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SOME IDENTITIES FOR BERNOULLI NUMBERS OF THE SECOND KIND ARISING FROM A NON-LINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper, we give explicit and new identities for the Bernoulli numbers of the second kind which are derived from a non-linear differential equation.

1. Introduction

For $r \in \mathbb{N}$, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \underbrace{\left(\frac{t}{e^t-1}\right)\times\cdots\times\left(\frac{t}{e^t-1}\right)}_{r\text{-times}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [1]-[16])}. \tag{1}$$

When $x=0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.

As is well known, the Bernoulli polynomials of the second kind are given by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [3, 5, 7, 14])}. \tag{2}$$

Indeed, $b_n(x) = B_n^{(n)}(x+1)$.

When $x=0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind.

The first few Bernoulli numbers of the second kind are $b_0 = 1$, $b_1 = \frac{1}{2}$, $b_2 = -\frac{1}{6}$, $b_3 = \frac{1}{4}$, $b_4 = -\frac{19}{30}$, $b_5 = \frac{9}{4}$, $\ldots$.

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From (2), we have

$$b_n(x)=\sum_{l=0}^{n}\binom{n}{l}b_l(x)_{n-l}\,,\quad (n\geq 0)\,, \tag{3}$$

where $(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\,x^l$, with $S_1(n,l)$ the Stirling number of the first kind.

Let $u \neq 1 \in \mathbb{C}$. Then the Frobenius-Euler polynomials are defined by generating function to be

$$\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(u|x)\frac{t^n}{n!},\quad \text{(see [1, 11, 15])}. \tag{4}$$

From (4), L. Carlitz gave the following identity:

$$\begin{aligned} H_n(x|\alpha)\,H_n(x|\beta) &= H_{m+n}(x|\alpha\beta)\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \\ &+\frac{\alpha(1-\beta)}{1-\alpha\beta}\sum_{r=0}^{m}\binom{m}{r}H_r(\alpha)\,H_{m+n-r}(x|\alpha\beta) \\ &+\frac{\beta(1-\beta)}{1-\alpha\beta}\sum_{s=0}^{n}\binom{n}{s}H_s(\beta)\,H_{m+n-s}(x|\alpha\beta)\,, \end{aligned} \tag{5}$$

where $\alpha,\beta\in\mathbb{C}$ with $\alpha\neq 1$, $\beta\neq 1$ and $\alpha\beta\neq 1$, $m,n\in\mathbb{Z}_{\geq 0}$ (see [11]).

In [11], the second author gave some new and interesting identities and formulas for the Frobenius-Euler polynomials of higher order which are derived from a non-linear differential equation.

In this paper, we develop some new method for obtaining identities related to Bernoulli numbers of the second kind arising from a non-linear differential equation. From our method, we derive new identities for the Bernoulli numbers of the second kind.

2. Some identities for Bernoulli numbers of the second kind

In this section, we assume that

$$F=F(t)=\frac{1}{\log(1+t)},\quad \text{and } F^N(t)=\underbrace{F\times\cdots\times F}_{n\text{-times}}\quad \text{for } N\in\mathbb{N}. \tag{6}$$

Thus, by (6), we get

$$F^{(1)}=\frac{dF(t)}{dt}=\frac{(-1)}{(\log(1+t))^2}\left(\frac{1}{1+t}\right)=\frac{(-1)}{1+t}F^2, \tag{7}$$

$$\begin{aligned} F^{(2)} &= \frac{d}{dt}F^{(1)}(t)=\frac{(-1)^2}{(1+t)^2}F^2+\frac{(-1)}{1+t}2F\cdot F^{(1)} \\ &= \frac{(-1)^2}{(1+t)^2}F^2+2\frac{(-1)^2}{(1+t)^2}F^3=\frac{(-1)^2}{(1+t)^2}\left(F^2+2F^3\right). \end{aligned} \tag{8}$$

