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HYPERSTABILITY OF THE GENERAL LINEAR FUNCTIONAL EQUATION

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Abstract. We give some results on hyperstability for the general linear equation. Namely, we show that a function satisfying the linear equation approximately (in some sense) must be actually the solution of it.

1. Introduction

Let X, Y be normed spaces over fields $\mathbb{F}, \mathbb{K}$, respectively. A function $f\colon X \to Y$ is linear provided it satisfies the functional equation

$$f(ax+by) = Af(x) + Bf(y), \quad x, y \in X, \tag{1}$$

where $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$. We see that for $a = b = A = B = 1$ in (1) we get the Cauchy equation while the Jensen equation corresponds to $a = b = A = B = \frac{1}{2}$. The general linear equation has been studied by many authors, in particular the results of the stability can be found in [5], [6], [8], [9], [10], [13], [14].

We present some hyperstability results for the equation (1). Namely, we show that, for some natural particular forms of φ, the functional equation (1) is φ-hyperstable in the class of functions $f\colon X \to Y$, i.e., each $f\colon X \to Y$ satisfying the inequality

$$\|f(ax+by) - Af(x) - Bf(y)\| \le \varphi(x, y), \quad x, y \in X,$$

must be linear. In this way we expect to stimulate somewhat the further research of the issue of hyperstability, which seems to be a very promising subject to study within the theory of Hyers-Ulam stability.

The hyperstability results concerning the Cauchy equation can be found in [2], the general linear in [12] with $\varphi(x, y) = \|x\|^p + \|y\|^p$, where $p < 0$. The Jensen equation was studied in [1] and there were received some hyperstability results for $\varphi(x, y) = c\|x\|^p\|y\|^q$, where $c \ge 0$, $p, q \in \mathbb{R}$, $p + q \notin \{0, 1\}$.

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The stability of the Cauchy equation involving a product of powers of norms was introduced by J. M. Rassias in [15], [16] and it is sometimes called Ulam-Găvruţa-Rassias stability. For more information about Ulam-Găvruţa-Rassias stability we refer to [7], [11], [17], [18], [19].

One of the method of the proof is based on a fixed point result that can be derived from [3] (Theorem 1). To present it we need the following three hypothesis:

(H1) X is a nonempty set, Y is a Banach space, $f_1,\dots,f_k\colon X\to X$ and $L_1,\dots,L_k\colon X\to\mathbb{R}_+$ are given.

(H2) $\mathcal{T}\colon Y^X\to Y^X$ is an operator satisfying the inequality

$$\big\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\big\|\le\sum_{i=1}^{k}L_i(x)\big\|\xi(f_i(x))-\mu(f_i(x))\big\|,\quad \xi,\mu\in Y^X,\ x\in X.$$

(H3) $\Lambda\colon {\mathbb{R}_+}^X\to{\mathbb{R}_+}^X$ is defined by

$$\Lambda\delta(x):=\sum_{i=1}^{k}L_i(x)\delta(f_i(x)),\quad \delta\in{\mathbb{R}_+}^X,\ x\in X.$$

Now we are in a position to present the above mentioned fixed point theorem.

Theorem 1.1. *Let hypotheses* (H1)–(H3) *be valid and functions* $\varepsilon\colon X\to\mathbb{R}_+$ *and* $\varphi: X\to Y$ *fulfil the following two conditions*

$$\big\|\mathcal{T}\varphi(x)-\varphi(x)\big\|\le\varepsilon(x),\quad x\in X,$$

$$\varepsilon^*(x):=\sum_{n=0}^{\infty}\Lambda^n\varepsilon(x)<\infty,\quad x\in X.$$

Then there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x)-\psi(x)\|\le\varepsilon^*(x),\quad x\in X.$$

Moreover,

$$\psi(x):=\lim_{n\to\infty}\mathcal{T}^n\varphi(x),\quad x\in X.$$

The next theorem shows that a linear function on $X\setminus\{0\}$ is linear on the whole X.

