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ON A CLASS OF TERNARY CYCLOTOMIC POLYNOMIALS

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Abstract. A cyclotomic polynomial $\Phi_n(x)$ is said to be ternary if $n = pqr$ for three distinct odd primes $p < q < r$. Let $A(n)$ be the largest absolute value of the coefficients of $\Phi_n(x)$. If $A(n) = 1$ we say that $\Phi_n(x)$ is flat. In this paper, we classify all flat ternary cyclotomic polynomials $\Phi_{pqr}(x)$ in the case $q \equiv \pm 1 \pmod p$ and $4r \equiv \pm 1 \pmod{pq}$.

1. Introduction

Let

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (k,n)=1}}^{n} (x - e^{\frac{2\pi i k}{n}}) = \sum_{j=0}^{\phi(n)} a(n,j)x^j$$

be the n-th cyclotomic polynomial, where ϕ is the Euler totient function. It can be shown that $a(n,j) \in \mathbb{Z}$. Let

$$A(n) = \max\{|a(n,j)| \mid 0 \le j \le \phi(n)\}$$

denote the largest absolute value of the coefficients of $\Phi_n(x)$. If $A(n) = 1$ we say that $\Phi_n(x)$ is *flat*. It turns out that for the purpose of determining $A(n)$, it suffices to consider squarefree and odd integers n. Clearly, if n has at most two distinct odd prime factors, then $A(n) = 1$.

However, the coefficients of *ternary* cyclotomic polynomials $\Phi_{pqr}(x)$, where $p < q < r$ are odd primes, become much more complicated, such as $a(3{\cdot}5{\cdot}7, 7) = -2$ and $a(5{\cdot}7{\cdot}11, 119) = -3$. The investigation about the coefficients of ternary cyclotomic polynomials have a long history and there are many references on this subject, see, for instance, [1-17, 19, 20, 21, 23, 24]. One interesting open problem involving this topic is to give a complete characterization of all flat ternary cyclotomic polynomials, but this appears very difficult. Throughout

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this paper, we assume that $p < q < r$ are odd primes (unless otherwise specified).

In 1978, Beiter [5] classified all flat cyclotomic polynomials of the form $\Phi_{3qr}(x)$. More precisely,

Proposition 1.1 (Beiter)**.** *Let* $3 < q < r$ *be primes such that* $r = (wq \pm 1)/h$, $1 < h \leq (q-1)/2$. *Then* $A(3qr) = 1$ *if and only if one of these conditions holds*: (1) $w \equiv 0$ *and* $h + q \equiv 0 \pmod 3$ *or* (2) $h \equiv 0$ *and* $w + r \equiv 0 \pmod 3$.

Currently, we know several families of flat ternary cyclotomic polynomials. In 2006, Bachman [2] showed that

$$A(pqr) = 1 \text{ if } p \geq 5,\ q \equiv -1 \pmod p \text{ and } r \equiv 1 \pmod{pq}. \tag{1.1}$$

This first established the existence of an infinite family of flat ternary cyclotomic polynomials. A generalization of (1.1) was later obtained by Flanagan [12] who showed $A(pqr) = 1$ if $p \geq 5$, $q \equiv \pm 1 \pmod p$ and $r \equiv \pm 1 \pmod{pq}$. In 2007, Kaplan [17] improved on these results by proving that

$$A(pqr) = 1 \text{ if } r \equiv \pm 1 \pmod{pq}. \tag{1.2}$$

In 2012, Elder [11] (arXiv:1207.5811v1) reproved (1.2) and derived the following result: Let $p < q < r$ be odd primes and w a positive integer such that $r \equiv \pm w \pmod{pq}$, $p \equiv 1 \pmod w$ and $q \equiv 1 \pmod{wp}$. Then $A(pqr) = 1$.

In 2010, Ji [16] showed that in the case $2r \equiv \pm 1 \pmod{pq}$, $A(pqr) = 1$ if and only if $p = 3$ and $q \equiv 1 \pmod 3$.

In this paper, we classify all flat ternary cyclotomic polynomials $\Phi_{pqr}(x)$ in the case $q \equiv \pm 1 \pmod p$ and $4r \equiv \pm 1 \pmod{pq}$. That is,

Theorem 1.2. *Let* $p < q < r$ *be odd primes such that* $q \equiv \pm 1 \pmod p$ *and* $4r \equiv \pm 1 \pmod{pq}$. *Then* $A(pqr) = 1$ *if and only if one of these conditions holds*:

(1) $p = 3$, $q > 7$ *and* $q \equiv -1 \pmod 3$ *or*

(2) $p = 5$, $q > 11$ *and* $q \equiv 1 \pmod 5$.

