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A GENERALIZED COMMON FIXED POINT THEOREM FOR TWO FAMILIES OF SELF-MAPS

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Abstract. Brief developments in metrical fixed point theory are covered and a significant generalization of recent results obtained in [18], [27], [32] and [33] is established through an extension of the property (EA) to two sequences of self-maps using the notions of weak compatibility and implicit relation.

1. Introduction

In this paper, (X, d) denotes a metric space, fx the image of $x \in X$ under a self-map f on X.

The well-known Banach contraction principle asserts that every contraction f on a complete metric space X with the choice

$$d(fx, fy) \leq qd(x, y) \quad \text{for all} \quad x, y \in X \quad \text{for some} \quad 0 < q < 1 \tag{1.1}$$

has a unique fixed point.

In 1968, Kannan [17] analyzed a substantially new contractive type condition to ensure the existence of fixed point for maps that have discontinuity in its domain. Kannan's result is effective in characterizing metric completeness [45], though it is independent of Banach's theorem.

Later, many generalizations of Banach's result were developed by weakening the contraction condition (1.1) using various linear, rational and general contractive type inequalities and relaxing the completeness of the metric space (*cf.* [3], [4], [6], [9], [10, 11, 12], [13], and so on).

An extensive collection of various types of contraction mappings and their comparative study, initiated by Rhoades [36], was further developed by Collaco and Silva [5], Kinces and Totok [19] and Rhoades [37]. One such an extension of Banach contranction mapping theorem to a family of self-maps was established by Kikina and Kikina [18] as follows:

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Theorem 1.1. *Let $f_1, f_2, \dots, f_k$ be self-maps on X satisfying the inequalities*:

$$\begin{aligned}(1.2)\qquad & [1+pd(x,y)]d(f_ix, f_{i+1}y) \\ & \le p[d(x,f_ix)d(y,f_{i+1}y) + d(x,f_{i+1}y)d(y,f_ix)] \\ & \quad + q\max\left\{d(x,y), d(x,f_ix), d(y,f_{i+1}y), \tfrac{1}{2}[d(x,f_{i+1}y)+d(y,f_ix)]\right\} \\ & \quad \textit{for all } x,y\in X,\ i=1,2,\dots,k,\end{aligned}$$

where $f_{k+1} = f_1$, $p \ge 0$ and $0 \le q < 1$. If X is complete, then the mappings $f_1, f_2, \dots, f_k$ will have a unique common fixed point.

We see that for $k = 1$ and $p = 0$, (1.2) is weaker than (1.1), and the result of Rhoades [36] follows from Theorem 1.1 when $k = 3$ and $p = 0$.

Motivated by the fact that a fixed point of any map can always be viewed as a common fixed point for it and the identity map, the scope of Banach's theorem was widened by extending it to two or more self-maps or nonself-maps with some boundary conditions under general contraction-type conditions (See various papers from [1] to [45] in the references).

This paper first covers brief developments in metrical fixed point theory. Then Theorem 1.1 is generalized through an extension of property (EA) to a pair of sequences of self-maps and the notions of weakly compatible self-maps and implicit relation. Interestingly, this will also be a generalization of recent results obtained in [27], [32] and [33].

2. Brief developments

Self-maps f and r on a metric space (X, d) are known to be commuting if $frx = rfx$ for all $x \in X$, where fr denotes the composition of f and r. As a weaker form of it, Sessa [39] introduced weakly commuting maps f and r on X with the choice:

$$(2.1)\qquad d(frx, rfx) \le d(fx, rx) \quad \text{for all} \ \ x \in X.$$

Many interesting results for commuting and weakly commuting mappings were established during 70's and 80's. One can refer to the works of Das and Naik [7], Jungck [14], Pant [24], Singh and Singh [44] etc. Weak commutativity was further generalized as compatible maps by Gerald Jungck [15] and as R-weakly commutating maps by Pant [25]. In fact, we have:

Definition 2.1. Self-maps f and r on X are said to be R-weakly commuting if

$$(2.2)\qquad d(frx, rfx) \le Rd(fx, rx) \quad \text{for all } x \in X \text{ for some } R > 0.$$

Writing $R = 1$ in (2.2), we get (2.1) and weak commutativity of f and r follows from their R-weak commutativity with $R = 1$, but the reverse implication is true only when $R \le 1$ as shown in [25].

Splitting the condition (2.2), Pathak et al. [28] gave:

Definition 2.2. Self-maps f and r on X are said to be R-weakly commuting of type (A_g) if

$$d(frx, rrx) \leq Rd(fx, rx) \quad \text{for all } x \in X \text{ for some } R > 0. \tag{2.3}$$

Interchanging the roles of f and r in Definition 2.2, we get:

Definition 2.3. Self-maps f and r on X are said to be R-weakly commuting of type (A_f) if

$$d(ffx, rfx) \leq Rd(fx, rx) \quad \text{for all } x \in X \text{ for some } R > 0. \tag{2.4}$$

In a comparative study of various weaker forms of commuting maps, Singh and Tomar [41] remarked that R-weak commutativity is independent of its two types given in Definition 2.2 and Definition 2.3.

