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FEKETE-SZEGÖ PROBLEM FOR CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract. For $1 \le \alpha < 2$, let $\mathcal{F}(\alpha)$ denote the class of locally univalent normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{\alpha}{2} - 1.$$

In the present paper, we shall obtain the sharp upper bound for Fekete-Szegö functional $|a_3 - \lambda a_2^2|$ for the complex parameter λ.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{H}$ denote the class of analytic functions in $\mathbb{D}$. Here we think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{B}$ denote the class of analytic functions $\omega : \mathbb{D} \to \mathbb{D}$ of the form

$$\omega(z) = \sum_{k=0}^{\infty} c_k z^k. \tag{1.1}$$

Then in view of Schwarz-Pick lemma, it is well-known that

$$|c_0| \le 1 \quad \text{and} \quad |c_1| \le 1 - |c_0|^2. \tag{1.2}$$

Let $\mathcal{A}$ denote the family of functions f in $\mathcal{H}$ normalized by $f(0) = 0$ and $f'(0) = 1$. A function f is said to be univalent in $\mathbb{D}$ if it is one-to-one in $\mathbb{D}$. Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. A domain $0 \in \Omega \subseteq \mathbb{C}$ is said to be starlike domain if the line segment joining 0 to any point in Ω must lie in Ω. A function $f \in \mathcal{A}$ is said to be starlike function if f maps $\mathbb{D}$ onto a domain $f(\mathbb{D})$ which is starlike with respect to the origin. We denote the class

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of univalent starlike functions in $\mathcal{A}$ by $\mathcal{S}^*$. It is well-known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A domain $\Omega \subseteq \mathbb{C}$ is said to be convex if it is starlike with respect to every point in Ω. A function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. We denote the class of convex univalent functions in $\mathbb{D}$ by $\mathcal{C}$. A function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

It is well-known that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a convex univalent function g and a number $\phi \in \mathbb{R}$ such that

$$\mathrm{Re}\left(e^{i\phi}\frac{f'(z)}{g'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

Let $\mathcal{K}$ denote the class of close-to-convex functions f in $\mathcal{A}$. It is well-known that every close-to-convex function is univalent in $\mathbb{D}$. For the basic properties of functions in these subclasses, we refer to [4].

2. Preliminaries

For $1 \le \alpha < 2$, let $\mathcal{F}(\alpha)$ denote the class of locally univalent functions f in the class $\mathcal{A}$ satisfying the condition

$$\mathrm{Re}\,(P_f(z)) > \frac{\alpha}{2} - 1 \quad \text{for } z \in \mathbb{D},$$

where

$$P_f(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

In particular by taking $\alpha = 1$, the class $\mathcal{F}(\alpha)$ reduces to the following class

$$\mathcal{F} := \mathcal{F}(1) = \left\{ f \in \mathcal{A} :\ \mathrm{Re}\,P_f(z) > -\frac{1}{2} \quad \text{for } z \in \mathbb{D} \right\}.$$

It is known that $\mathcal{F} \subseteq \mathcal{K}$ (see [20, 21]). In [22], the region of variability for functions in the class $\mathcal{F}$ has been studied. The sharp arclength for functions in $\mathcal{F}(\alpha)$ has been investigated in [23]. For the recent investigation about the class $\mathcal{F}$ we refer to [16]. For a detailed discussion about these classes we refer to [20, 21].

For a locally univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$, the quantities

$$T_f(z) := \frac{f''(z)}{f'(z)} \quad \text{and} \quad S_f(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

represent the pre-Schwarzian and Schwarzian derivative respectively (see [9, 17]). The quantity $a_3 - a_2^2$ represents $\frac{1}{6}S_f(0)$. For a real (or more generally, a complex) number λ, the coefficient functional $\phi_\lambda(f) = a_3 - \lambda a_2^2$ for functions f in the class $\mathcal{A}$ plays vital role in the theory of univalent functions. For instance, maximizing $|\phi_\lambda(f)|$ over the class $\mathcal{S}$ or on its subclasses is called the Fekete-Szegö problem. By means of Loewner's method, M. Fekete and G. Szegö [5] obtained the following estimate