2. Some lemmas

To prove Theorem 1.2, several lemmas will be useful. First we have

Lemma 2.1. *Let* $p < q$ *be odd primes and* s, t *be positive integers such that* $pq + 1 = ps + qt$. *Put* $\Phi_{pq}(x) = \sum_{j=0}^{\phi(pq)} a(pq, j)x^j$, *then*

$$a(pq, j) = \begin{cases} 1 & \text{if } j = up + vq \text{ for some } 0 \leq u \leq s-1,\ 0 \leq v \leq t-1; \\ -1 & \text{if } j = up + vq + 1 \text{ for some } 0 \leq u \leq q-s-1, \\ & \quad 0 \leq v \leq p-t-1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a proof see e.g. Lam and Leung [18] or Thangadurai [22]. □

The next two lemmas, due to Kaplan [17], play an important role in our proof.

Lemma 2.2 (Kaplan)**.** *Let* $\Phi_m(x) = \sum_{j=0}^{\phi(m)} a(m,j)x^j$ *and* $p < q < r$ *be odd primes. Let* $n \geq 0$ *be an integer and* $f(i)$ *be the unique value* $0 \leq f(i) \leq pq-1$ *such that*

$$rf(i) + i \equiv n \pmod{pq}. \tag{2.1}$$

(1) *Then*

$$\sum_{i=0}^{p-1} a(pq, f(i)) = \sum_{j=q}^{q+p-1} a(pq, f(j)).$$

(2) *Set*

$$a^*(pq, l) = \begin{cases} a(pq, l) & \text{if } rl \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(pqr, n) = \sum_{i=0}^{p-1} a^*(pq, f(i)) - \sum_{j=q}^{q+p-1} a^*(pq, f(j)).$$

Lemma 2.3 (Kaplan)**.** *Let* $p < q < r$ *be odd primes. Then for any prime* $s > q$ *such that* $s \equiv \pm r \pmod{pq}$*,* $A(pqr) = A(pqs)$*.*

3. Proof of Theorem 1.2

By Lemma 2.3, it suffices to consider primes r such that $4r \equiv 1 \pmod{pq}$. The proof will be split into the following three parts.

3.1. $p \geq 7$.

We will use Lemma 2.2 to specify a coefficient $a(pqr, n)$ which has absolute value greater than one. Our first goal here is to show that:

Fact 1. *Let* $7 \leq p < q < r$ *be primes such that* $q \equiv 1 \pmod{p}$ *and* $4r \equiv 1 \pmod{pq}$*.*

(1) *If* $p \equiv 1 \pmod 4$*, then* $a(pqr, pqr + pr - 5qr + q + r + 1) = 2$*.*

(2) *If* $p \equiv 3 \pmod 4$*, then* $a(pqr, pqr - 5qr + q + r + 1) = 2$*.*

Proof. Let $q = kp + 1$. Then $pq + 1 = p \cdot (q - k) + q \cdot 1$ and we can write the conclusion of Lemma 2.1 as

$$a(pq, j) = \begin{cases} 1 & \text{if } j = up \text{ for some } 0 \leq u \leq q-k-1; \\ -1 & \text{if } j = up + vq + 1 \text{ for some } 0 \leq u \leq k-1,\ 0 \leq v \leq p-2; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

(1) Let $n = pqr + pr - 5qr + q + r + 1$. By substituting n into congruence (2.1) and using $0 \leq f(i) \leq pq - 1$, we have

$$f(i) = pq + p - q - 4i + 5 \tag{3.2}$$

for $0 \le i \le p-1$ and $q \le i \le q+p-1$. Then one readily verifies that

$$rf(0) > \cdots > rf(p-1) > rf(q) > n > rf(q+1) > \cdots > rf(q+p-1).$$

Thus it follows from Lemma 2.2(2) that

$$a(pqr,n) = -\sum_{j=1}^{p-1} a(pq, f(q+j)). \tag{3.3}$$

Let $1 \le j \le p-1$. Then $f(q+j) \equiv -4j \not\equiv 0 \pmod p$. Hence, by (3.1), $a(pq, f(q+j)) \ne 1$ and the quantity $a(pq, f(q+j))$ takes on one of two values: 0, -1.

Note that $p \equiv 1 \pmod 4$. Now we claim that $a(pq, f(q+j)) = -1$ if and only if $j = 1$ or $j = \frac{3p+5}{4}$.