Definition 2.4. Self-maps f and r on X are said to be compatible if

$$\lim_{n\to\infty} d(frx_n, rfx_n) = 0 \tag{2.5}$$

whenever there exists a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = z \quad \text{for some} \quad z \in X. \tag{2.6}$$

In view of the asymptotic condition (2.5), compatible maps were also known as asymptotically commuting. It is remarked from [32] that a pair (f, g) of self-maps can be weakly commuting but there may not be any sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the choice (2.6). Such maps are vacuously compatible. In this paper, we adopt the convention of nonvacuous compatibility. On the other hand, self-maps f and g are noncompatible if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with (2.6) but $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$ or $+\infty$.

Pathak and Khan [30] characterized and compared different types of compatibility by splitting the asymptotic condition (2.5) in various ways, and proved that the compatibility of any type for (f, r) is equivalent to their compatibility, provided both f and r are continuous. Compatibility and its types find nice applications in the context of boundary value problems, number theoretic problems and dynamical programming e.g., see [20, 23, 29, 43].

Though the fixed point theory for a single map without continuity is traced back to Kannan [17], the following notion was introduced in [26] in the study of common fixed points for noncompatible and discontinuous maps:

Definition 2.5. Self-maps f and r on X are reciprocally continuous at $z \in X$ if for any sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice (2.6), we have

$$\lim_{n\to\infty} frx_n = fz \text{ and } \lim_{n\to\infty} rfx_n = rz. \tag{2.7}$$

And f and r are reciprocally continuous if and only if they are reciprocally continuous at each $z \in X$.

This was further weakened by Pant et al. [27] as follows:

Definition 2.6. Self-maps f and r on X are weakly reciprocally continuous at $z \in X$ if for any sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice (2.6), we have

$$\lim_{n\to\infty} frx_n = fz \quad \text{or} \quad \lim_{n\to\infty} rfx_n = rz. \tag{2.8}$$

And f and r are weakly reciprocally continuous if and only if they are weakly reciprocally continuous at each $z \in X$.

With these ideas, Pant et al. [27] proved:

Theorem 2.1. *Let f and r be weakly reciprocally continuous self-maps on X satisfying the inclusion $f(X) \subset r(X)$ and the inequality*

$$d(fx, fy) \le ad(rx, ry) + bd(fx, rx) + cd(ry, fy) \quad \text{for all } x, y \in X, \tag{2.9}$$

where a, b and c are nonnegative real numbers with $a + b + c < 1$. Suppose that X is complete and f and r are either compatible or R-weakly commuting of type (A_g) or (A_f). Then f and r have a unique common fixed point.

Recently, the author with Sivarama Prasad [32] proved:

Theorem 2.2. *Let f, g and r be self-maps on X satisfying the inequality*

$$d(fx, gy) \le q \max\left\{d(rx, ry), d(rx, fx), d(ry, gy), \frac{d(rx, gy) + d(ry, fx)}{2}\right\} \quad \text{for all } x, y \in X, \tag{2.10}$$

where $0 < q < 1$. Suppose that either (f, r) or (g, r) satisfies the property (EA) and $r(X)$ is a complete subspace of X. If either (f, r) or (g, r) is R-weakly commuting of type (A_g) or (A_f), then f, g and r have a unique common fixed point.

Definition 2.7. A point $x \in X$ is called a coincidence point for self-maps f and r if $fx = rx = y$, and y is a point of coincidence of f and r in this case.

It may be noted that the existence of coincidence point is not necessary for a pair of maps to be commuting.

Example 2.1. Let $X = \mathbb{R}$ with usual metric $d(x, y) = |x - y|$ for all $x \in X$. Define $fx = x + a$ and $rx = x + b$ where $a \neq b$. Then $frx = rfx = x + a + b$ but f and r have no coincidence point.

Definition 2.8. Self-maps f and r which commute at their coincidence points are called coincidentally commuting [8], weakly compatible [16], partially commuting [38] or compatible type (N) [40].

Definition 2.9. Let $\phi : [0, \infty) \to [0, \infty)$ be a contractive modulus with the choice $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$. A contractive modulus ϕ is upper semicontinuous (abbreviated as *usc*) if and only if $\limsup_{n\to\infty} \phi(t_n) \le \phi(t_0)$ for all $t = t_0$ and all $\langle t_n \rangle_{n=1}^{\infty} \subset [0, \infty)$ with $\lim_{n\to\infty} t_n = t_0$.