From (7) and (8), we can derive the following equations:

$$F^{(3)} = \frac{dF^{(2)}}{dt} = \frac{(-1)^3 2}{(1+t)^3}\left(F^2 + 2F^3\right) + \frac{(-1)^2}{(1+t)^2}\left(2FF^{(1)} + 6F^2F^{(1)}\right) \tag{9}$$

$$= \frac{(-1)^3}{(1+t)^3}\left(2F^2 + 4F^3\right) + \frac{(-1)^3}{(1+t)}\left(2F^3 + 6F^4\right)$$

$$= \frac{(-1)^3}{(1+t)^3}\left(2F^2 + 6F^3 + 6F^4\right),$$

and

$$F^{(4)} = \frac{dF^{(3)}}{dt} = \frac{(-1)^4 3}{(1+t)^4}\left(2F^2 + 6F^3 + 6F^4\right) \tag{10}$$

$$+ \frac{(-1)^3}{(1+t)^3}\left(4FF^{(1)} + 18F^2F^{(1)} + 24F^3F^{(1)}\right)$$

$$= \frac{(-1)^4}{(1+t)^4}\left(6F^2 + 18F^3 + 18F^4\right) + \frac{(-1)^4}{(1+t)^4}\left(4F^3 + 18F^4 + 24F^5\right)$$

$$= \frac{(-1)^4}{(1+t)^4}\left(6F^2 + 22F^3 + 36F^4 + 24F^5\right).$$

Continuing this process, we set

$$F^{(N)} = \frac{(-1)^N}{(1+t)^N} \sum_{i=2}^{N+1} a_{i-1}(N) F^i, \tag{11}$$

where

$$F^{(N)} = \frac{d^N F(t)}{dt^N} \quad \text{and } N \in \mathbb{N}.$$

Now, we will determine the coefficients $a_{i-1}(N)$ in (11). Taking the derivative of (11) with respect to t, we have

(12)

$$F^{(N+1)} = \frac{(-1)^{N+1} N}{(1+t)^{N+1}} \sum_{i=2}^{N+1} a_{i-1}(N) F^i + \frac{(-1)^N}{(1+t)^N} \sum_{i=2}^{N+1} a_{i-1}(N)\, iF^{i-1}F^{(1)}$$

$$= \frac{(-1)^{N+1} N}{(1+t)^{N+1}} \sum_{i=2}^{N+1} a_{i-1}(N) F^i + \frac{(-1)^{N+1}}{(1+t)^{N+1}} \sum_{i=2}^{N+1} i a_{i-1}(N) F^{i+1}$$

$$= \frac{(-1)^{N+1} N}{(1+t)^{N+1}} \sum_{i=2}^{N+1} a_{i-1}(N) F^i + \frac{(-1)^{N+1}}{(1+t)^{N+1}} \sum_{i=3}^{N+1} (i-1)\, a_{i-2}(N) F^i$$

$$= \frac{(-1)^{N+1}}{(1+t)^{N+1}} \Big\{ N a_1(N) F^2 + \sum_{i=3}^{N+1} \left((i-1)\, a_{i-2}(N) + N a_{i-1}(N)\right) F^i$$

$$+ a_N(N)(N+1) F^{N+2} \Big\}.$$

Replacing N by $N+1$ in (11), we get

$$F^{(N+1)} = \frac{(-1)^{N+1}}{(1+t)^{N+1}} \sum_{i=2}^{N+2} a_{i-1}(N+1) F^i. \tag{13}$$

From (12) and (13), we have

$$\begin{aligned} &\sum_{i=2}^{N+1} a_{i-1}(N+1) F^i \\ &= N a_1(N) F^2 + \sum_{i=3}^{N+1} \left((i-1) a_{i-2}(N) + N a_{i-1}(N)\right) F^i \\ &\quad + (N+1) a_N(N) F^{N+2}. \end{aligned} \tag{14}$$

By comparing the coefficients on both sides in (14), we get

$$a_1(N+1) = N a_1(N), \quad a_{N+1}(N+1) = (N+1) a_N(N), \tag{15}$$

and

$$a_{i-1}(N+1) = (i-1) a_{i-2}(N) + N a_{i-1}(N), \quad (3 \leq i \leq N+1). \tag{16}$$

From (1), (12) and (13), we have

$$-\frac{t}{1+t} F^2 = F^{(1)} = -\frac{1}{1+t} a_1(1) F^2. \tag{17}$$

Thus, by (17), we get $a_1(1) = 1$. By (15), we see that

$$a_1(N+1) = N a_1(N) = N(N-1) a_1(N-1) = \cdots = N! a_1(1) = N!, \tag{18}$$

and

$$\begin{aligned} a_{N+1}(N+1) &= (N+1) a_N(N) = (N+1) N a_{N-1}(N-1) \\ &= \cdots = (N+1) N \cdots 2 a_1(1) = (N+1)!. \end{aligned} \tag{19}$$

