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MAPPING PRESERVING NUMERICAL RANGE OF OPERATOR PRODUCTS ON C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. Denote by $W(a)$ the numerical range of an element $a \in \mathcal{A}$. We show that the condition $W(ax) = W(bx), \forall x \in \mathcal{A}$ implies that $a = b$. Using this, among other results, it is proved that if $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective map such that $W(\phi(a)\phi(b)\phi(c)) = W(abc)$ for all a, b and $c \in \mathcal{A}$, then $\phi(1) \in Z(B)$ and the map $\psi = \phi(1)^2\phi$ is multiplicative.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra with unit 1 and let $S(\mathcal{A})$ be the state space of $\mathcal{A}$, i.e., $S(\mathcal{A}) = \{\varphi \in \mathcal{A}' : \varphi \geq 0, \varphi(1) = 1\}$ (here $\mathcal{A}'$ is the topological dual of $\mathcal{A}$). For each $a \in \mathcal{A}$, the algebraic numerical range $V(a)$ and numerical radius $v(a)$ are defined by

$$V(a) = \{f(a) : f \in S(\mathcal{A})\} \quad \text{and} \quad v(a) = \sup_{z \in V(a)} |z|.$$

By the Gelefand-Naimark theorem, every C^*-algebra may be viewed as a closed $*$-subalgebra of $B(H)$ where $B(H)$ denotes the algebra of all bounded linear operators acting on a Hilbert space H. It is well known that $V(a)$ is the closure of $W(a)$ and $v(a) = w(a) = \sup_{\lambda \in W(a)} |\lambda|$, where $W(a) = \{(at, t) : t \in H, \|t\| = 1\}$ and $(,)$ denotes the inner product. Here $W(a)$ is called the usual numerical range of the operator a.

In the last few decades, there has been a considerable interest in the problem of characterization of maps that preserves the numerical range or the numerical radius, see for instance the papers [4, 12, 13, 15] and the references therein. Notice that, based on the aforesaid, preserving the usual numerical range W implies the preservation of the spacial numerical range V. Therefore, we will concentrate our study henceforth on W. Recently, Hou and Di described in [9] surjective maps on the algebra $B(H)$ which preserves the numerical range of

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the product. Namely, they characterized surjective mappings which satisfy one of the following conditions

$$W(\phi(a)\phi(b)) = W(ab), \tag{1a}$$

$$W(\phi(a)^*\phi(b)) = W(a^*b), \tag{1b}$$

$$W(\phi(a)\phi(b)\phi(a)) = W(aba), \tag{1c}$$

for every a and b in $B(H)$. In this paper, we extend these results by completely describing additive and surjective maps $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras satisfying (1a) or (1b) for every $a, b \in \mathcal{A}$. Concerning the condition (1c), we consider a more general case. More precisely, we show that if ϕ is surjective and satisfy $W(\phi(a)\phi(b)\phi(c)) = W(abc), \forall a, b$ and $c \in \mathcal{A}$ (without the additivity assumption), then the map $\psi = \phi(1)^2\phi$ is multiplicative and preserves the set of self-adjoint elements. It is worth noticing that our proofs differ from those of [7] and [9] since we do not assume that $\mathcal{A}$ contains rank one operators. At last, observe that the proof we put forward is much simpler.

The outline of the paper is as follows. Firstly, we show that if a and b in $\mathcal{A}$ are such that $W(ax) = W(bx)$ for every $x \in \mathcal{A}$, then the two operators a and b coincide. This result is used several times in our proofs. Namely, it helps us to show that if ϕ is additive and satisfies (1a) or (1b), then $\phi(1) \in \mathcal{Z}(\mathcal{B})$, where $Z(\mathcal{B})$ stands for the center of $\mathcal{B}$, and $\phi(1)\phi$ is a Jordan $*$-isomorphism. This characterization also allows us to show that if a map ϕ is surjective and satisfies $W(abc) = W(\phi(a)\phi(b)\phi(c))$ whenever a, b and c are in $\mathcal{A}$, then the map $\phi(1)^2\phi$ is multiplicative and therefore ϕ has standard forms when $\mathcal{A}$ and $\mathcal{B}$ are the algebras of all bounded linear operators acting on a Hilbert space.