With this, Singh and Mishra [42] proved the following result:

Theorem 2.3. *Let f, g and r be self-maps on X satisfying the inclusions*

$$f(X) \subset r(X) \quad \text{and} \quad g(X) \subset r(X) \tag{2.11}$$

and the contractive-type condition

$$d(fx, gy) \le \phi\Bigg(\max\Bigg\{ d(rx, ry), d(fx, rx), d(gy, ry), \frac{d(gy, rx) + d(fx, ry)}{2} \Bigg\}\Bigg) \quad \text{for all } x, y \in X. \tag{2.12}$$

Suppose that one of $f(X)$, $g(X)$ and $r(X)$ is a complete subspace of X. If (f, r) and (g, r) are weakly compatible, then the three maps f, g and r will have a unique common fixed point.

Theorem 2.3 was generalized by the author with Swatmaram in [33] using the notion of asymptotic regularity under a weaker form of the inequality (2.12), when the contractive modulus ϕ is nondecreasing:

Theorem 2.4. *Let f, g and r be self-maps on X satisfying the inclusions* (2.11) *and the inequality*

$$d(fx, gy) \le \phi\big(d(rx, ry), d(rx, fx), d(ry, gy), d(rx, gy), d(ry, fx)\big) \quad \text{for all } x, y \in X, \tag{2.13}$$

where ϕ is a nondecreasing and usc contractive modulus.

Given $x_0 \in X$, suppose that

(a) *the pair (f, g) is a.r. at x_0 with respect to r in the sense that there is an (f, g) orbit with respect to r with the choice*

$$y_{2n-1} = fx_{2n-2} = rx_{2n-1}, \quad y_{2n} = gx_{2n-1} = rx_{2n} \quad \text{for } n \ge 1 \tag{2.14}$$

such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

(b) *one of $f(X)$, $g(X)$ and $r(X)$ is orbitally complete at x_0, that is every Cauchy sequence in some orbit* (2.14) *converges in one of $f(X)$, $g(X)$ and $r(X)$ respectively.*

Then f, g and r will have a common coincidence point. Further, if either (f, r) or (g, r) is a weakly compatible pair, then f, g and r will have a unique common fixed point.

To establish a significant generalization of Theorem 1.1, Theorem 2.1, Theorem 2.2 and Theorem 2.4 in the next section, we need following useful notions:

Definition 2.10. Self-maps with choice (2.6) are called tangential maps [38].

This notion was rediscovered in [1] as follows:

Definition 2.11. Self-maps f and r on X satisfy the property (EA) [1] if (2.6) holds good for some $\langle x_n \rangle_{n=1}^{\infty} \subset X$.

It is interesting to note that nonvacuously compatible, compatible maps of all types and noncompatible maps are included in the class of self-maps satisfying the property (EA).

Restricting the commutativity to a smallest subset of the domain of maps, Singh and Tomar [41] did a nice comparative study of various weaker forms of commuting maps. In fact, it was observed from [41] that compatibility and all its types, and R-weak commutativity and its types imply the weak compatibility. Since two self-maps fail to be weakly compatible only if they have a coincidence point at which they do not commute, weak compatibility is the minimal condition for the maps to have a common fixed point.

The following is an easy consequence for weakly compatible maps:

Lemma 2.1. *If self-maps f and r are weakly compatible, then their point of coincidence with respect to a coincidence point will also be a coincidence point for them.*

From the above discussion, we see that weak compatibility and property (EA) are weaker conditions of compatibility and all its types. Pathak et al. [28] proved that both these notations are independent of each other.

We support this fact with the following examples:

Example 2.2 (Ex. 1.4, [2])**.** Let $X = [0, \infty)$ with usual metric $d(x, y) = |x-y|$ for all $x \in X$. Define $fx = x^2$ and $rx = x + 2$. Choose $x_n = 2 + \frac{1}{n}$ for $n \geq 1$. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = 4,$$

so that f and r satisfy the property (EA). But f and r do not commute at their coincidence point, namely 2 and hence are not weakly compatible.

Example 2.3. Let $X = (1, \infty)$ with usual metric $d(x, y) = |x - y|$ for all $x \in X$. Define $fx = x^3$ and $rx = x^2$. Then $d(fx, gx) = x^2(x - 1) = 0$ if and only if $x = 0, 1$. But $0, 1 \notin X$ Thus f and g do not have a coincidence point at all, though $fgx = gfx = x^5$ for all $x \in X$. That is f and r are vacuously weakly compatible. But there is no sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X satisfying (2.6).

Definition 2.12. A class $\wp$ of self-maps f on X satisfies the property (EA) [22] if there is a $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fx_n = z \quad \text{for some } z \in X \quad \text{for each } f \in \wp. \tag{2.15}$$

In particular if $\wp$ consists of only two maps f and r, (2.15) reduces to (2.6).

The property (EA) was extended to two pairs of self-maps by Liu et al. [21] as given below:

Definition 2.13. The pairs (f,r) and (g,s) of self-maps on X share the common property (EA) if there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} ry_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} sy_n = u \quad \text{for some } u \in X. \tag{2.16}$$

Pathak and Shahzad [31] called (f,g) tangential with respect to (r,s) if (2.16) holds good for some sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in X.