$$\left|a_3 - \lambda a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{2.1}$$

for $f \in \mathcal{S}$ in the case that λ is a real parameter, $0 \leq \lambda < 1$. Although Koebe function $z/(1-z)^2$ is an extremal for many problems in the class $\mathcal{S}$ and various subclasses of $\mathcal{S}$, it is interesting to see that Koebe function fails to be an extremal for Fekete-Szegö problem for $0 < \lambda < 1$. The inequality (2.1) is sharp in the following sense that for each λ in $[0, 1)$, there exists a function $f \in \mathcal{S}$ for which equality holds.

In 1987, W. Koepf [10] solved Fekete-Szegö problem for functions that are close-to-convex. In the same year, Koepf [11] generalized Fekete-Szegö problem for functions that are close-to-convex of order β. In 1985, Pfluger [18] employed the variational method to give yet another treatment of the Fekete-Szegö inequality, including a description of the image domains under extremal functions. In 1986, Jenkins [7] obtained Fekete-Szegö inequality (2.1) by means of his general coefficient theorem [6]. Using Jenkins' method, Pfluger [19] has shown that

$$\left|a_3 - \lambda a_2^2\right| \leq 1 + 2\left|\exp\left(\frac{-2\lambda}{1-\lambda}\right)\right| \tag{2.2}$$

holds for complex λ in the unit disk $\mathbb{D}$ with $\operatorname{Re}\left(\frac{1}{1-\lambda}\right) \geq 1$. The interesting fact is that the inequality (2.2) is sharp if and only if λ is in a certain "pear" shaped subdomain of the disk given by

$$\lambda = 1 - \frac{(u + i\theta v)}{(u^2 + v^2)}, \quad -1 \leq \theta \leq 1$$

where

$$u = 1 - \log(\cos\tau) \quad \text{and} \quad v = \tan\tau - \tau, \quad 0 < \tau < \pi/2.$$

Ma and Minda [13, 14, 15] gave a complete answer to the Fekete-Szegö problem for the classes of strongly close-to-convex functions and strongly starlike functions. Motivated by the work of Ma and Minda (see [14]), a new method for solving the Fekete-Szegö problem for classes of close-to-convex functions defined in terms of subordination has been investigated by Choi, Kim and Sugawa [3]. Fekete-Szegö problem has long and rich history in the literature (see for instance, [1, 2, 8, 12]).

In the present paper, we shall solve the Fekete-Szegö problem for functions in the class $\mathcal{F}(\alpha)$ with complex parameter λ. In particular (by taking $\alpha = 1$) we shall obtain the sharp bound for the Fekete-Szegö functional for functions in the class $\mathcal{F}$.

3. Main results

Theorem 3.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}(\alpha)$ *and* $\lambda \in \mathbb{C}$. *Then we have*

$$\tag{3.2} |a_3 - \lambda {a_2}^2| \leq \begin{cases} \frac{1}{6}(4-\alpha)\left|1 + \frac{1}{2}(4-\alpha)(2-3\lambda)\right| & \text{for } \left|\lambda - \frac{10-2\alpha}{3(4-\alpha)}\right| \geq \frac{2}{3(4-\alpha)} \\ \frac{1}{6}(4-\alpha) & \text{for } \left|\lambda - \frac{10-2\alpha}{3(4-\alpha)}\right| < \frac{2}{3(4-\alpha)}. \end{cases}$$

The inequality (3.2) *is sharp.*

Proof. Let $f \in \mathcal{F}(\alpha)$ be given by

$$\tag{3.3} f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Then Re $(1 + zf''(z)/f'(z)) > \frac{\alpha}{2} - 1$ for $z \in \mathbb{D}$, which is equivalent to

$$\mathrm{Re}\left(2 - \frac{\alpha}{2} + z\frac{f''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