From (18) and (19), we have

$$\begin{aligned} &a_1(1) = 0! = 1, \quad a_1(2) = 1!, \quad a_1(3) = 2!, \ldots, a_1(N) = (N-1)!, \\ &a_1(1) = 1! = 1, \quad a_2(2) = 2!, \quad a_3(3) = 3!, \ldots, a_N(N) = N!. \end{aligned} \tag{20}$$

That is, the matrix $(a_i(j))_{1 \leq i,j \leq N}$ is given by

$$\begin{bmatrix} 0! & 1! & 2! & 3! & \cdots & \cdots & (N-1)! \\ & 2! & & & & & \\ & & 3! & & & & \\ & & & \ddots & & & \\ & & 0 & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & N! \end{bmatrix}.$$

By (16), we get

$$a_2(N+1) = 2a_1(N) + Na_2(N). \tag{21}$$

Thus, from (21), we have

$$\begin{aligned}
&a_2(N+1) \\
&= 2a_1(N) + Na_2(N) = 2(N-1)! + Na_2(N) \\
&= 2(N-1)! + N(2a_1(N-1) + (N-1)a_2(N-1)) \\
&= 2(N-1)! + 2N(N-2)! + N(N-1)a_2(N-1) \\
&= 2N!\left(\frac{1}{N} + \frac{1}{N-1}\right) + (N)_2\, a_2(N-1) \\
&= 2N!\left(\frac{1}{N} + \frac{1}{N-1}\right) + (N)_2\,(2a_1(N-2) + (N-2)a_2(N-2)) \\
&= 2N!\left(\frac{1}{N} + \frac{1}{N-1}\right) + (N)_2\,(2(N-3)! + (N-2)a_2(N-2)) \\
&= 2N!\left(\frac{1}{N} + \frac{1}{N-1} + \frac{1}{N-2}\right) + (N)_3\, a_2(N-2) = \cdots \\
&= 2N!\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{N-(N-2)}\right) \\
&\quad + (N)_{N-1}\, a_2(N-(N-2)) \\
&= 2N!\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{2} + 1 - 1\right) + N!a_2(2) \\
&= 2N!(H_N - 1) + 2!N! = 2N!H_N,
\end{aligned} \tag{22}$$

where $H_N = H_{N,1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$ is the harmonic number.

From (22), we have

$$\begin{aligned}
&a_3(N+1) \\
&= 3a_2(N) + Na_3(N) \\
&= 3 \cdot 2 \cdot (N-1)!H_{N-1} + Na_3(N) \\
&= 3!(N-1)!H_{N-1} + Na_3(N) \\
&= 3!(N-1)!H_{N-1} + N\{3a_2(N-1) + (N-1)a_3(N-1)\} \\
&= 3!(N-1)!H_{N-1} + N\{3 \cdot 2(N-2)!H_{N-2} + (N-1)a_3(N-1)\} \\
&= 3!(N-1)!H_{N-1} + 3!N(N-2)!H_{N-2} + N(N-1)a_3(N-1) \\
&= 3!N!\left(\frac{H_{N-1}}{N} + \frac{H_{N-2}}{N-1}\right) + N(N-1)a_3(N-1) \\
&= 3!N!\left(\frac{H_{N-1}}{N} + \frac{H_{N-2}}{N-1}\right) \\
&\quad + N(N-1)\{3a_2(N-2) + (N-2)a_3(N-2)\}
\end{aligned} \tag{23}$$

$$
\begin{aligned}
&= 3!N!\left(\frac{H_{N-1}}{N}+\frac{H_{N-2}}{N-1}\right) \\
&\quad + N\left(N-1\right)\left\{3!\left(N-3\right)!H_{N-3}+\left(N-2\right)a_3\left(N-2\right)\right\} \\
&= 3!N!\left(\frac{H_{N-1}}{N}+\frac{H_{N-2}}{N-1}+\frac{H_{N-3}}{N-2}\right)+N\left(N-1\right)\left(N-2\right)a_3\left(N-2\right) \\
&= \cdots \\
&= 3!N!\left(\frac{H_{N-1}}{N}+\frac{H_{N-2}}{N-1}+\cdots+\frac{H_{N-(N-2)}}{N-(N-3)}\right)+\left(N\right)_{N-2}3! \\
&= 3!N!H_{N,2},
\end{aligned}
$$