In recent years, the idea of inserting implicit relation in the contraction-type condition, due to Popa [34, 35], has attracted the researchers because of its capacity to cover several contractive conditions and unify fixed point theorems.

In this paper, $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ denotes a lower semicontinuous implicit function with the choice:

(P_a) $\psi(l,0,l,0,0,l) > 0$ for all $l > 0$,
(P_b) $\psi(l,l,0,0,l,l) > 0$ for all $l > 0$.

Example 2.4. Set

$$\psi(l_1,l_2,l_3,l_4,l_5,l_6) = (1+pl_2)l_1 - p(l_3l_4 + l_5l_6) - q\max\left\{l_2,l_3,l_4,\tfrac{l_5+l_6}{2}\right\},$$

where p and q have the same choice as given in Theorem 1.1. Then

$$\begin{aligned}(P_a)\quad \psi(l,0,l,0,0,l) &= (1+p.0)l - p(l.0+0.l) - q\max\left\{0,l,0,\tfrac{0+l}{2}\right\}\\ &= (1-q)l > 0 \quad \text{for all} \quad l>0,\\ (P_b)\quad \psi(l,l,0,0,l,l) &= (1+p.l)l - p(0.0+l.l) - q\max\left\{l,0,0,\tfrac{l+l}{2}\right\}\\ &= (1-q)l > 0 \quad \text{for all} \quad l>0.\end{aligned}$$

With $p = 0$ in Example 2.4, we have:

Example 2.5. Let

$$\psi(l_1,l_2,l_3,l_4,l_5,l_6) = l_1 - q\max\left\{l_2,l_3,l_4,\tfrac{l_5+l_6}{2}\right\},\ 0 \le q < 1.$$

Example 2.6. Let $\psi(l_1,l_2,l_3,l_4,l_5,l_6) = l_1 - [al_2 + bl_3 + cl_4 + e(l_5+l_6)]$, where a,b,c and e are nonnegative numbers with $a+b+c+2e < 1$.
Then

(P_a) $\psi(l,0,l,0,0,l) = (1-b-e)l > 0$ for all $l > 0$,
(P_b) $\psi(l,l,0,0,l,l) = l - (a+2e)l = (1-a-2e)l > 0$ for all $l > 0$.

Example 2.7. Let $\psi(l_1,l_2,l_3,l_4,l_5,l_6) = l_1 - \phi\big(\max\{l_2,l_3,l_4,l_5,l_6\}\big)$, where ϕ is a nondecreasing and usc contractive modulus. Then

(P_a) $\psi(l,0,l,0,0,l) = l - \phi\big(\max\{0,l,0,0,l\}\big) = l - \phi(l) > 0$ for all $l > 0$,
(P_b) $\psi(l,l,0,0,l,l) = l - \phi\big(\max\{l,0,0,l,l\}\big) = l - \phi(l) > 0$ for all $l > 0$.

With $\phi(t) = qt$, in Example 2.7 where $0 \le q < 1$, we get:

Example 2.8. Let $\psi(l_1,l_2,l_3,l_4,l_5,l_6) = l_1 - q\max\{l_2,l_3,l_4,l_5,l_6\},\ 0 \le q < 1.$

3. Main result and discussion

First, we introduce extensions of the ideas discussed in earlier sections to a pair of sequences of self-maps:

Given an integer $k > 0$, consider the sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ of self-maps on X such that $f_{k+i} = f_i$ and $g_{k+i} = g_i$ for all i.

Definition 3.1. The sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ share the property (EA) or the sequence $\{f_i\}_{i=1}^{\infty}$ is tangential with respect to the sequence $\{g_i\}_{i=1}^{\infty}$ if there exist associated sequences $\langle x_n^{(i)} \rangle_{n=1}^{\infty}$ in X $i = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} f_i x_n^{(i)} = \lim_{n\to\infty} g_i x_n^{(i)} = z, \ i = 1, 2, \ldots \quad \text{for some } \ z \in X. \tag{3.1}$$

For $k = 1$ and $x_n^{(1)} = x_n$ and $f_1 = f$, $g_1 = g$, we see that Definition 3.1 reduces to Definition 2.11 and if $g_i = f_i$ for all i, (3.1) reduces to (2.15) with $\wp = \{f_i\}_{i=1}^{\infty}$. Further for $k = 2$, $x_n^{(1)} = x_n$, $x_n^{(2)} = y_n$ and $f_1 = f$, $f_2 = r$, $g_1 = g$, $g_2 = s$ in Definition 3.1, we get Definition 2.13.