If $a(pq, f(q+j)) = -1$, according to (3.1), then there must exist $0 \le u \le k-1$ and $0 \le v \le p-2$ such that

$$f(q+j) = up + vq + 1. \tag{3.4}$$

By using (3.2) and taking (3.4) modulo q, we have

$$up - p + 4j - 4 \equiv 0 \pmod q. \tag{3.5}$$

From $0 \le u \le k-1$ and $1 \le j \le p-1$, we infer that $-q < up-p+4j-4 < 2q$, and thus, by (3.5), $up - p + 4j - 4 = 0$ or q. Since $p \equiv 1 \pmod 4$, we have $j = 1$, if $up - p + 4j - 4 = 0$; and $j = \frac{3p+5}{4}$, if $up - p + 4j - 4 = q$.

Conversely, if $j = 1$, then $f(q+1) = p + (p-5)q + 1$, and thus, by (3.1), $a(pq, f(q+1)) = -1$; if $j = \frac{3p+5}{4}$, then $f(q + \frac{3p+5}{4}) = (k-2)p + (p-6)q + 1$, and, by (3.1) again, $a(pq, f(q + \frac{3p+5}{4})) = -1$, as desired.

Hence, combining our claim with (3.3) gives $a(pqr, n) = 2$.

(2) Let $n = pqr - 5qr + q + r + 1$. Proceeding as before, applying n to congruence (2.1), we have

$$f(i) = pq - q - 4i + 5 \tag{3.6}$$

for $0 \le i \le p-1$ and $q \le i \le q+p-1$. This yields

$$rf(0) > \cdots > rf(p-1) > rf(q) > n > rf(q+1) > \cdots > rf(q+p-1).$$

So, by Lemma 2.2(2),

$$a(pqr,n) = -\sum_{j=1}^{p-1} a(pq, f(q+j)). \tag{3.7}$$

Let $1 \le j \le p-1$. Then $f(q+j) \equiv -4j \not\equiv 0 \pmod p$, and thus, in view of (3.1), $a(pq, f(q+j)) = 0$ or -1.

Note that $p \equiv 3 \pmod 4$. Now we claim that $a(pq, f(q+j)) = -1$ if and only if $j = 1$ or $j = \frac{p+5}{4}$.

If $a(pq, f(q+j)) = -1$, by (3.1), then

$$f(q+j) = up + vq + 1, \tag{3.8}$$

where $0 \le u \le k-1$ and $0 \le v \le p-2$. By using (3.6) and taking (3.8) modulo q, we have

$$up + 4j - 4 \equiv 0 \pmod{q}.$$

Since $0 \le u \le k-1$ and $1 \le j \le p-1$, we deduce that $0 \le up + 4j - 4 < 3q$, and therefore $up + 4j - 4 = 0$, q or $2q$. If $up + 4j - 4 = 2q$, then $2j - 3 \equiv 0 \pmod{p}$ and thus $j = \frac{p+3}{2}$. While this gives $u = 2k - 2$, a contradiction to $0 \le u \le k-1$. It is easy to prove that if $up + 4j - 4 = q$, then $j = \frac{p+5}{4}$; and if $up + 4j - 4 = 0$, then $j = 1$.

On the other hand, if $j = 1$, then $f(q+1) = (p-5)q + 1$, and by (3.1), $a(pq, f(q+1)) = -1$; if $j = \frac{p+5}{4}$, then $f(q + \frac{p+5}{4}) = (k-1)p + (p-6)q + 1$ and thus $a(pq, f(q + \frac{p+5}{4}) = -1$, as claimed.

Consequently, by (3.7), we get $a(pqr, n) = 2$. □

Next we prove that:

Fact 2. *Let $7 \le p < q < r$ be primes such that $q \equiv -1 \pmod{p}$ and $4r \equiv 1 \pmod{pq}$.*

(1) *If $p \equiv 1 \pmod 4$, then $a(pqr, 3qr + q + \frac{3p-3}{4}) = -2$.*

(2) *If $p \equiv 3 \pmod 4$ and $p > 7$, then $a(pqr, pr + 3qr + q + \frac{p-3}{4}) = -2$.*

(3) *If $p = 7$, then $a(7qr, 3qr + 7r + 1) = 2$.*

Proof. Let $q = kp - 1$. Then $pq + 1 = p \cdot k + q \cdot (p-1)$. Similarly, we rewrite the conclusion of Lemma 2.1 in the form

