)k_{i+1}}$. Note that the length of the code equals to the number of columns of the generator matrix. Since a runs through all elements of the ring and there is no zero column, the number of columns is

$$p^{le}\left(\left(p^l\right)^{e(k_1-1)}\left(p^l\right)^{\sum\limits_{i=1}^{e-1}(e-i)k_{i+1}}\right) - 1 = \left|C_{k_1,\ldots,k_e}\right| - 1 = n.$$

Observe that the rows consisting of only the elements of the ideal $\left(\gamma^i\right)$ contain equally often the elements of the ideal $\left(\gamma^i\right)$ for all $i = 0, \ldots, e-1$ due to the construction nature of the generator matrix. Then the homogeneous weight of a row consisting of only the elements of the ideal $\left(\gamma^i\right)$ for all $i = 0, \ldots, e-1$ is

$$\begin{aligned}
&\frac{\left|C_{k_1,\ldots,k_e}\right|}{p^{l(e-i)}}\left(p^{l(e-1)}\left(p^l-1\right) + \left(p^{l(e-i)} - p^l\right)\left(p^l-1\right)p^{l(e-2)}\right) \\
=\ &\left|C_{k_1,\ldots,k_e}\right|\left(p^l-1\right)p^{l(e-2)}\left(\frac{p^l}{p^{l(e-i)}} + \frac{p^{l(e-i)} - p^l}{p^{l(e-i)}}\right) \\
=\ &\left|C_{k_1,\ldots,k_e}\right|\left(p^l-1\right)p^{l(e-2)}.
\end{aligned}$$

Therefore, the homogeneous weight of the rows of the generator matrix does not depend on i.

To complete the proof, it remains to show that all codewords of the linear code $C_{k_1,\ldots,k_e}$ have the same weight $w_0 = p^{l(e-2)}\left(p^l-1\right)\left|C_{k_1,\ldots,k_e}\right|$. Set $t = \sum\limits_{i=1}^{e} k_i$ and define the map σ from $R^{k_1} \times (\gamma)^{k_2} \times \cdots \times \left(\gamma^{e-1}\right)^{k_e}$ to R by

$$\sigma\left(x_1, \ldots, x_{k_1}, x_{k_1+1}, \ldots, x_{k_1+k_2}, \ldots, x_{1+\sum\limits_{i=1}^{e-1} k_i}, \ldots, x_{\sum\limits_{i=1}^{e} k_i}\right) = \sum_{i=1}^{t} c_i x_i,$$

where $c_i \in R$ for all $i \in \{1, \ldots, t\}$. Observe that σ is an R-module homomorphism and so it is clear that $\mathrm{Im}\sigma = \left(\gamma^i\right)$ for some $i \in \{0, 1, \ldots, e\}$. Since there is no zero column of the generator matrix $G_{(k_1+\cdots+k_e)\times n}$, this is possible only when $c_i = 0$ for all i then $\mathrm{Im}\sigma = (\gamma^e) = \{0\}$. Since $R^{k_1} \times (\gamma)^{k_2} \times \cdots \times \left(\gamma^{e-1}\right)^{k_e}/\mathrm{Ker}\sigma \cong \mathrm{Im}\sigma = \left(\gamma^i\right)$, each residue class of $R^{k_1} \times (\gamma)^{k_2} \times \cdots \times \left(\gamma^{e-1}\right)^{k_e}$

with respect to mod $\mathrm{Ker}\sigma$ corresponds to a distinct element of the ideal $\left(\gamma^i\right)$. Then, it is not difficult to see that any codeword of the linear code $C_{k_1,\ldots,k_e}$ has exactly $\frac{\left|R^{k_1}\right|\left|(\gamma)^{k_2}\right|\cdots\left|\left(\gamma^{e-1}\right)^{k_e}\right|}{\left|(\gamma^i)\right|}$ times the nonzero elements of the ideal $\left(\gamma^i\right)$. Note that $\left|R^{k_1}\right|\left|(\gamma)^{k_2}\right|\cdots\left|\left(\gamma^{e-1}\right)^{k_e}\right| = |C_{k_1,\ldots,k_e}|$. Hence, by the above observation, the proof is completed. □