Our main result is the following:

Theorem 3.1. *For fixed positive integer k, let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be two sequences of self-maps on X with $f_{k+i} = f_i$ and $g_{k+i} = g_i$ for all i sharing the property (EA) and satisfying the following implicit conditions*

$$\begin{aligned}&\psi(d(f_i x, f_{i+1} y), d(g_i x, g_{i+1} y), d(f_i x, g_i x), d(f_{i+1} y, g_{i+1} y),\\ &d(g_i x, f_{i+1} y), d(f_i x, g_{i+1} y)) \le 0 \ \ \textit{for all} \ \ x, y \in X, \ i = 1, 2, 3, \ldots.\end{aligned} \tag{3.2}$$

For each i, suppose that one of the following conditions holds good:

(c) *g_i is onto;*
(d) *$g_i(X)$ is closed;*
(e) *$f_i(X)$ is closed and $f_i(X) \subset g_i(X)$.*

If each (f_i, g_i) is weakly compatible, then each pair (f_i, g_i) has a coincidence point, which will also be a coincidence point for the remaining pairs and hence is a common coincidence point for all the maps $\{f_i, g_i : \ i = 1, 2, \ldots\}$. In fact, this common coincidence point will be their unique common fixed point.

Proof. In view of the condition that $f_{k+i} = f_i$ and $g_{k+i} = g_i$ for all i, we realize that both the sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ reduce to k-tuples $(f_1, f_2, \ldots, f_k)$ and $(g_1, g_2, \ldots, g_k)$ respectively and (3.2) contains k inequalities only.

It is not hard to show that the limit z in (3.1) will be a common fixed point for all f_i and g_i, $i = 1, 2, \ldots, k$ whenever it is their common coincidence point.

In fact, suppose that

$$f_1 z = f_2 z = \cdots = f_k z = g_1 z = g_2 z = \cdots = g_k z. \tag{3.3}$$

Then with $x = x_n^{(i)}$ and $y = z$, (3.2) gives

$$\begin{aligned}&\psi(d(f_i x_n^{(i)}, f_{i+1} z), d(g_i x_n^{(i)}, g_{i+1} z), d(f_i x_n^{(i)}, g_i x_n^{(i)}),\\ &d(f_{i+1} z, g_{i+1} z), d(g_i x_n^{(i)}, f_{i+1} z), d(f_i x_n^{(i)}, g_{i+1} z)) \le 0.\end{aligned}$$

Applying the limit as $n \to \infty$ in this and using (3.1), (3.3) and the lower semicontinuity of ψ, we get

$$\psi(d(z, f_i z), d(z, f_i z), 0, 0, d(z, f_i z), d(z, f_i z)) \leq 0,$$

which will be against the choice (P_b) if $d(z, f_i z) > 0$. Therefore, $d(z, f_i z) = 0$ or $f_i z = z$ for all i and hence z is a common fixed point for f_i and g_i, $i = 1, 2, \ldots, k$.

In view of the cyclical invariance of the conditions of the theorem, it is enough to prove that the limit z is a coincidence point for (f_1, g_1) and hence for the remaining pairs (f_j, g_j), $j = 2, 3, \ldots, k$.

Let g_1 be onto, we have

$$z = g_1 p_1 \quad \text{for some } p_1 \in X. \tag{3.4}$$

Writing $x = p_1$ and $y = x_n^{(2)}$ in the first inequality of (3.2), we get

$$\psi(d(f_1 p_1, f_2 x_n^{(2)}), d(g_1 p_1, g_2 x_n^{(2)}), d(f_1 p_1, g_1 p_1), d(f_2 x_n^{(2)}, g_2 x_n^{(2)}), d(g_1 p_1, f_2 x_n^{(2)}), d(f_1 p_1, g_2 x_n^{(2)})) \leq 0.$$

Proceeding the limit as $n \to \infty$ and using (3.1), (3.4) and the lower semicontinuity of ψ, this yields

$$\psi(d(f_1 p_1, g_1 p_1), 0, d(f_1 p_1, g_1 p_1), 0, 0, d(f_1 p_1, g_1 p_1)) \leq 0,$$

which would contradict (P_a) if $d(f_1 p_1, g_1 p_1) > 0$.

Thus we must have $f_1 p_1 = g_1 p_1 = z$.

Since g_i is onto, we get in a similar way as above that $f_i p_i = g_i p_i = z$ for $i = 2, 3, \ldots, k$. Therefore, weak compatibility of all the pairs imply that

$$f_i g_i(p_i) = g_i f_i(p_i) \quad \text{or} \quad f_i z = g_i z \quad \text{for} \quad i = 1, 2, 3, \ldots, k. \tag{3.5}$$

Now writing $x = y = z$ in the first inequality of (3.2), we have

$$\psi(d(f_1 z, f_2 z), d(g_1 z, g_2 z), d(f_1 z, g_1 z), d(f_2 z, g_2 z), d(g_1 z, f_2 z), d(f_1 z, g_2 z)) \leq 0.$$

Using (3.5) in this, we get

$$\psi(d(f_1 z, f_2 z), d(f_1 z, f_2 z), 0, 0, d(f_1 z, f_2 z), d(f_1 z, f_2 z)) \leq 0.$$

This would be against the choice (P_b) if $d(f_1 z, f_2 z) > 0$. Thus $d(f_1 z, f_2 z) = 0$ or $f_1 z = f_2 z = g_1 z = g_2 z$.