Let

$$\tag{3.4} \widetilde{P}_f(z) = \frac{\left(2 - \frac{\alpha}{2} + z\frac{f''(z)}{f'(z)}\right)}{(2 - \frac{\alpha}{2})} \quad \text{for } z \in \mathbb{D}.$$

Then clearly $\widetilde{P}_f(0) = 1$ and $\mathrm{Re}\,\widetilde{P}_f(z) > 0$ in $\mathbb{D}$. Therefore there exists an analytic function $\omega \in \mathcal{B}$ of the form (1.1) such that

$$\tag{3.5} \widetilde{P}_f(z) = \frac{1 + z\omega(z)}{1 - z\omega(z)}.$$

From (3.4) and (3.5) we see that

$$\left(2 - \frac{\alpha}{2} + z\frac{f''(z)}{f'(z)}\right)(1 - z\omega(z)) = (1 + z\omega(z))(2 - \alpha/2),$$

which is equivalent to

$$\tag{3.6} zf''(z)(1 - z\omega(z)) = (4-\alpha)z\omega(z)f'(z).$$

By simplifying (3.6) using the series representations in (1.1) and (3.3) we obtain

$$\tag{3.7} \begin{cases} zf''(z)(1 - z\omega(z)) = 2a_2 z + (6a_3 - 2a_2c_0)z^2 + \cdots \text{ and} \\ (4-\alpha)z\omega(z)f'(z) = (4-\alpha)c_0 z + (4-\alpha)(2a_2c_0 + c_1)z^2 + \cdots . \end{cases}$$

On comparing the coefficients of z and z^2 using (3.6) and (3.7), we obtain $a_2 = \frac{1}{2}(4-\alpha)c_0$ and

$$\tag{3.8} 6a_3 = 2a_2c_0 + (4-\alpha)(2a_2c_0 + c_1).$$

Using the representation of $a_2 = \frac{1}{2}(4-\alpha)c_0$ in (3.8), we obtain

$$a_3 = \frac{1}{6}(4-\alpha)\left((5-\alpha)c_0^2 + c_1\right).$$

Thus we consider

$$\begin{aligned}
a_3 - \lambda a_2^2 &= \frac{(4-\alpha)}{6}\left((5-\alpha)c_0^2 + c_1\right) - \frac{\lambda}{4}(4-\alpha)^2 c_0^2 \\
&= \frac{(4-\alpha)}{6}\left(c_1 + \left((5-\alpha) - \frac{3}{2}\lambda(4-\alpha)\right)c_0^2\right) \\
&= \frac{(4-\alpha)}{6}\left(c_1 + \left(1 + \frac{1}{2}(4-\alpha)(2-3\lambda)\right)c_0^2\right) \\
&= \frac{(4-\alpha)}{6}\left(c_1 + \gamma(\alpha)c_0^2\right),
\end{aligned} \tag{3.9}$$

where $\gamma(\alpha) = 1 + \frac{1}{2}(4-\alpha)(2-3\lambda)$. By applying the inequalities (1.2) in (3.9), we see that

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| &\le \frac{(4-\alpha)}{6}\left(|c_1| + |\gamma(\alpha)||c_0|^2\right) \\
&\le \frac{(4-\alpha)}{6}\left(1 - |c_0|^2 + |\gamma(\alpha)||c_0|^2\right) \\
&\le \frac{(4-\alpha)}{6}\left(1 + (|\gamma(\alpha)| - 1)|c_0|^2\right).
\end{aligned}$$

Hence we obtain

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} \frac{1}{6}(4-\alpha)|\gamma(\alpha)| & \text{for } |\gamma(\alpha)| \ge 1 \\ \frac{1}{6}(4-\alpha) & \text{for } |\gamma(\alpha)| < 1. \end{cases} \tag{3.10}$$

We note that $|\gamma(\alpha)| < 1$ if and only if $\left|\lambda - \frac{10-2\alpha}{3(4-\alpha)}\right| < \frac{2}{3(4-\alpha)}$. Finally, (3.10) reduces to