Theorem 3.4. *Let $C_{k_1,\ldots,k_e}$ be a one-homogeneous weight code over R of length n and constant weight w_0. Then, there exists a positive integer π such that $n = \pi\frac{|C_{k_1,\ldots,k_e}|-1}{p^l-1}$ and $w_0 = \pi p^{l(e-2)}\,|C_{k_1,\ldots,k_e}|$.*

Proof. By Theorem 3.2, we write

$$p^{l(e-2)}\left(p^l-1\right)|C_{k_1,\ldots,k_e}|\,n = w_0\left(|C_{k_1,\ldots,k_e}|-1\right).$$

Since $\left(p^{l(e-2)}\,|C_{k_1,\ldots,k_e}|\,, \frac{|C_{k_1,\ldots,k_e}|-1}{p^l-1}\right) = 1$, we conclude that there exists a positive integer π such that $n = \pi\frac{|C_{k_1,\ldots,k_e}|-1}{p^l-1}$ and $w_0 = \pi p^{l(e-2)}\,|C_{k_1,\ldots,k_e}|$. □

Theorem 3.4 says that it is possible to derive more one-homogeneous weight codes from one-homogeneous weight code with the generator matrix $G_{(k_1+\cdots+k_e)\times n}$ given in Proposition 3.3. Before giving a method to derive more one-homogeneous weight codes from the one-homogeneous weight code having the generator matrix $G_{(k_1+\ldots+k_e)\times n}$ as in Proposition 3.3, we state the following definition.

Definition. Let n be a nonnegative integer and let A be any matrix. Then,

$$A^n = (\underbrace{A|\;A|\;\cdots|\;A}_{n\text{-times}}).$$

Theorem 3.5. *Let $\pi = \frac{t}{p^l-1}$, where $t = \sum_{i=1}^{p^l-1} n_i$ and n_i is nonnegative integer for all $i = 1,\ldots,p^l-1$. Then, there exists a family of one-Hamming weight codes over $\mathbb{F}_{p^l}$ with the parameters*

$$\left[\pi\left(|C_{k_1,\ldots,k_e}|-1\right)p^{l(e-1)}, \sum_{i=0}^{e-1}(e-i)\,k_{i+1}, \pi\,|C_{k_1,\ldots,k_e}|\left(p^l-1\right)p^{l(e-2)}\right]_{p^l}.$$

Proof. Take $C_{k_1,\ldots,k_e}$ as a code having the generator matrix $G_{(k_1+\cdots+k_e)\times n}$ in Proposition 3.3 over R. Then, $C_{k_1,\ldots,k_e}$ is a one-homogeneous weight code over R of length $|C_{k_1,\ldots,k_e}|-1$. Observe that p^l-1 divides both the length $|C_{k_1,\ldots,k_e}|-1$ and the numbers of nonzero elements in each ideal of R. In this case, we can partition the rows of the generator matrix $G_{(k_1+\cdots+k_e)\times n}$ into p^l-1 equal parts such that all parts have the same number zero divisors and units and they split each ideal as well. Let $A_1, A_2, \ldots, A_{p^l-1}$ be all parts of the generator matrix $G_{(k_1+\cdots+k_e)\times n}$. It is easy to see that each part A_i generates a one-homogeneous weight code over R of length $\frac{|C_{k_1,\ldots,k_e}|-1}{p^l-1}$ and of the nonzero weight $\frac{|C_{k_1,\ldots,k_e}|(p^l-1)p^{l(e-2)}}{p^l-1}$. Let $\hat{C}_{k_1,\ldots,k_e}$ be a code with a generator matrix $\hat{G}$ given by

$$\left(A_1^{n_1} \middle| \, A_2^{n_2} \middle| \cdots \middle| \, A_{p^l-1}^{n_{p^l-1}} \right),$$

where n_i's are nonnegative integers for all $i = 1, \ldots, p^l - 1$. Then, $\hat{C}_{k_1,\ldots,k_e}$ is a one-homogeneous weight code over R of length $\pi\left(|C_{k_1,\ldots,k_e}| - 1\right)$ and of nonzero weight $\pi\,|C_{k_1,\ldots,k_e}|\left(p^l - 1\right)p^{l(e-2)}$. Hence, by the Gray map φ, we obtain a family of one-Hamming weight codes over $\mathbb{F}_{p^l}$ with the parameters as desired. □