Writing $x = y = z$ in the second, third, …inequalities of (3.2) and using (3.5) and proceeding as above, it follows that

$$f_2 z = f_3 z = \cdots = f_k z = g_2 z = g_3 z = \cdots = g_k z.$$

In other words, z is a common coincidence point and hence a common fixed point for all f_i, g_i, $i = 1, 2, \ldots, k$, in view of the argument done at the beginning of the proof.

Suppose that $g_1(X)$ is closed. Then in view of (3.1), we find that $\{g_1 x_n^{(1)}\}_{n=1}^{\infty}$ is a Cauchy sequence and $z \in g_1(X)$ so that (3.4) holds good. The remaining proof similarly follows from the previous case.

Suppose that $f_1(X)$ is closed. Since $f_1(X) \subset g_1(X)$, a common fixed point for all f_i, g_i follows from the previous case.

Finally, to establish the uniqueness of the common fixed point z, let z' also be a common fixed point of f_i, g_i, $i = 1, 2, \ldots, k$. Then from (3.2) we see that

$$\psi(d(f_i z, f_{i+1} z'), d(g_i z, g_{i+1} z'), d(f_i z, g_i z), d(f_{i+1} z', g_{i+1} z'), d(g_i z, f_{i+1} z'), d(f_i z, g_{i+1} z')) \leq 0 \quad \text{for} \quad i = 1, 2, 3, \ldots, k$$

or

$$\psi(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z, z')) \leq 0,$$

which would again will be against the choice (P_b) if $z \neq z'$. Hence we must have $z = z'$. That is, the common fixed point z is unique. □

Corollary 3.1. *For fixed positive integer k, let $\{f_i\}_{i=1}^{\infty}$ be a sequence of self-maps on X with $f_{k+i} = f_i$ satisfy the following inequalities*

$$\psi(d(f_i x, f_{i+1} y), d(x, y), d(f_i x, x), d(f_{i+1} y, y), d(x, f_{i+1} y), d(f_i x, y)) \leq 0 \quad \textit{for} \quad i = 1, 2, 3, \ldots, k \tag{3.6}$$

for all $x, y \in X$. Given $x_0 \in X$, suppose that there are points x_n in X with

$$f_i x_{k(n-1)+(i-1)} = x_{k(n-1)+i} \quad \textit{for} \quad i = 1, 2, 3, \ldots, k, \; n = 1, 2, \ldots, \tag{3.7}$$

and

$$\lim_{n \to \infty} x_n = z \quad \textit{for some } z \in X. \tag{3.8}$$

Then $f_1, f_2, \ldots, f_k$ will have a unique common fixed point.

Proof. Taking $g_i = I_X$ for all i in Theorem 3.1, we see that each g_i is onto. Since I_X is known to commute with each f_i, each pair (f_i, I_X) is weakly compatible. Define

$$x_n^{(i)} = x_{k(n-1)+(i-1)} \quad \text{for all } n, \; i = 1, 2, 3, \ldots, k. \tag{3.9}$$

Then (3.7) and (3.8) imply that (f_i, I_X) share the common property (EA). Hence a unique common fixed point is ensured by Theorem 3.1. □

To show that Theorem 1.1 is a particular case of Corollary 3.1, consider ψ as in Example 2.4, where p and q have the same choice as given in Theorem 1.1. Then the inequalities given in (1.2) are particular cases of the relations (3.6).

Given $x_0 \in X$, where X is a complete metric space. In the following few lines, we establish that $\langle x_n \rangle_{n=1}^{\infty}$ defined in (3.7) is a Cauchy sequence in X.

For this we require

$$d(x_n, x_{n+1}) \leq q \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \quad \text{for all } n \geq 2. \tag{3.10}$$

In fact, taking $x = x_{k(n-1)+i}$ and $y = x_{k(n-1)+i+1}$ in ith inequality of (1.2), we have

$$[1 + p d(x_{k(n-1)+i}, x_{k(n-1)+i+1})] d(f_i x_{k(n-1)+i}, f_{i+1} x_{k(n-1)+i+1})$$

$$
\begin{aligned}
&\le p[d(x_{k(n-1)+i}, f_i x_{k(n-1)+i}) d(x_{k(n-1)+i+1}, f_{i+1} x_{k(n-1)+i+1}) \\
&\quad + d(x_{k(n-1)+i}, f_{i+1} x_{k(n-1)+i+1}) d(y, f_i x_{k(n-1)+i})] \\
&\quad + q \max \Big\{ d(x_{k(n-1)+i}, x_{k(n-1)+i+1}), d(x_{k(n-1)+i}, f_i x_{k(n-1)+i}), \\
&\qquad d(x_{k(n-1)+i+1}, f_{i+1} x_{k(n-1)+i+1}), \\
&\qquad \tfrac{1}{2}[d(x_{k(n-1)+i}, f_{i+1} x_{k(n-1)+i+1}) + d(x_{k(n-1)+i+1}, f_i x_{k(n-1)+i})] \Big\}
\end{aligned}
$$