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} \frac{1}{6}(4-\alpha)\left|1 + \frac{1}{2}(4-\alpha)(2-3\lambda)\right| & \text{for } \left|\lambda - \frac{10-2\alpha}{3(4-\alpha)}\right| \ge \frac{2}{3(4-\alpha)} \\ \\ \frac{1}{6}(4-\alpha) & \text{for } \left|\lambda - \frac{10-2\alpha}{3(4-\alpha)}\right| < \frac{2}{3(4-\alpha)}. \end{cases}$$

To show the first inequality in (3.2) is sharp, we consider

$$\begin{aligned}
f_0(z) &= \frac{1}{3-\alpha}\left(\frac{1}{(1-z)^{(3-\alpha)}} - 1\right) \\
&= z + \frac{1}{2}(4-\alpha)z^2 + \frac{1}{6}(20 - 9\alpha + \alpha^2)z^3 + \cdots.
\end{aligned} \tag{3.11}$$

From (3.11), it is easy to see that

$$P_{f_0}(z) = 1 + z\frac{f_0''(z)}{f_0'(z)} = 1 + (4-\alpha)\frac{z}{1-z}.$$

Therefore, $\operatorname{Re} P_{f_0}(z) > \frac{\alpha}{2} - 1$ for $z \in \mathbb{D}$. This shows that $f_0 \in \mathcal{F}(\alpha)$. From (3.11), we have $a_2 := a_2(f_0) = \frac{1}{2}(4-\alpha)$ and $a_3 := a_3(f_0) = \frac{1}{6}(4-\alpha)(5-\alpha)$. A simple computation shows that

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{1}{6}(4-\alpha)(5-\alpha) - \frac{\lambda}{4}(4-\alpha)^2 \\ &= \frac{(4-\alpha)}{6}\left(5-\alpha-\frac{3}{2}\lambda(4-\alpha)\right) \\ &= \frac{(4-\alpha)}{6}\left(1+\frac{1}{2}(4-\alpha)(2-3\lambda)\right). \end{aligned}$$

This shows that the first inequality in (3.2) is sharp. To show the second inequality in (3.2) is sharp, we consider the function $f_1(z)$ defined by

$$\begin{aligned} f_1'(z) &= \frac{1}{(1-z^2)^{(2-\alpha/2)}} \\ &= 1 + \frac{1}{2}(4-\alpha)z^2 + \frac{1}{8}(24-10\alpha+\alpha^2)z^4 + \cdots. \end{aligned}$$

It is easy to see that $f_1(z) = z + \frac{1}{6}(4-\alpha)z^3 + \frac{1}{40}(24-10\alpha+\alpha^2)z^5 + \cdots$. A simple computation shows that

$$P_{f_1}(z) = 1 + z\frac{f_1''(z)}{f_1'(z)} = 1 + (4-\alpha)\frac{z^2}{1-z^2}.$$

Therefore $f_1 \in \mathcal{F}(\alpha)$ because $\operatorname{Re} P_{f_1}(z) > \frac{\alpha}{2} - 1$ for $z \in \mathbb{D}$. From the series representation of $f_1(z)$, we have $a_2 := a_2(f_1) = 0$ and $a_3 := a_3(f_1) = \frac{1}{6}(4-\alpha)$. This shows that the second inequality in (3.2) is sharp. $\square$

In particular by letting $\alpha = 1$ in Theorem 3.1 we obtain the Fekete-Szegö inequality for functions in the class $\mathcal{F}$.

Corollary 3.12. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}$ and $\lambda \in \mathbb{C}$. Then we have*

$$|a_3 - \lambda {a_2}^2| \le \begin{cases} \left|2 - \frac{9}{4}\lambda\right| & \text{for } \left|\lambda - \frac{8}{9}\right| \ge \frac{2}{9} \\ \frac{1}{2} & \text{for } \left|\lambda - \frac{8}{9}\right| < \frac{2}{9}. \end{cases} \tag{3.13}$$

The inequality (3.13) *is sharp.*

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