Theorem 3.6. *The codes having the parameters given in Theorem 3.5 are optimal.*

Proof. For the proof, it is enough to show that they attain the Griesmer bound. Let $x = \sum_{i=0}^{e-1}(e-i)\,k_{i+1}$ and $a = \sum_{i=1}^{p^l-1} n_i$. Observe that

$$|C_{k_1,\ldots,k_e}| = p^{lx}$$

and

$$\pi\,|C_{k_1,\ldots,k_e}|\left(p^l - 1\right)p^{l(e-2)} = ap^{l(e+x-2)}.$$

Then

$$\begin{aligned}
\sum_{i=0}^{x-1}\left\lceil \frac{ap^{l(e+x-2)}}{\left(p^l\right)^i} \right\rceil &= ap^{l(e+x-2)} + ap^{l(e+x-3)} + \cdots + ap^{l(e-1)} \\
&= ap^{l(e-1)}\left(p^{l(x-1)} + p^{l(x-2)} + \cdots + 1\right) \\
&= \frac{ap^{l(e-1)}\left(p^{lx} - 1\right)}{p^l - 1} \\
&= \pi\left(|C_{k_1,\ldots,k_e}| - 1\right)p^{l(e-1)} \\
&= n.
\end{aligned}$$

□

Now we present some examples that illustrate the findings of the previous results.

Example 3.7. Let $C_{k_1=1,k_2=1}$ be a code over the ring $\mathbb{F}_2[u]/\left(\xi(u)^2\right)$, where $\xi(u)$ is an irreducible polynomial over $F_2[u]$ of degree 2. Suppose that

$$C_{k_1=1,k_2=1}$$

has the generator matrix $G = (G_1|\,G_2|\,G_3)$, where

$$G_1 = \begin{pmatrix} 0 & \xi(u) + \mathbf{u}\xi(u) & \mathbf{1} & \mathbf{1} + \xi(u) & \mathbf{1} + \mathbf{u}\xi(u) & \mathbf{u} + \xi(u) + \mathbf{u}\xi(u) \\ \xi(u) & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & \mathbf{u}\xi(u) & \mathbf{u} & \mathbf{u} + \xi(u) & \mathbf{1} + \mathbf{u} + \mathbf{u}\xi(u) & \mathbf{1} + \mathbf{u} + \xi(u) + \mathbf{u}\xi(u) \\ u\xi(u) & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta \end{pmatrix},$$

Table 1. An infinite family of optimal one-Hamming weight codes obtained by the Gray map.

n_1	n_2	n_3	$\varphi\left(C_{k_1=1,k_2=1}\right)$
1	0	0	$[84,3,64]_4$
1	1	0	$[168,3,128]_4$
1	1	1	$[252,3,192]_4$
2	1	1	$[336,3,256]_4$
2	2	1	$[420,3,320]_4$
2	2	2	$[504,3,384]_4$
3	2	2	$[588,3,448]_4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$G_3=\begin{pmatrix} 0 & \xi(u) & \mathbf{1+u} & \mathbf{1+u}+\xi(u) & \mathbf{u}+\mathbf{u}\xi(u) & \mathbf{1}+\xi(u)+\mathbf{u}\xi(u) \\ \xi(u)+u\xi(u) & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta \end{pmatrix},$$

and $\mathbf{G}_\beta=\begin{pmatrix} 0 & \xi(u) & u\xi(u) & \xi(u)+u\xi(u) \end{pmatrix}$. According to Proposition 3.3, $C_{k_1=1,k_2=1}$ is a one-homogeneous weight code over the ring $\mathbb{F}_2[u]/\left(\xi(u)^2\right)$ of length $n=63$ and nonzero weight $w_0=192$. By Theorem 3.5, it is seen that the each of the parts G_1, G_2 and G_3 generate a one-homogeneous weight code over the ring $\mathbb{F}_2[u]/\left(\xi(u)^2\right)$ of length $n=31$ and nonzero weight $w_0=64$. Moreover, a code having the generator matrix $(G_1^{n_1}|\,G_2^{n_2}|\,G_3^{n_3})$ is a one-homogeneous weight code. Hence, by Gray map φ, we can obtain more one-Hamming weight codes over $\mathbb{F}_4$ with respect to n_1, n_2 and n_3, some of which parameters are given in Table 1.