which on using (3.7) and then simplifying gives

$$
\begin{aligned}
&d(x_{k(n-1)+i+1}, x_{k(n-1)+i+2}) \\
&\le q \max \big\{ d(x_{k(n-1)+i}, x_{k(n-1)+i+1}), d(x_{k(n-1)+i+1}, f_i x_{k(n-1)+i+2}), \\
&\qquad \tfrac{1}{2} d(x_{k(n-1)+i}, x_{k(n-1)+i+2}) \big\}.
\end{aligned} \tag{3.11}
$$

Now from the triangle inequality, we see that

$$
\begin{aligned}
&d(x_{k(n-1)+i}, x_{k(n-1)+i+2}) \\
&\le q \max \big\{ d(x_{k(n-1)+i}, x_{k(n-1)+i+1}), d(x_{k(n-1)+i+1}, f_i x_{k(n-1)+i+2}) \big\}
\end{aligned}
$$

so that

$$
\begin{aligned}
&\tfrac{1}{2}[d(x_{k(n-1)+i}, x_{k(n-1)+i+2})] \\
&\le q \max \big\{ d(x_{k(n-1)+i}, x_{k(n-1)+i+1}), d(x_{k(n-1)+i+1}, f_i x_{k(n-1)+i+2}) \big\}.
\end{aligned}
$$

With this (3.11) becomes

$$
\begin{aligned}
&d(x_{k(n-1)+i+1}, x_{k(n-1)+i+2}) \\
&\le q \max \big\{ d(x_{k(n-1)+i}, x_{k(n-1)+i+1}), d(x_{k(n-1)+i+1}, f_i x_{k(n-1)+i+2}) \big\}.
\end{aligned}
$$

Since this holds for all $i = 1, 2, \ldots, k$, (3.10) follows for all n.

Now, if $d(x_m, x_{m+1}) > d(x_{m-1}, x_m)$, then $d(x_m, x_{m+1}) > 0$ and (3.10) would imply that

$$d(x_n, x_{n+1}) \le q d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

a contradiction, since $q < 1$. Therefore

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n) \quad \text{for all } n$$

so that (3.10) reduces to

$$d(x_n, x_{n+1}) \le q d(x_{n-1}, x_n) \quad \text{for all } n \ge 2. \tag{3.12}$$

Repeated application of (3.12) gives

$$d(x_n, x_{n+1}) \le q^{n-1} d(x_1, x_2) \quad \text{for all } n \ge 2.$$

Therefore for $m > n$, we get

$$
\begin{aligned}
&d(x_m, x_n) \\
&\le \underbrace{d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)}_{m-n \text{ terms}}
\end{aligned}
$$

$$\leq q^{m-1}d(x_1,x_2)+q^{m-2}d(x_1,x_2)+\cdots+q^{n}d(x_1,x_2)+q^{n-1}d(x_1,x_2)$$
$$= q^{n-1}d(x_1,x_2)[1+q+\cdots+q^{m-n-2}+q^{m-n-1}].$$

Proceeding the limit as $n\to\infty$, the above inequality yields $d(x_m,x_n)\to 0$. In other words, $\langle x_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, it converges to some point z in X.

Therefore the unique common fixed point follows from Corollary 3.1.

Now let $k=1$, $f_1=f$, $g_1=r$ and consider ψ as in Example 2.8 with $0\leq q<1$ in Theorem 3.1. Then we immediately have:

Corollary 3.2. *Let f and r be self-maps on X satisfying the property (EA) and the contraction condition*

$$d(fx,fy)\leq q\max\{d(rx,ry),d(fx,rx),d(ry,fy),d(rx,fy),d(ry,fx)\}\quad \text{for all } x,y\in X. \tag{3.13}$$

Suppose that one of the following conditions holds good:

(f) *r is onto;*
(g) *$r(X)$ is closed;*
(h) *$f(X)$ is closed and $f(X)\subset r(X)$.*

If (f,r) is weakly compatible, then f and r will have a unique common fixed point.

Next we prove that Theorem 2.1 is a particular case of Corollary 3.2:

Suppose that the conditions of Theorem 2.1 hold good. We first observe that inequality (3.13) is weaker than (2.9). It may be noted as earlier that R-weak commutativity of (f,r) of either type implies their weak compatibility. Let $x_0\in X$. In view of the inclusion $f(X)\subset r(X)$, we can choose points x_1, $x_2,\ldots,x_n,\ldots$ in X such that $y_n=fx_{n-1}=rx_n$ for all $n\geq 1$. By a routine iterative procedure, it is easy to show that $\langle y_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, $y_n\to p$ for some $p\in X$ as $n\to\infty$, which in turn implies that f and r satisfy the property (EA). Then the unique common fixed point follows from Corollary 3.2. Thus Corollary 3.2 is a generalization of Theorem 2.1, where the compatibility and weak reciprocal continuity of the pair (f,r) are chipped in the conditions (f-h).