2. Preliminaries

In this section, we collect some properties of the numerical range needed in the sequel. Let two unital C^*-algebras $\mathcal{A} \subset B(H)$ and $\mathcal{B} \subset B(K)$ be given. By $\mathrm{Sp}(a)$ (resp. $r(a)$) we denote the spectrum (resp. the spectral radius) of an element $a \in \mathcal{A}$. Since it does not lead to misunderstanding, we shall denote the norms in both algebras by the same symbols $\|\cdot\|$. We denote by $\mathcal{H}(\mathcal{A})$ the set of self adjoint elements defined by $\mathcal{H}(\mathcal{A}) = \{a \in \mathcal{A} : a = a^*\}$. It is well-known that $a \in \mathcal{H}(\mathcal{A})$ if and only if $W(a) \subset \mathbb{R}$. Further, an element $a \in \mathcal{A}$ is positive if and only if $W(a) \subset \mathbb{R}_+$ (or equivalently $a = a^*$ and $\mathrm{Sp}(a) \subset \mathbb{R}_+$), where $\mathbb{R}_+$ denotes the set of positive real numbers. In the case where $\mathcal{A} = C(K)$ for some Hausdorff compact space K we have $W(a) \subset V(a) = \mathrm{co}(a(K))$ for each $a \in C(K)$, see [16, Theorem 6]. Here co stands for closed convex hull. We summarize some other basic properties of the numerical range on the following lemma. One may see [2, 8] for more information.

Lemma 2.1.

(i) $\|a\| = w(a) = r(a)$ *for every* $a \in \mathcal{A}$ *such that* $aa^* = a^*a$.
(ii) $W(a) = \{\lambda\} \Longleftrightarrow a = \lambda 1$, *for every* $a \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$.

Finally, recall that a linear map $\psi : \mathcal{A} \to \mathcal{B}$ is called *unital* if $\psi(1) = 1$, and it is said to be a *Jordan homomorphism* if $\psi(a^2) = \psi(a)^2$ for all $a \in \mathcal{A}$. Equivalently, the map ψ is a Jordan homomorphism if and only if $\psi(ab+ba) = \psi(a)\psi(b) + \psi(b)\psi(a)$ for all a and b in $\mathcal{A}$. We also recall that the map ψ is said to be *self-adjoint* provided that $\psi(a^*) = \psi(a)^*$ for all $a \in \mathcal{A}$. Self-adjoint Jordan homomorphisms are called *Jordan $*$-homomorphisms*, and by a Jordan $*$-isomorphism, we mean a bijective $*$-homomorphism.

3. Main results and proofs

We start with the following introductory results, which may be of independent interest. We give a characterization of elements $a, b \in \mathcal{A}$ satisfying $W(ax) = W(bx), \forall x \in \mathcal{A}$ or $w(ax) = w(bx), \forall x \in \mathcal{H}(\mathcal{A})$. It is worth observing that the authors in [3] have recently considered the question whether the equality $\mathrm{Sp}(ax) = \mathrm{Sp}(bx)$ for every $x \in \mathcal{A}$, where $a, b \in \mathcal{A}$ are fixed elements, implies $a = b$. An affirmative answer has been obtained for some classes of algebras, including C^*-algebras.

We begin with the following proposition which gives necessary conditions which ensure that $a = b$ if $w(ax) = w(bx), \forall x \in \mathcal{H}(\mathcal{A})$. The argument of the proof is borrowed from [11, Lemma 3.4] by slight some modifications. We present it here for the sake of completeness.