(3.9)
$$a(pq, j) = \begin{cases} 1 & \text{if } j = up + vq \text{ for some } 0 \le u \le k-1,\ 0 \le v \le p-2; \\ -1 & \text{if } j = up + 1 \text{ for some } 0 \le u \le q-k-1; \\ 0 & \text{otherwise.} \end{cases}$$

(1) Note that $p \equiv 1 \pmod 4$. Let $n = 3qr + q + \frac{3p-3}{4}$. By using (2.1) and $0 \le f(i) \le pq - 1$, we have

$$f(i) = 3p + 7q - 4i - 3 \tag{3.10}$$

for $0 \le i \le p-1$ and $q \le i \le q + p - 1$. We then infer that $rf(i) > n$ whenever $i \in \{0, 1, \ldots, p-1\} \cup \{q, q+1, \ldots, q + \frac{3p-7}{4}\}$, and $rf(i) < n$ whenever $i \in \{q + \frac{3p-3}{4}, \ldots, q + p - 1\}$. From Lemma 2.2(2), we derive that

$$a(pqr, n) = -\sum_{j=\frac{3p-3}{4}}^{p-1} a(pq, f(q+j)). \tag{3.11}$$

Since $f(q + \frac{3p-3}{4}) = 3q$ and $f(q + p - 1) = (k-1)p + 2q$, by (3.9), we have $a(pq, f(q + \frac{3p-3}{4})) = a(pq, f(q+p-1)) = 1$. Then Eq. (3.11) becomes

$$a(pqr, n) = -2 - \sum_{j=\frac{3p+1}{4}}^{p-2} a(pq, f(q+j)).$$

Now we claim that $a(pq, f(q+j)) = 0$ for all $\frac{3p+1}{4} \leq j \leq p-2$. Since $f(q+j) \equiv -4j-6 \not\equiv 1 \pmod p$, $a(pq, f(q+j)) \neq -1$ by (3.9). If $a(pq, f(q+j)) = 1$, according to (3.9), there must exist $0 \leq u \leq k-1$ and $0 \leq v \leq p-2$ such that

$$f(q+j) = up + vq. \tag{3.12}$$

By using (3.10) and taking (3.12) modulo q, we infer that

$$up - 3p + 4j + 3 \equiv 0 \pmod q. \tag{3.13}$$

But the conditions $0 \leq u \leq k-1$ and $\frac{3p+1}{4} \leq j \leq p-2$ imply

$$0 < up - 3p + 4j + 3 < q.$$

This is a contradiction to (3.13) and proves our claim. Therefore $a(pqr, n) = -2$.

(2) Note that $p \equiv 3 \pmod 4$. Let $n = pr + 3qr + q + \frac{p-3}{4}$. Using congruence (2.1), we get

$$f(i) = 2p + 7q - 4i - 3, \tag{3.14}$$

where $0 \leq i \leq p-1$ and $q \leq i \leq q+p-1$. It can easily be verified that $rf(i) > n$ whenever $i \in \{0, 1, \ldots, p-1\} \cup \{q, q+1, \ldots, q+\frac{p-7}{4}\}$, and $rf(i) < n$ whenever $i \in \{q+\frac{p-3}{4}, \ldots, q+p-1\}$. In particular, on noting that $f(q+\frac{p-3}{4}) = p+3q$ and $f(q+p-1) = (k-2)p + 2q$, we infer from (3.9) that $a(pq, f(q+\frac{p-3}{4})) = a(pq, f(q+p-1)) = 1$. So, by Lemma 2.2(2),

$$a(pqr, n) = -2 - \sum_{j=\frac{p+1}{4}}^{p-2} a(pq, f(q+j)). \tag{3.15}$$

Now we claim that $a(pq, f(q+j)) = 0$ for all $\frac{p+1}{4} \leq j \leq p-2$. Since $f(q+j) \equiv -4j-6 \not\equiv 1 \pmod p$, by (3.9), we have $a(pq, f(q+j)) \neq -1$. If $a(pq, f(q+j)) = 1$, then there exist $0 \leq u \leq k-1$ and $0 \leq v \leq p-2$ such that $f(q+j) = up+vq$. By using (3.14) and taking this equality modulo q, we have

$$up - 2p + 4j + 3 \equiv 0 \pmod q. \tag{3.16}$$

Due to $0 \leq u \leq k-1$ and $\frac{p+1}{4} \leq j \leq p-2$, we have $-q < up-2p+4j+3 < 2q$, and thus, by (3.16), $up - 2p + 4j + 3 = 0$ or q. It is straightforward to verify that both of these two cases are impossible. Hence we prove our claim and infer that $a(pqr, n) = -2$.