where we put

$$
H_{N,2}=\frac{H_{N-1}}{N}+\frac{H_{N-2}}{N-1}+\cdots+\frac{H_1}{2}+\frac{H_0}{1},\quad H_0=H_{0,1}=0. \tag{24}
$$

By (16) and (23), we also get

$$
\begin{aligned}
&a_4\left(N+1\right) \\
&= 4a_3\left(N\right)+Na_4\left(N\right) \\
&= 4\cdot 3!\left(N-1\right)!H_{N-1,2}+Na_4\left(N\right) \\
&= 4!\left(N-1\right)!H_{N-1,2}+Na_4\left(N\right) \\
&= 4!\left(N-1\right)!H_{N-1,2} \\
&\quad + N\left\{4!\left(N-2\right)!H_{N-2,2}+\left(N-1\right)a_4\left(N-1\right)\right\} \\
&= 4!N!\left\{\frac{H_{N-1,2}}{N}+\frac{H_{N-2,2}}{N-1}\right\}+\left(N\right)_2a_4\left(N-1\right) \\
&= 4!N!\left\{\frac{H_{N-1,2}}{N}+\frac{H_{N-2,2}}{N-1}\right\} \\
&\quad + \left(N\right)_2\left\{4!\left(N-3\right)!H_{N-3,2}+\left(N-2\right)a_4\left(N-2\right)\right\} \\
&= 4!N!\left\{\frac{H_{N-1,2}}{N}+\frac{H_{N-2,2}}{N-1}+\frac{H_{N-3,2}}{N-2}\right\}+\left(N\right)_3a_4\left(N-2\right) \\
&= \cdots \\
&= 4!N!\left\{\frac{H_{N-1,2}}{N}+\frac{H_{N-2,2}}{N-1}+\cdots+\frac{H_{N-(N-3),2}}{N-(N-4)}\right\}+\left(N\right)_{N-3}a_4\left(4\right) \\
&= 4!N!H_{N,3},
\end{aligned} \tag{25}
$$

where we put

$$
H_{N,3}=\frac{H_{N-1,2}}{N}+\frac{H_{N-2,2}}{N-1}+\cdots+\frac{H_{0,2}}{1},\quad H_{0,2}=0. \tag{26}
$$

Note that

$$
H_{1,2}=\frac{H_0}{1}=0,\quad H_{2,2}=\frac{H_1}{2}=\frac{1}{2}.
$$

Continuing this process, we have

$$a_j(N) = j!\,(N-1)!H_{N-1,j-1}, \quad (j \in \mathbb{N}), \tag{27}$$

where we define

$$H_{N,1} = H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}, \tag{28}$$

$$\begin{aligned} H_{N,j} &= \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{0,j-1}}{1}, \\ H_{0,j-1} &= 0 \quad (2 \le j \le N). \end{aligned} \tag{29}$$

Therefore, by (11) and (27), we obtain the following theorem.

Theorem 1. *For $N \in \mathbb{N}$, let us consider the following non-linear differential equation with respect to t:*

$$F^{(N)}(t) = \frac{(-1)^N}{(1+t)^N} \sum_{j=2}^{N+1} (j-1)!\,(N-1)!H_{N-1,j-2}F^j, \tag{30}$$

where

$H_{N,0} = 1$ *for all N,*

$H_{N,1} = H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$,

$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{0,j-1}}{1}, \quad H_{0,j-1} = 0 \quad (2 \le j \le N).$

Then $F = F(t) = \frac{1}{\log(1+t)}$ is a solution of (30).