Proposition 3.1. *Let $\mathcal{A}$ be an unital C^*-algebra and $a, b \in \mathcal{A}$ be two positive elements such that $ab = ba$. If $w(ax) = w(bx)$ for every $x \in \mathcal{H}(\mathcal{A})$, then $a = b$.*

Proof. Let $\mathcal{B}$ be the unital C^*-algebra $\mathcal{B}$ generated by a and b. Since, $ab = ba$, this algebra is commutative. Henceforth, without loss of generality we may suppose that $\mathcal{A}$ is a commutative C^*-algebra. On the other hand, it is well known that every positive element in a C^*-algebra has unique square root, then to prove that $a = b$, it suffices to show that $a^2 = b^2$. Suppose to the contrary that $a^2 \neq b^2$. Since $a^2 - b^2$ is self-adjoint, there exists a non-zero $\beta \in \mathrm{Sp}(a^2 - b^2)$. We may assume that $\beta > 0$ (otherwise, we could replace $a^2 - b^2$ by $b^2 - a^2$). Let $\alpha = \frac{1}{2} \sup \mathrm{Sp}(a^2 - b^2) > 0$, and consider the real valued continuous function f defined on the spectrum of $a^2 - b^2$ such that $f(2\alpha) = 1, 0 \leq f(\lambda) \leq 1, \forall \lambda \in \mathrm{Sp}(a^2 - b^2)$ and $f(\lambda) = 0 \iff \lambda \leq \alpha$. Put $x_1 = f(a^2 - b^2)$ and $g(\lambda) = \lambda f(\lambda)^2, \forall \lambda \in \mathrm{Sp}(a^2 - b^2)$. So, using functional calculus (see [6, Theorem 2.9]) and the fact that $x_1(a^2 - b^2)x_1$ is self adjoint, we get $w(x_1(a^2 - b^2)x_1) = r(x_1(a^2 - b^2)x_1) = \sup_{\lambda \in \mathrm{Sp}(a^2 - b^2)} |g(\lambda)| = 2\alpha$. In addition by using the same argument, it is easily shown that $x_1(a^2 - b^2)x_1 \geq \alpha x_1^2$. Now for $t \in H$ such that $\|t\| = 1$, let us define the positive linear form φ_t by $\varphi_t(a) = (at, t), \forall a \in \mathcal{A}$. Since $\|b\| x_1^2 - x_1 b^2 x_1$ is positive, we have $\|b\| \varphi_t(x_1^2) \geq \varphi_t(x_1 b^2 x_1) \geq 0$. On account of $x_1(a^2 - b^2)x_1 \geq \alpha x_1^2$, it follows that $\varphi_t(x_1 a^2 x_1) \geq (1 + \frac{\alpha}{\|b\|})\varphi_t(x_1 b^2 x_1)$. Since $0 \leq \varphi_t(x_1 a^2 x_1) \leq w(x_1 a^2 x_1)$, we infer that $w(x_1 a^2 x_1) \geq (1 + \frac{\alpha}{\|b\|}) w(x_1 b^2 x_1)$. Accordingly $\|a x_1\| \geq \sqrt{1 + \frac{\alpha}{\|b\|}} \|b x_1\| > \|b x_1\|$. This obviously contradicts the hypothesis of the proposition, since $x_1 \in$

$\mathcal{H}(\mathcal{A})$ and by Lemma 2.1, we have $\|ax_1\| = w(ax_1) = w(bx_1) = \|bx_1\|$. Thus $a = b$ as desired. □

The next two propositions are crucial for the rest of the paper. They give a characterization of elements $a, b \in \mathcal{A}$ satisfying $W(ax) = W(bx)$ for every $x \in \mathcal{A}$ (or $\mathcal{H}(\mathcal{A})$).

Proposition 3.2. *Let $\mathcal{A}$ be an unital C^*-algebra and $a, b \in \mathcal{A}$. If $W(ax) = W(bx), \forall x \in \mathcal{A}$, then $a = b$.*