The following example provides a pair of self-maps for which a common fixed point can be determined by Corollary 3.2 but not by Theorem 2.1:

Example 3.1. Let $X=[2,\infty)$ with the usual metric $d(x,y)=|x-y|$. Then X is complete. Define $f,r:X\to X$ by $f2=2$, $fx=6$ for $2<x\leq 5$, $fx=\frac{x+5}{5}$ for $x>5$ and $r2=2$, $rx=12$ for $2<x\leq 5$, $rx=\frac{x+1}{3}$ for $x>5$. Then $r(X)=X$, that is r is onto, and f and r commute at their coincidence point $x=2$. That is f and r are weakly compatible. Further the inequality (3.13) holds good with $q=\frac{29}{30}$.

Write $x_n=5+\frac{1}{n}$ for all $n=1,2,3,\ldots$. Then $\lim\limits_{n\to\infty} fx_n=\lim\limits_{n\to\infty} rx_n=2$ so that f and r satisfy the property (EA). Therefore by Corollary 3.2, f and r

have a unique common fixed point. Indeed, 2 is the only common fixed point for f and r.

However, (2.9) fails. For instance with $a = \frac{2}{5}$, $b = \frac{1}{15}$ and $c = \frac{1}{2}$, we find that $d(f35, f2) = 6 > 10a + 4b + 0.c = ad(r35, r2) + bd(fx, rx) + cd(r2, f2)$. Hence Theorem 2.1 cannot be employed to find the common fixed point, though they are R-weakly commuting of type (A_g) or (A_f) and X is complete. In other words, Corollary 3.2 is a proper generalization of Theorem 2.1. More over, f and r are neither compatible nor weakly reciprocally continuous since $\lim_{n\to\infty} frx_n = \lim_{n\to\infty} f\left(2 + \frac{1}{5n}\right) = 6 \neq f2$ and $\lim_{n\to\infty} rfx_n = \lim_{n\to\infty} g\left(2 + \frac{1}{5n}\right) = 12 \neq g2$. This reveals that compatibility and weak reciprocal continuity can be dropped in Theorem 2.1 to obtain a common fixed point.

With $k = 2$, $f_1 = f$, $f_2 = g$ and $g_1 = g_2 = r$, we have:

Corollary 3.3. *Let f, g and r be self-maps on X satisfying one of the inequalities*

$$\psi\big(d(fx, gy)d(rx, ry), d(fx, rx), d(gy, ry), d(rx, gy), d(fx, ry)\big) \leq 0, \tag{3.14}$$

$$\psi\big(d(gx, fy)d(rx, ry), d(gx, rx), d(fy, ry), d(rx, fy), d(gx, ry)\big) \leq 0, \tag{3.15}$$

for all $x, y \in X$. Suppose that either (f, r) or (g, r) satisfies the property (EA) and that one of the following conditions holds good:

(i) $r(X)$ *is closed*;
(j) $f(X)$ *is closed and* $f(X) \subset r(X)$;
(k) $g(X)$ *is closed and* $g(X) \subset r(X)$.

If either (f, r) or (g, r) is weakly compatible, then f, g and r will have a unique common fixed point.

It is not difficult to prove that the weak compatibility and the property (EA) of either pair is sufficient in Corollary 3.3 to obtain a fixed point under either of the inequalities (3.14) and (3.15).

Now using the implicit relation given in Example 2.5, we see that the inequality (3.14) reduces to (2.10). Since every complete subspace of X is closed and R-weak commutativity of (f, r) of either type implies their weak compatibility, the common fixed point of f, g and r can be obtained by Corollary 3.3. In other words, Theorem 2.2 is a particular case of Corollary 3.3.

Finally we assert that Corollary 3.3 is a significant generalization of Theorem 2.4. In fact, we use the implicit relation given in Example 2.7 so that (3.14) reduces to (2.13). Given $x_0 \in X$, suppose that (f, g) is *ar* at x_0 with respect to r. Then $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ where y_n is defined in (2.14). From the proof of Theorem 2.4, we find that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $r(X)$ be orbitally complete at x_0. Then it follows that $\lim_{n\to\infty} y_n = rp$. Using (2.13) it is easily shown that (f, r) and (g, r) satisfy the property (EA).

Other two cases that $f(X)$ and $g(X)$ are orbitally complete can similarly be handled. The unique common fixed point finally follows from Corollary 3.3. Hence Theorem 2.4 is a particular case of Corollary 3.3.

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