(3) Let $n = 3qr + 7r + 1$. By using $rf(i) + i \equiv n \pmod{7q}$, we obtain $f(i) = 3q - 4i + 11$ for $0 \leq i \leq 6$; and $f(q+j) = 6q - 4j + 11$ for $0 \leq j \leq 6$. So

$$rf(q) > \cdots > rf(q+6) > rf(0) > n > rf(1) > \cdots > rf(6).$$

Note that $f(1) = 7 + 3q$ and $f(6) = (k-2)7 + 2q$. By (3.9), it can easily be checked that $a(7q, f(1)) = a(7q, f(6)) = 1$ and $a(7q, f(i)) = 0$ for $i = 2, 3, 4, 5$. Then it follows from Lemma 2.2(2) that $a(7qr, n) = 2$. □

3.2. $p = 3$.

Let $3 < q < r$ be primes such that $4r \equiv 1 \pmod{3q}$. The aim is to show that $A(3qr) = 1$ if and only if $q > 7$ and $q \equiv -1 \pmod 3$.

Indeed, considering Proposition 1.1 with $h = 4$, we obtain that for $q \geq 11$ and $4r \equiv 1 \pmod{3q}$, $A(3qr) = 1$ if and only if $q \equiv -1 \pmod 3$.

It remains to consider $q = 5$ and $q = 7$. Note that $4 \cdot 19 \equiv 1 \pmod{3 \cdot 5}$ and $4 \cdot 37 \equiv 1 \pmod{3 \cdot 7}$. By using the PARI/GP system or [1], we have $A(3\cdot 5\cdot 19) = A(3\cdot 7\cdot 37) = 2$. In view of Lemma 2.3, we infer that $A(3\cdot 5\cdot r) = 2$ when $4r \equiv 1 \pmod{3 \cdot 5}$ and $A(3 \cdot 7 \cdot r) = 2$ when $4r \equiv 1 \pmod{3 \cdot 7}$.

3.3. $p = 5$.

(1) Let $5 < q < r$ be primes such that $q \equiv -1 \pmod 5$ and $4r \equiv 1 \pmod{5q}$. We will prove

$$a(5qr, 2qr + 3) = 2.$$

Let $n = 2qr + 3$. By using $rf(i) + i \equiv n \pmod{5q}$, we deduce that $f(i) = 2q - 4i + 12$ for $0 \leq i \leq 4$ and $f(q + j) = 3q - 4j + 12$ for $0 \leq j \leq 4$. So

$$rf(q) > \cdots > rf(q+4) > rf(0) > rf(1) > rf(2) > n > rf(3) > rf(4). \tag{3.17}$$

Let $q = 5k - 1$. Then $f(3) = 2q$ and $f(4) = (k - 1)5 + q$. By using Lemma 2.1, we have $a(5q, f(3)) = a(5q, f(4)) = 1$. It follows from Lemma 2.2(2) and (3.17) that $a(5qr, n) = 2$.

(2) Let $5 < q < r$ be primes such that $q \equiv 1 \pmod 5$ and $4r \equiv 1 \pmod{5q}$. The purpose is to show that

$$A(5qr) = \begin{cases} 2 & \text{if } q = 11; \\ 1 & \text{otherwise.} \end{cases} \tag{3.18}$$

Observe that $4 \cdot 179 \equiv 1 \pmod{5 \cdot 11}$. By using the PARI/GP system or [1], we have $A(5 \cdot 11 \cdot 179) = 2$. So, by Lemma 2.3, $A(5 \cdot 11 \cdot r) = 2$ for primes r with $4r \equiv 1 \pmod{5 \cdot 11}$.

Now it remains to show $A(5qr) = 1$ in the case $q > 11$, $q \equiv 1 \pmod 5$ and $4r \equiv 1 \pmod{5q}$. Note that Lemma 2.2 yields

$$a(5qr, n) = \sum_{i=0}^{4} a^*(5q, f(i)) + \sum_{i=q}^{q+4} \Big(- a^*(5q, f(i))\Big), \tag{3.19}$$

where $f(i) \equiv r^{-1}(n - i) \pmod{5q}$, $0 \leq f(i) \leq 5q - 1$, and

$$a^*(5q, f(i)) = \begin{cases} a(5q, f(i)) & \text{if } rf(i) \leq n; \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