Proof. Firstly, assume that $a = a^*$. Since $W(b) = W(a) \subset \mathbb{R}$, then $b^* = b$. On the other hand, by observing that $W(a^2) = W(ba)$ and the fact that a^2 is self adjoint, we infer that $(ba)^* = ba$. By taking into account that $(ba)^* = a^*b^*$ and that a and b are self adjoint, we get $ab = ba$. We prove now that $a = b$. Let $\mathcal{B}$ be the C^*-algebra $\mathcal{B}$ generated by a and b. Since a and b are self adjoint and satisfy $ab = ba$, this algebra is commutative. Hence, it can be identified with $C(K)$, the algebra of all continuous functions on a compact Hausdorff space K. Observe also that $W(ax) = W(bx), \forall x \in \mathcal{B}$. We claim that the two functions a and b have the same sign (both positive, or both negative on K). Indeed, assuming that there exists $t_1 \in K$ such that $a(t_1) > 0$ and $b(t_1) < 0$. Therefore there exists an open set U_1 such that $a(t) > 0$ and $b(t) < 0$ for all $t \in U_1$. By Urysohn's lemma, one can find a continuous function $c_1 : K \to [0, 1]$ satisfying $c(t_1) = 1$ and $\operatorname{supp}(c) \subset U_1$. On the other hand $W(ac) \subset \operatorname{co}(ac(K))$ and $W(bc) \subset \operatorname{co}(bc(K))$. Then, we get $W(ac) \subset [0, +\infty)$ and $W(bc) \subset (-\infty, 0]$. Observe that the case where $W(ac) = W(bc) = \{0\}$, does not occur since $ac \neq 0$. Therefore $W(ac) \neq W(bc)$, contrary to our assumption. Thus a and b have the same sign as suggested above. Consequently, without loss of generality, we can suppose that a, b are positive on U. From the condition $W(ax) = W(bx), \forall x \in \mathcal{B}$, we infer that $w(ax) = w(bx), \forall x \in \mathcal{B}$. It follows from Proposition 3.1 that $a = b$. We return now to the general case; i.e., if $a \in \mathcal{A}$ is arbitrary. Observe that, by assumption, we have, $W(aa^*x) = W(ba^*x)$ and $W(ab^*x) = W(bb^*x), \forall x \in \mathcal{A}$. Based on the aforesaid, we infer that $aa^* = ba^*$ and $ab^* = bb^*$. Accordingly, $(a - b)(a^* - b^*) = 0$, which implies that $a = b$. This ends the proof. □

Proposition 3.3. *If $\mathcal{A}$ is a unital C^*-algebra and a and $b \in \mathcal{H}(\mathcal{A})$. If $W(ax) = W(xb), \forall x \in \mathcal{H}(\mathcal{A})$ or $W(ax) = W(bx), \forall x \in \mathcal{H}(\mathcal{A})$, then $a = b$.*

Proof. We give the proof for the condition $W(ax) = W(xb), \forall x \in \mathcal{H}(\mathcal{A})$. For the other condition the proof is similar. By using a similar reasoning as above, we can easily prove that $ab = ba$. Considering the commutative C^*-algebras $\mathcal{B}$ generated by a, b and by taking into account that $W(ax) = W(bx), \forall x \in \mathcal{H}(\mathcal{B})$. By a similar reasoning as in the proof of the above proposition, we can show that a and b are either both positive or negative. We infer by means of Proposition 3.1 that $a = b$. □

Remark 3.4. The results of Propositions 3.2 and 3.3 are still valid if we replace the numerical range by its closure. That is if two elements a and b satisfy $V(ax) = V(bx)$ for every $x \in \mathcal{A}$ (or in $\mathcal{H}(\mathcal{A})$), then a similar argument can be used to show that $a = b$.

At this juncture, we are in a position to characterize surjective mapping satisfying

$$W(\phi(a)\phi(b)\phi(c)) = W(abc) \text{ for all } a, b \text{ and } c \in \mathcal{A}, \tag{2a}$$

$$W(\phi(a)\phi(b)\phi(c)) = W(abc) \text{ for all } a, b \text{ and } c \in \mathcal{H}(\mathcal{A}). \tag{2b}$$

Theorem 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective mapping satisfying* (2a). *Then $\phi(1) \in Z(\mathcal{B})$, $\phi(1)^3 = 1$, and ϕ satisfies $\phi(ab) = \phi(1)^2\phi(a)\phi(b)$ for all a and $b \in \mathcal{A}$. In particular, the mapping $\psi = \phi(1)^2\phi$ is multiplicative and preserves self-adjoint elements* (*i.e., $\psi(a) \in \mathcal{H}(\mathcal{B})$ whenever $a \in \mathcal{H}(\mathcal{A})$*).