Theorem 1.2. *Let* X,Y *be normed spaces over* $\mathbb{F}$, $\mathbb{K}$, *respectively,* $a,b\in\mathbb{F}\setminus\{0\}$, $A,B\in\mathbb{K}$. *If a function* $f\colon X\to Y$ *satisfies*

(2) $$f(ax+by)=Af(x)+Bf(y),\quad x,y\in X\setminus\{0\},$$

then there exist an additive function $g\colon X\to Y$ *satisfying conditions*

(3) $$g(bx)=Bg(x)\ \textit{and}\ g(ax)=Ag(x),\quad x\in X$$

and a vector $\beta\in Y$ *with*

(4) $$\beta=(A+B)\beta$$

such that

(5) $$f(x) = g(x) + \beta, \quad x \in X.$$

Conversely, if a function $f\colon X \to Y$ *has the form* (5) *with some* $\beta \in Y$*, an additive* $g\colon X \to Y$ *such that* (3) *and* (4) *hold, then it satisfies the equation* (1) *(for all* $x, y \in X$*).*

Proof. Assume that f fulfils (2). Replacing x by bx and y by $-ax$ in (2) we get

(6) $$f(0) = Af(bx) + Bf(-ax), \quad x \in X \setminus \{0\}.$$

Next with x replaced by bx and y by ax in (2) we have

(7) $$f(2abx) = Af(bx) + Bf(ax), \quad x \in X \setminus \{0\}.$$

Let $f = f_e + f_o$, where f_e, f_o denote the even and the odd part of f, respectively. It is obvious that f_e, f_o satisfy (2), (6) and (7).

First we show that f_o is additive. According to (6) and (7) for the odd part of f we have

$$Af_o(bx) = Bf_o(ax), \quad x \in X$$

and

$$f_o(2abx) = Af_o(bx) + Bf_o(ax), \quad x \in X.$$

Thus

(8) $$f_o(x) = 2Bf_o\Big(\frac{x}{2b}\Big) = 2Af_o\Big(\frac{x}{2a}\Big), \quad x \in X.$$

By (8) and (2)

(9) $$\begin{aligned} f_o(x) + f_o(y) &= 2Af_o\Big(\frac{x}{2a}\Big) + 2Bf_o\Big(\frac{y}{2b}\Big) = 2f_o\Big(a\frac{x}{2a} + b\frac{y}{2b}\Big) \\ &= 2f_o\Big(\frac{x+y}{2}\Big), \quad x, y \in X \setminus \{0\}. \end{aligned}$$

Fix $z \in X \setminus \{0\}$ and write $X_z := \{pz : \ p > 0\}$. Then X_z is a convex set, there exist an additive map $g_z\colon X_z \to Y$ and a constant $\beta_z \in Y$ such that

$$f_o(x) = g_z(x) + \beta_z, \quad x \in X_z.$$

We observe that

$$\begin{aligned} g_z(pz) + \beta_z = f_o(pz) &= f_o\Big(\frac{3pz - pz}{2}\Big) = \frac{f_o(3pz) - f_o(pz)}{2} \\ &= \frac{g_z(3pz) - g_z(pz)}{2} = g_z(pz), \quad p > 0, \end{aligned}$$

which means that $\beta_z = 0$. Hence

$$f_o\Big(\frac{1}{2}z\Big) = g_z\Big(\frac{1}{2}z\Big) = \frac{1}{2}g_z(z) = \frac{1}{2}f_o(z).$$

Therefore with (9) we obtain

$$f_o\Big(\frac{x+y}{2}\Big) = \frac{f_o(x) + f_o(y)}{2}, \quad x, y \in X$$

and as $f_o(0) = 0$, f_o is additive. Using additivity of f_o and (8) we obtain

$$f_o(bx) = 2Bf_o\Big(\frac{x}{2}\Big) = Bf_o(x), \quad x \in X$$

and

$$f_o(ax) = 2Af_o\Big(\frac{x}{2}\Big) = Af_o(x), \quad x \in X,$$

which means that (3) holds with $g = f_o$.

Using (6) and (7) for the even part of f we obtain

$$f_e(0) = f_e(2abx), \quad x \in X \setminus \{0\},$$

which means that f_e is a constant function and (4) holds with $\beta := f_e(x)$.

For the proof of the converse, assume that a function $f\colon X \to Y$ has the form (5) with some $\beta \in Y$, an additive $g\colon X \to Y$ such that (3) and (4) hold. Then for all $x, y \in X$

$$\begin{aligned} f(ax+by) &= g(ax+by) + \beta = g(ax) + g(by) + \beta \\ &= Ag(x) + Bg(y) + (A+B)\beta \\ &= Af(x) + Bf(y), \end{aligned}$$

which finishes the proof. □

2. Hyperstability results

Theorem 2.1. *Let X, Y be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K} \setminus \{0\}$, $c \geq 0$, $p, q \in \mathbb{R}$, $p + q < 0$ and $f\colon X \to Y$ satisfies*

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq c\|x\|^p\|y\|^q, \quad x, y \in X \setminus \{0\}. \tag{10}$$

Then f is linear.