Example 3.8. Let $C_{k_1=1,k_2=2}$ be a code over $\mathbb{Z}_9$ with the generator matrix $G=(G_1|\,G_2)$, where

$$G_1=\begin{pmatrix} \mathbf{0} & \mathbf{3} & \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{G}_{\alpha_1} & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta \end{pmatrix},\ G_2=\begin{pmatrix} \mathbf{0} & \mathbf{6} & \mathbf{5} & \mathbf{7} & \mathbf{8} \\ \mathbf{G}_{\alpha_2} & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta & \mathbf{G}_\beta \end{pmatrix},$$

where

$$G_{\alpha_1}=\begin{pmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 6 \end{pmatrix},\ G_{\alpha_2}=\begin{pmatrix} 0 & 6 & 6 & 6 \\ 6 & 0 & 3 & 6 \end{pmatrix}$$

and

$$\mathbf{G}_\beta=\begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 3 & 6 & 6 & 6 \\ 0 & 3 & 6 & 0 & 3 & 6 & 0 & 3 & 6 \end{pmatrix}.$$

According to Proposition 3.3, $C_{k_1=1,k_2=2}$ is a one-homogeneous weight code over $\mathbb{Z}_9$ of length $n=80$ and nonzero weight $w_0=162$. By Theorem 3.5, it is seen that each of the parts G_1 and G_2 generate a one-homogeneous weight code over $\mathbb{Z}_9$ of length $n=40$ and nonzero weight $w_0=81$. Also, a code having the generator matrix $(G_1^{n_1}|\,G_2^{n_2})$ is a one-homogeneous weight code over $\mathbb{Z}_9$. Hence, by the Gray map φ, we can obtain more one-Hamming weight codes

TABLE 2. An infinite family of optimal one-Hamming weight codes obtained by the Gray map.

n_1	n_2	$\varphi\left(C_{k_1=1,k_2=2}\right)$
1	0	$[120,4,81]_3$
1	1	$[240,4,162]_3$
2	1	$[360,4,243]_3$
2	2	$[480,4,324]_3$
3	2	$[600,4,405]_3$
3	3	$[720,4,486]_3$
4	3	$[840,4,567]_3$
$\vdots$	$\vdots$	$\vdots$

over $\mathbb{Z}_3$ with respect to $n_1$and n_2, some of which parameters are given in Table 2.

In the following two examples, we illustrate that Theorem 3.5 is a refinement of Theorem 3.6 in [17].

Example 3.9. Let $C_{k_1=1,k_2=1}$ be a code over the ring $\mathbb{F}_5[u]/\left(u^2\right)$ with the generator matrix $G=(G_1|\,G_2\,|G_3\,|G_4\,)$, where

$$G_1=\begin{pmatrix} 0 & \mathbf{u} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1+u} \\ u & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha \end{pmatrix},$$

$$G_2=\begin{pmatrix} 0 & \mathbf{2u} & \mathbf{2+u} & \mathbf{3+u} & \mathbf{4+u} & \mathbf{1+2u} & \mathbf{2+2u} \\ 2u & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha \end{pmatrix},$$

$$G_3=\begin{pmatrix} 0 & \mathbf{3u} & \mathbf{3+2u} & \mathbf{4+2u} & \mathbf{1+3u} & \mathbf{2+3u} & \mathbf{3+3u} \\ 3u & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha \end{pmatrix},$$

$$G_4=\begin{pmatrix} 0 & \mathbf{4u} & \mathbf{4+3u} & \mathbf{1+4u} & \mathbf{2+4u} & \mathbf{3+4u} & \mathbf{4+4u} \\ 4u & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha & \mathbf{G}_\alpha \end{pmatrix},$$

and $G_\alpha=\begin{pmatrix} 0 & u & 2u & 3u & 4u \end{pmatrix}$. According to Proposition 3.3, $C_{k_1=1,k_2=1}$ is a one-homogeneous weight code over the ring $\mathbb{F}_5[u]/\left(u^2\right)$ of length $n=124$ and nonzero weight $w_0=500$. By Theorem 3.5, it is seen that each of the parts G_1, G_2, G_3 and G_4 generate a one-homogeneous weight code over the ring $\mathbb{F}_5[u]/\left(u^2\right)$ of length $n=31$ and nonzero weight $w_0=125$. Moreover, a code having the generator matrix $(G_1^{n_1}|\,G_2^{n_2}\,|G_3^{n_3}\,|G_4^{n_4}\,)$ is a one-homogeneous weight code over the ring $\mathbb{F}_5[u]/\left(u^2\right)$. Hence, by the Gray map φ, we can obtain more one-Hamming weight codes over $\mathbb{F}_5$ with respect to n_1, n_2 and n_3, some of which parameters are given in Table 3.