Proof. Set $u = \phi(1)$. Take $a = b = c = 1$ in (2a), we obtain $W(u^3) = W(1) = \{1\}$. Thus $u^3 = 1$ and hence u is invertible. Given $a, b \in \mathcal{A}$ such that $\phi(a) = \phi(b)$. By (2a), we have $W(ac) = W(u\phi(a)\phi(c)) = W(u\phi(b)\phi(c)) = W(bc)$ for every $c \in \mathcal{A}$. By Proposition 3.2, we infer that $a = b$ and ϕ is bijective as desired. Also, we have $W(u\phi(a)\phi(b)) = W(1ab) = W(a1b) = W(\phi(a)u\phi(b))$. Thus we get $W(u\phi(a)\phi(b)) = W(\phi(a)u\phi(b)), \forall b \in \mathcal{A}$. Since ϕ is a bijection and on account of Proposition 3.2, we get $u\phi(a) = \phi(a)u$, $\forall a \in \mathcal{A}$. That is to say that $u \in Z(\mathcal{B})$. To end the proof, observe that

$$\begin{aligned} W(\phi(a)\phi(b)\phi(c)) &= W(abc) = W(1(ab)c) \\ &= W(u\phi(ab)\phi(c)), \forall c \in \mathcal{A}. \end{aligned}$$

By recalling that ϕ is bijective, again Proposition 3.2 implies that $\phi(ab) = u^{-1}\phi(a)\phi(b) = u^2\phi(a)\phi(b)$. Now, put $\psi = u^2\phi$. We have

$$\psi(a)\psi(b) = u^2\phi(a)u^2\phi(b) = u\phi(a)\phi(b) = u^2\phi(ab) = \psi(ab).$$

Finally, observe that ϕ preserves self-adjoint elements because for every self adjoint element $a \in \mathcal{A}$, we have $W(\psi(a)) = W(\phi(1)^2\phi(a)) = W(a) \subset \mathbb{R}$. The proof is thus complete. $\square$

Corollary 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective mapping satisfying*

$$W(\phi(a_1)\phi(a_2)\cdots\phi(a_p)) = W(a_1a_2\cdots a_p) \text{ for all } a_1a_2\cdots a_p \in \mathcal{A} \tag{3}$$

for some integer $p \in \mathbb{N}$ with $p \geq 3$. Then $\phi(1) \in Z(\mathcal{B})$, $\phi(1)^p = 1$ and $\phi(1)^{p-1}\phi(a_1a_2\cdots a_p) = \phi(a_1)\phi(a_2)\cdots\phi(a_p)$ for all $a_1a_2\cdots a_p \in \mathcal{A}$.

Proof. It is obvious that $\phi(1)^p = 1$. If $\phi(a) = \phi(b)$ for some $a, b \in \mathcal{A}$. Since $W(\phi(a)\phi(x)\phi(1)^{p-2}) = W(\phi(b)\phi(x)\phi(1)^{p-2})$, by (3), it yields that $W(ax) =$

$W(bx)$ for every $x \in \mathcal{A}$. By Proposition 3.2, we get $a = b$. Hence ϕ is a bijection. Define $\psi = \phi(1)^{p-1}\phi$. We see that $\psi(1) = 1$ and

$$\psi(a)\psi(b)\psi(c) = \phi(1)^{p-1}\phi(a)\phi(1)^{p-1}\phi(b)\phi(1)^{p-1}\phi(c) = \phi(1)^{p-3}\phi(a)\phi(b)\phi(c).$$

From (3), we can deduce that $W(\psi(a)\psi(b)\psi(c)) = W(abc)$, $\forall a, b, c \in \mathcal{A}$. By Theorem 3.5, the results follows. $\square$

Based, on Theorem 3.5, we know that if ϕ satisfy (2a), then the mapping $\psi = \phi(1)^2\phi$ is multiplicative. The question of when a multiplicative map is additive was attacked by several authors. For instance, if ψ is a bijective map on a standard operator algebra, Molnàr showed in [14] that if ϕ satisfies $\psi(ABA) = \psi(A)\psi(B)\psi(A)$, then ψ is additive. Hence, based on the aforesaid, when the algebras $\mathcal{A}$ and $\mathcal{B}$ are the algebras of all bounded linear operators acting on some Hilbert spaces, Theorem 3.5 can be refined as follows.