Let $s = \frac{4q+1}{5}$ and $t = 1$. Then $5q + 1 = 5s + qt$. Applying Lemma 2.1 for these $p < q$, s, t, we get

$$a(5q, f(i)) = \begin{cases} 1 & \text{if } f(i) \equiv 0 \pmod 5 \text{ and } 0 \le f(i) \le 4q-4; \\ -1 & \text{if } f(i) \equiv 1 \pmod 5 \text{ and } 1 \le f(i) \le q-5; \\ -1 & \text{if } f(i) \equiv 2 \pmod 5 \text{ and } q+1 \le f(i) \le 2q-5; \\ -1 & \text{if } f(i) \equiv 3 \pmod 5 \text{ and } 2q+1 \le f(i) \le 3q-5; \\ -1 & \text{if } f(i) \equiv 4 \pmod 5 \text{ and } 3q+1 \le f(i) \le 4q-5; \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

For simplicity, we will write $a_{f(i)} := a(5q, f(i))$ in the rest of this paper if there are no confusion arising from doing so.

For any given $n \in [0, \phi(5qr)]$, the value of $f(i)$ is uniquely defined, since $rf(i) + i \equiv n \pmod{5q}$ and we have

$$\begin{aligned} & & f(q) &\equiv f(0) + q \pmod{5q}, \\ f(1) &\equiv f(0) - 4 \pmod{5q}, & f(q+1) &\equiv f(0) + q - 4 \pmod{5q}, \\ f(2) &\equiv f(0) - 8 \pmod{5q}, & f(q+2) &\equiv f(0) + q - 8 \pmod{5q}, \\ f(3) &\equiv f(0) - 12 \pmod{5q}, & f(q+3) &\equiv f(0) + q - 12 \pmod{5q}, \\ f(4) &\equiv f(0) - 16 \pmod{5q}, & f(q+4) &\equiv f(0) + q - 16 \pmod{5q}. \end{aligned} \tag{3.22}$$

In order to use (3.19) and (3.20), we need to determine for which i, $rf(i) \le n$. Now according to the value of $f(0)$, we give the following tables. The first row of each table is the inequality about $rf(i)$ for $i \in \{0, 1, 2, 3, 4, q, q+1, q+2, q+3, q+4\}$. The values of $a_{f(i)}$ are obtained by using (3.21) and (3.22).

In the following tables, let $\overline{f(0)}$ be the unique integer such that $0 \le \overline{f(0)} \le 4$ and $\overline{f(0)} \equiv f(0) \pmod 5$.

TABLE 1. $0 \le \overline{f(0)} \le 3$

	$rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0)$								
$\overline{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$
0	0	0	0	0	0	0	0	0	−1	1
1	0	0	0	0	1	0	0	−1	1	−1
2	0	0	0	0	0	0	−1	1	0	0
3	0	0	0	0	0	−1	1	0	0	0

TABLE 2. $4 \le f(0) \le 7$

	$rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1)$								
$f(0)$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$
4	0	0	0	−1	0	0	0	0	0	1
5	0	0	0	0	1	0	0	−1	1	−1
6	0	0	0	1	0	0	−1	1	−1	0
7	0	0	0	0	0	−1	1	0	0	0

Table 3. $8 \le f(0) \le 11$

	$rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2)$									
$f(0)$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$
8	0	0	0	−1	0	0	0	0	0	1
9	0	0	−1	0	1	0	0	0	1	−1
10	0	0	0	1	0	0	−1	1	−1	0
11	0	0	1	0	0	−1	1	−1	0	0

Table 4. $12 \le f(0) \le 15$

	$rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3)$									
$f(0)$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$
12	0	0	0	−1	0	0	0	0	0	1
13	0	0	−1	0	1	0	0	0	1	−1
14	0	−1	0	1	0	0	0	1	−1	0
15	0	0	1	0	0	−1	1	−1	0	0

Table 5. $16 \le f(0) \le q-1$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$									
$\overline{f(0)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	1	0	0	−1	1	−1	0	0	0
1	1	0	0	−1	0	−1	0	0	0	1
2	0	0	−1	0	1	0	0	0	1	−1
3	0	−1	0	1	0	0	0	1	−1	0
4	−1	0	1	0	0	0	1	−1	0	0