From (2), we note that

$$F(t) = \frac{1}{\log(1+t)} = \sum_{n=1}^{\infty} b_n \frac{t^{n-1}}{n!} + \frac{1}{t} = \sum_{n=0}^{\infty} \frac{b_n}{n+1} \frac{t^n}{n!} + \frac{1}{t}. \tag{31}$$

Thus, by (31), we get

$$\begin{aligned} F^{(N-1)} &= \frac{d^{N-1}}{dt^{N-1}} \left(\frac{1}{\log(1+t)} \right) \\ &= \sum_{n=N-1}^{\infty} \frac{b_{n+1}}{n+1} \frac{t^{n-N+1}}{(n-N+1)!} + \frac{1}{t^N} (-1)^{N-1} (N-1)! \\ &= \sum_{n=0}^{\infty} \frac{b_{n+N}}{n+N} \frac{t^n}{n!} + \frac{1}{t^N} (-1)^{N-1} (N-1)!. \end{aligned} \tag{32}$$

From (32), we have

$$t^N F^{(N-1)} = \sum_{n=N-1}^{\infty} \frac{b_{n+1}}{n+1} \frac{t^{n+1}}{(n-N+1)!} + (-1)^{N-1} (N-1)! \tag{33}$$

$$= \sum_{n=N}^{\infty} \frac{b_n}{n} \frac{t^n}{(n-N)!} + (-1)^{N-1} (N-1)!.$$

Thus, by (33), we get

(34)

$$\begin{aligned}
&(1+t)^N t^{N+1} F^{(N)}(t) \\
&= (1+t)^N \sum_{n=N+1}^{\infty} \frac{b_n}{n} \frac{t^n}{(n-N-1)!} + (-1)^N N! (1+t)^N \\
&= \left(\sum_{l=0}^{\infty} \binom{N}{l} t^l \right) \left(\sum_{m=N+1}^{\infty} \frac{b_m}{m} \frac{t^m}{(m-N-1)!} \right) + (-1)^N N! \sum_{n=0}^{\infty} \binom{N}{n} t^n \\
&= \sum_{n=N+1}^{\infty} \left(\sum_{l=0}^{n-N-1} \binom{N}{l} \frac{b_{n-l}}{n-l} n \cdots (n-l-N) \right) \frac{t^n}{n!} + (-1)^N N! \sum_{n=0}^{N} (N)_n \frac{t^n}{n!}.
\end{aligned}$$

The higher-order Bernoulli numbers of the second kind is defined by the generating function to be

$$\left(\frac{t}{\log(1+t)} \right)^k = \sum_{n=0}^{\infty} b_n^{(k)} \frac{t^n}{n!}, \quad \text{(see [3, 5, 7, 14])}.$$

Indeed, we note that $b_n^{(k)} = B_n^{(n-k+1)}(1)$.

From Theorem 1, we have

(35)

$$\begin{aligned}
&(1+t)^N t^{N+1} F^{(N)}(t) \\
&= (-1)^N \sum_{j=2}^{N+1} (j-1)! (N-1)! H_{N-1,j-2} t^{N+1} F^j \\
&= (-1)^N \sum_{j=2}^{N+1} (j-1)! (N-1)! H_{N-1,j-2} \left(\frac{t}{\log(1+t)} \right)^j t^{N+1-j} \\
&= (-1)^N \sum_{j=0}^{N-1} (N-j)! (N-1)! H_{N-1,N-1-j} t^j \sum_{m=0}^{\infty} b_m^{(N+1-j)} \frac{t^m}{m!} \\
&= (-1)^N \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\min\{n,N-1\}} (N-j)! (N-1)! H_{N-1,N-1-j} \frac{b_{n-j}^{(N+1-j)} n!}{(n-j)!} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ (-1)^N \sum_{j=0}^{\min\{n,N-1\}} (N-j)! (N-1)! \right. \\
&\quad \left. \times H_{N-1,N-1-j} n(n-1) \cdots (n-j+1) b_{n-j}^{(N+1-j)} \right\} \frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (34) and (35), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$(-1)^N \sum_{j=0}^{\min\{n,N-1\}} (N-j)!\,(N-1)! H_{N-1,N-1-j}\,(n)_j\, b_{n-j}^{(N+1-j)}$$
$$= \begin{cases} (-1)^N N!\,(N)_n & \text{if } 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \binom{N}{l} \frac{b_{n-l}}{n-l}\,(n)_{l+N+1} & \text{if } n \geq N+1. \end{cases}$$

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