Corollary 3.7. *Let H and K be complex Hilbert spaces and let $\phi : B(H) \to B(K)$ be a surjective map (without the assumption of additivity). Then ϕ satisfies Eq.* (2a) *if and only if there is a unitary operator $U : H \to K$ such that ϕ is of the form $\phi(A) = \varepsilon UAU^*$ for all $A \in B(H)$, where $\varepsilon^3 = 1$.*

Proof. Checking the 'if' part is straightforward, and we therefore will only deal with the 'only if' part. Assume that ϕ satisfies (2a). By Theorem 3.5, we have that $\phi(1) \in Z(B(K))$ and $\phi(1)^3 = 1$. Since the algebra $B(K)$ has a trivial center, then $u = \phi(1) = \varepsilon.1$, where ε is a complex number such that $\varepsilon^3 = 1$. Also according to Theorem 3.5, the map $\psi = u^2\phi$, is multiplicative and $\psi(1) = 1$. Consequently, by [14] it is additive. Finally, we have shown that ψ is an algebra isomorphism which preserves self-adjoint elements. Thus, by [4] ψ takes the following form: $\psi(A) = UAU^*$ for all $A \in B(H)$ where U is unitary. $\square$

For mapping $\phi : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{B})$ satisfying (2b), we have a similar result which follows.

Theorem 3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. Let $\phi : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{B})$ be a surjective mapping satisfying* (2b)*. Then $\phi(1) \in Z(\mathcal{H}(\mathcal{B}))$, $\phi(1)^3 = 1$, and $\phi = \phi(1)\psi$, where ψ is multiplicative.*

Proof. The proof is similar to that of Theorem 3.5 by invoking Proposition 3.3. The details are omitted. $\square$

As a special case of Theorem 3.8 we derive the following result.

Theorem 3.9. *Consider the case where $\mathcal{A} = B(H)$ and $\mathcal{B} = B(K)$ for some complex Hilbert spaces H and K. Let $\phi : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{B})$ be a surjective map. Then, ϕ satisfies* (2b) *if and only if there exists a unitary or conjugate unitary operator U and $\varepsilon \in \mathbb{C}$ such that $\phi(A) = \varepsilon UAU^*$ for all $A \in \mathcal{H}(\mathcal{A})$ and $\varepsilon^3 = 1$.*

Proof. The sufficiency is easy to see. Indeed, this follows from the well-known fact that if U is a unitary or conjugate unitary operator, then $W(UAU^*) = W(A)$ for every $A \in \mathcal{A}$. Conversely, suppose that ϕ satisfies Eq. (2b) for every $A \in \mathcal{H}(\mathcal{A})$. Theorem 3.8, implies that $\psi = \phi(1)^2\phi$ is multiplicative and $\psi(1) = 1$. Therefore, by [1, Theorem 2.1] there exists a unitary or conjugate unitary operator U such that $\psi(A) = UAU^*$ for all $A \in \mathcal{H}(\mathcal{A})$. To end the proof observe that $\phi = u\psi = \phi(1)\psi$ and in particular ϕ is linear. Since, by Theorem 3.8 we have $\phi(1) \in Z(\mathcal{H}(\mathcal{B}))$, we infer that $\phi(1) \in Z(\mathcal{B})$. Therefore, $\phi(1) = \varepsilon.1$, where ε is a complex number such that $\varepsilon^3 = 1$. Thus, completing the proof. □

We give now the following theorem which characterizes surjective maps satisfying (1a) in the case of C^*-algebras. This result has been also proved in [7, Theorem 2.1.] for the Hilbert space operators case but without the extra condition that ϕ is additive. It would be interesting to remove the additive assumption in Theorem 3.10 below. We are not able to do that at present.