Table 6. $q \le f(0) \le q+12$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
q	0	0	0	−1	0	0	0	0	0	1
$q+1$	1	0	−1	0	1	−1	0	0	1	−1
$q+2$	0	−1	0	1	0	0	0	1	−1	0
$q+3$	−1	0	1	0	0	0	1	−1	0	0
$q+4$	0	0	0	0	−1	1	0	0	0	0
$q+5$	0	1	0	−1	0	0	−1	0	0	1
$q+6$	1	0	−1	0	1	−1	0	0	1	−1
$q+7$	0	−1	0	1	0	0	0	1	−1	0
$q+8$	−1	0	0	0	0	0	1	0	0	0
$q+9$	0	0	1	0	−1	1	0	−1	0	0
$q+10$	0	1	0	−1	0	0	−1	0	0	1
$q+11$	1	0	−1	0	1	−1	0	0	1	−1
$q+12$	0	−1	0	0	0	0	0	1	0	0

TABLE 7. $q+13 \le f(0) \le 2q-1$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$								
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	1	0	−1	1	0	−1	0	0
1	0	1	0	−1	0	0	−1	0	0	1
2	1	0	−1	0	0	−1	0	0	1	0
3	0	−1	0	0	1	0	0	1	0	−1
4	−1	0	0	1	0	0	1	0	−1	0

TABLE 8. $2q \le f(0) \le 2q+12$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$								
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
$2q$	0	0	−1	0	0	0	0	0	1	0
$2q+1$	1	−1	0	0	1	−1	0	1	0	−1
$2q+2$	−1	0	0	1	0	0	1	0	-1	0
$2q+3$	0	0	1	0	−1	1	0	−1	0	0
$2q+4$	0	0	0	−1	0	0	0	0	0	1
$2q+5$	0	1	−1	0	0	0	−1	0	1	0
$2q+6$	1	−1	0	0	1	−1	0	1	0	−1
$2q+7$	−1	0	0	1	0	0	1	0	−1	0
$2q+8$	0	0	0	0	−1	1	0	0	0	0
$2q+9$	0	0	1	−1	0	0	0	−1	0	1
$2q+10$	0	1	−1	0	0	0	−1	0	1	0
$2q+11$	1	−1	0	0	1	−1	0	1	0	−1
$2q+12$	−1	0	0	0	0	0	1	0	0	0

TABLE 9. $2q+13 \le f(0) \le 3q-1$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$								
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	0	1	−1	1	0	0	−1	0
1	0	0	1	−1	0	0	0	−1	0	1
2	0	1	−1	0	0	0	−1	0	1	0
3	1	−1	0	0	0	−1	0	1	0	0
4	−1	0	0	0	1	0	1	0	0	−1

Table 10. $3q \le f(0) \le 3q+12$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
$3q$	0	−1	0	0	0	0	0	1	0	0
$3q+1$	0	0	0	0	1	−1	1	0	0	−1
$3q+2$	0	0	0	1	−1	1	0	0	−1	0
$3q+3$	0	0	1	−1	0	0	0	−1	0	1
$3q+4$	0	0	−1	0	0	0	0	0	1	0
$3q+5$	0	0	0	0	0	0	−1	1	0	0
$3q+6$	0	0	0	0	1	−1	1	0	0	−1
$3q+7$	0	0	0	1	−1	1	0	0	−1	0
$3q+8$	0	0	0	−1	0	0	0	0	0	1
$3q+9$	0	0	0	0	0	0	0	−1	1	0
$3q+10$	0	0	0	0	0	0	−1	1	0	0
$3q+11$	0	0	0	0	1	−1	1	0	0	−1
$3q+12$	0	0	0	0	−1	1	0	0	0	0

Table 11. $3q+13 \le f(0) \le 4q-1$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$									
$\overline{f(0)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	0	0	0	1	0	0	0	−1
1	0	0	0	0	0	0	0	0	−1	1
2	0	0	0	0	0	0	0	−1	1	0
3	0	0	0	0	0	0	−1	1	0	0
4	0	0	0	0	0	−1	1	0	0	0

Table 12. $4q \le f(0) \le 4q+3$

	$rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q)$									
$f(0)$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$
$4q$	0	0	0	0	0	1	0	0	0	−1
$4q+1$	0	0	0	0	0	0	0	0	−1	1
$4q+2$	0	0	0	0	0	0	0	−1	1	0
$4q+3$	0	0	0	0	0	0	−1	1	0	0

Table 13. $4q+4 \le f(0) \le 4q+7$

	$rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1)$									
$f(0)$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$
$4q+4$	0	0	0	0	0	1	0	0	0	−1
$4q+5$	0	0	0	0	0	0	0	0	−1	1
$4q+6$	0	0	0	0	0	0	0	−1	1	0
$4q+7$	0	0	0	0	0	0	−1	1	0	0