Example 3.10. Let $C_{k_1=1,k_2=0,k_3=0}$ be a code over the ring $\mathbb{F}_7[u]/\left(u^3\right)$ with the generator matrix $G=(G_1|\,G_2\,|G_3\,|G_4\,|G_5\,|G_6\,)$, where each of G_i is a row matrix and has exactly one nonzero element from minimal ideal $\left(u^2\right)$

TABLE 3. An infinite family of optimal one-Hamming weight codes obtained by the Gray map.

n_1	n_2	n_3	n_4	$\varphi\left(C_{k_1=1,k_2=1}\right)$
1	0	0	0	$[155, 3, 125]_5$
1	1	0	0	$[310, 3, 250]_5$
1	1	1	0	$[465, 3, 375]_5$
1	1	1	1	$[620, 3, 500]_5$
2	1	1	1	$[775, 3, 625]_5$
2	2	1	1	$[930, 3, 750]_5$
2	2	2	1	$[1085, 3, 875]_5$
2	2	2	2	$[1240, 3, 1000]_5$
⋮	⋮	⋮	⋮	⋮

TABLE 4. An infinite family of optimal one-Hamming weight codes obtained by the Gray map.

n_1	n_2	n_3	n_4	n_5	n_6	$\varphi\left(C_{k_1=1,k_2=0,k_3=0}\right)$
1	0	0	0	0	0	$[2793, 3, 2401]_7$
1	1	0	0	0	0	$[5586, 3, 4802]_7$
1	1	1	0	0	0	$[8379, 3, 7023]_7$
1	1	1	1	0	0	$[11172, 3, 9604]_7$
1	1	1	1	1	0	$[13965, 3, 12005]_7$
1	1	1	1	1	1	$[16758, 3, 14406]_7$
2	1	1	1	1	1	$[19551, 3, 16807]_7$
2	2	1	1	1	1	$[22344, 3, 19208]_7$
2	2	2	1	1	1	$[25137, 3, 21609]_7$
2	2	2	2	1	1	$[27930, 3, 24010]_7$
⋮	⋮	⋮	⋮	⋮	⋮	⋮

and the same number nonzero element from each of ideal (u) and (1). According to Proposition 3.3, $C_{k_1=1,k_2=0,k_3=0}$ is a one-homogeneous weight code over the ring $\mathbb{F}_7[u]/\left(u^3\right)$ of length $n = 342$ and of nonzero weight $w_0 = 14406$. By Theorem 3.5, it is seen that each of the parts G_i generates a one-homogeneous weight code over the ring $\mathbb{F}_7[u]/\left(u^3\right)$ of length $n = 57$ and of nonzero weight $w_0 = 2401$. Also, a code having the generator matrix $(G_1^{n_1}|\,G_2^{n_2}\,|G_3^{n_3}\,|G_4^{n_4}\,|G_5^{n_5}\,|G_6^{n_6})$ is a one-homogeneous weight code over the ring $\mathbb{F}_7[u]/\left(u^3\right)$. Hence, by the Gray map φ , we can obtain more one-Hamming weight codes over $\mathbb{F}_7$ with respect to $n_i's$, some of which parameters are given in Table 4.

4. Conclusion

We study the structures and the algebraic properties of linear codes of constant-weight over finite chain rings and we present some explicit constructions of constant weight codes over finite chain rings and their residue fields. By the Gray map, we derive a family of optimal one-Hamming weight codes over the residue field. Moreover, by the proposed generalized method, we derive more optimal one-Hamming weight p-ary linear codes over the ring $R = \mathbb{F}_{p^k}[u]/(u^s)$ than previously obtained by Shi *et al.* in [17] .

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