Theorem 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective and additive map satisfying* (1a). *Then ϕ is a Jordan $*$-isomorphism followed by a left multiplication by a fixed element $u \in Z(\mathcal{B})$ with $u^2 = 1$, where $Z(\mathcal{B})$ stands for the center of $\mathcal{B}$.*

Proof. Firstly, we prove that ϕ is bijective. It suffice to show that it is injective. Let $a, b \in \mathcal{A}$ such that $\phi(a) = \phi(b)$. By (1a), we have $W(\phi(a)\phi(c)) = W(ac) = W(\phi(b)\phi(c)) = W(bc), \forall c \in \mathcal{A}$. By Proposition 3.2, it yields that $a = b$. Hence we have proved that ϕ is injective. Take $a = b = 1$ in (1a), we obtain $W(\phi(1)^2) = W(1) = \{1\}$. Whence $\phi(1)$ is invertible and $\phi(1)^2 = 1$. We show now that ϕ is linear. Let $\lambda \in \mathbb{C}$. By (1a) we have

$$\begin{aligned} W(\lambda\phi(a)\phi(b)) &= \lambda W(\phi(a)\phi(b)) = \lambda W(ab) \\ &= W(\lambda a)b) = W(\phi(\lambda a)\phi(b)), \forall a, b \in \mathcal{A}. \end{aligned}$$

Whence, by Proposition 3.2, it yields that $\phi(\lambda a) = \lambda\phi(a), \forall a \in \mathcal{A}$. Since ϕ is additive, we infer that ϕ is a linear bijection. Now, take $a, b \in \mathcal{A}$ such that $ab = 0$. By (1a), yields that $\phi(a)\phi(b) = 0$. Hence [5, Lemma 4.4], implies that $\phi(1)\phi(a) = \phi(a)\phi(1), \forall a \in \mathcal{A}$. Together with the bijectivity of ϕ, this implies that $\phi(1) \in Z(\mathcal{B})$.

Finally, we show that ϕ has the asserted form. Set $\psi = u\phi$. It suffices to show that ψ is C^*-isomorphism. It is obvious that $\psi(1) = u^2 = 1$ and $W(\psi(a)\psi(b)) = W(ab), \forall a, b \in \mathcal{A}$. Thus, we have proved that ψ is a linear isomorphism satisfying $W(\psi(a)) = W(a)$ for every $a \in \mathcal{A}$. By [15, Theorem 3.1], the result follows. □

Finally, we turn to the second type of preserver problems involving involution. We have the following result.

Theorem 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective and additive map satisfying* (1b). *Then, $\phi(1)$ is unitary and $\phi = \phi(1)\psi$, where ψ is a Jordan $*$-isomorphism.*

Proof. Firstly, observe that by (1b), we have

$$\|\phi(a)\|^2 = \|\phi(a)\phi(a)^*\| = w(\phi(a)\phi(a)^*) = w(aa^*) = \|aa^*\|^2, \forall a \in \mathcal{A}.$$

Taking the square root, we obtain $\|\phi(a)\| = \|a\|$, which yields that ϕ is an isometry and hence a bijection. Now, let $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$. For all $b \in \mathcal{A}$, we have

$$\begin{aligned} W\left((\lambda\phi(a))^*\phi(b)\right) &= \overline{\lambda}W\left(\phi(a)^*\phi(b)\right) = \overline{\lambda}W(a^*b) \\ &= W\left((\lambda a)^*b\right) = W\left((\phi(\lambda a))^*\phi(b)\right). \end{aligned}$$

Using Proposition 3.2, we infer that $(\lambda\phi(a))^* = (\phi(\lambda a))^*$. Accordingly, $\lambda\phi(a) = \phi(\lambda a)$. In consequence of this, ϕ is a linear bijection. Thus, we have proved that ϕ is a linear isomorphism between two C^*-algebras which are isometric. By [10, Theorem 7], ϕ is a Jordan $*$-isomorphism followed by left multiplication by the fixed unitary operator $\phi(1)$. $\square$

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