Table 14. $4q+8 \le f(0) \le 4q+11$

	$rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2)$									
$f(0)$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$
$4q+8$	0	0	0	0	0	1	0	0	0	-1
$4q+9$	0	0	0	0	0	0	0	0	-1	1
$4q+10$	0	0	0	0	0	0	0	-1	1	0
$4q+11$	0	0	0	0	0	0	-1	1	0	0

Table 15. $4q+12 \le f(0) \le 4q+15$

	$rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3)$									
$f(0)$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$
$4q+12$	0	0	0	0	0	1	0	0	0	-1
$4q+13$	0	0	0	0	0	0	0	0	-1	1
$4q+14$	0	0	0	0	0	0	0	-1	1	0
$4q+15$	0	0	0	0	0	0	-1	1	0	0

Table 16. $4q+16 \le f(0) \le 5q-1$

	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$\overline{f(0)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
0	0	0	0	0	0	1	0	0	0	-1
1	0	0	0	0	0	0	0	0	-1	1
2	0	0	0	0	0	0	0	-1	1	0
3	0	0	0	0	0	0	-1	1	0	0
4	0	0	0	0	0	-1	1	0	0	0

Let $\Sigma = \{0, 1, 2, 3, 4, q, q+1, q+2, q+3, q+4\}$.

(I) If $rf(i) > n$ holds for all $i \in \Sigma$, by Lemma 2.2(2), we have $a(5qr, n) = 0$;

(II) If $rf(i) \le n$ holds for all $i \in \Sigma$, by Lemma 2.2, we also obtain $a(5qr, n) = 0$.

Otherwise, there must exist two neighboring symbols $rf(\ell_1)$ and $rf(\ell_2)$ in the first row of the corresponding table such that

$$rf(\ell_1) > n \ge rf(\ell_2).$$

If $0 \le \ell_2 \le 4$ (or $q \le \ell_2 \le q+4$), the value of $a(5qr, n)$ is given by computing the sum of values from $a_{f(\ell_2)}$ (or $-a_{f(\ell_2)}$) to the end of the relevant row. Let us illustrate it with the following examples:

Example 3.1. Let $q = 31$, $r = 349$ and $n = 1396$. Then $a(5qr, n) = 1$.

Proof. Note that $4r \equiv 1 \pmod{5q}$ and $n = 1396$. By using $rf(0) \equiv n \pmod{5q}$, we have $f(0) = 4$. According to (3.22) and Table 2, we obtain

$$rf(2) > rf(3) > rf(4) > rf(q) > \cdots > rf(q+4) > n \ge rf(0) > rf(1),$$

namely, $\ell_1 = q+4$ and $\ell_2 = 0$. Then $a(5qr, n)$ is equal to the sum of the values from $a_{f(0)}$ to the end of the third row in Table 2. That is

$$a(5qr, n) = a_{f(0)} + a_{f(1)} = 0 + 1 = 1. \qquad \square$$

Example 3.2. Let $q = 61$, $r = 229$ and $n = 47009$. Then $a(5qr, n) = -1$.

Proof. It is clear that $4r \equiv 1 \pmod{5q}$. By using $rf(0) \equiv n \pmod{5q}$, we have $f(0) = 156$. So $2q + 13 \le f(0) \le 3q - 1$ and $1 = \overline{f(0)} \equiv f(0) \pmod 5$. According to (3.22) and Table 9, we have

$$rf(q) > rf(q+1) > rf(q+2) > n > rf(q+3) > rf(q+4) > rf(0) > \cdots > rf(4),$$

namely, $\ell_1 = q + 2$ and $\ell_2 = q + 3$. Then $a(5qr, n)$ is equal to the sum of the values from $-a_{f(q+3)}$ to the end of the fourth row in Table 9. So we obtain

$$a(5qr, n) = (-1) + 0 + 0 + 0 + (-1) + 0 + 1 = -1. \qquad \square$$

It is a routine matter to check that the sum of values, from anywhere to the end of the row in all tables, is equal to -1, 0 or 1. Hence, $a(5qr, n) \in \{-1, 0, 1\}$ for all $n \in [0, \phi(5qr)]$. That is to say, $A(5qr) = 1$ in the case where $q > 11$, $q \equiv 1 \pmod 5$ and $4r \equiv 1 \pmod{5q}$. This establishes the validity of (3.18).

Finally, the proof of Theorem 1.2 is completed by using what we have proved and Lemma 2.3.

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