Proof. First we notice that without loss of generality we can assume that Y is a Banach space, because otherwise we can replace it by its completion.

Since $p + q < 0$, one of p, q must be negative. Assume that $q < 0$. We observe that there exists $m_0 \in \mathbb{N}$ such that

$$\Big|\frac{1}{A}\Big||a+bm|^{p+q} + \Big|\frac{B}{A}\Big|m^{p+q} < 1 \quad \text{for } m \geq m_0. \tag{11}$$

Fix $m \geq m_0$ and replace y by mx in (10). Thus

$$\|f(ax+bmx) - Af(x) - Bf(mx)\| \leq c\|x\|^p\|mx\|^q, \quad x \in X \setminus \{0\}$$

and

$$\Big\|\frac{1}{A}f((a+bm)x) - \frac{B}{A}f(mx) - f(x)\Big\| \leq \frac{c}{|A|}m^q\|x\|^{p+q}, \quad x \in X \setminus \{0\}. \tag{12}$$

Write

$$\begin{aligned} \mathcal{T}\xi(x) &:= \frac{1}{A}\xi((a+bm)x) - \frac{B}{A}\xi(mx), \\ \varepsilon(x) &:= \frac{c}{|A|}m^q\|x\|^{p+q}, \quad x \in X \setminus \{0\}, \end{aligned}$$

then (12) takes the form

$$\|\mathcal{T}f(x) - f(x)\| \le \varepsilon(x), \quad x \in X \setminus \{0\}.$$

Define

$$\Lambda\eta(x) := \Big|\frac{1}{A}\Big|\eta((a+bm)x) + \Big|\frac{B}{A}\Big|\eta(mx), \quad x \in X \setminus \{0\}.$$

Then it is easily seen that Λ has the form described in (H3) with $k = 2$ and $f_1(x) = (a+bm)x$, $f_2(x) = mx$, $L_1(x) = \frac{1}{|A|}$, $L_2(x) = |\frac{B}{A}|$ for $x \in X \setminus \{0\}$.

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$

$$\begin{aligned}
&\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \\
&= \Big\|\frac{1}{A}\xi((a+bm)x) - \frac{B}{A}\xi(mx) - \frac{1}{A}\mu((a+bm)x) + \frac{B}{A}\mu(mx))\Big\| \\
&\le \Big|\frac{1}{A}\Big|\|(\xi-\mu)((a+bm)x)\| + \Big|\frac{B}{A}\Big|\|(\xi-\mu)(mx)\| \\
&= \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|,
\end{aligned}$$

so (H2) is valid.

By (11) we have

$$\begin{aligned}
\varepsilon^*(x) &:= \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \\
&= \sum_{n=0}^{\infty} \frac{c}{|A|} m^q \Big(\Big|\frac{1}{A}\Big||a+bm|^{p+q} + \Big|\frac{B}{A}\Big| m^{p+q}\Big)^n \|x\|^{p+q} \\
&= \frac{\frac{c}{|A|} m^q \|x\|^{p+q}}{1 - |\frac{1}{A}||a+bm|^{p+q} - |\frac{B}{A}| m^{p+q}}, \quad x \in X \setminus \{0\}.
\end{aligned}$$

Hence, according to Theorem 1.1 there exists a unique solution $F\colon X \setminus \{0\} \to Y$ of the equation

$$F(x) = \frac{1}{A}F((a+bm)x) - \frac{B}{A}F(mx), \quad x \in X \setminus \{0\}$$

such that

$$\|f(x) - F(x)\| \le \frac{\frac{c}{|A|} m^q \|x\|^{p+q}}{1 - |\frac{1}{A}||a+bm|^{p+q} - |\frac{B}{A}| m^{p+q}}, \quad x \in X \setminus \{0\}.$$

Moreover,

$$F(x) := \lim_{n\to\infty} (\mathcal{T}^n f)(x), \quad x \in X \setminus \{0\}.$$

We show that

$$\begin{aligned}
(13) \qquad &\|\mathcal{T}^n f(ax+by) - A\mathcal{T}^n f(x) - B\mathcal{T}^n f(y)\| \\
&\le c\Big(\Big|\frac{1}{A}\Big||a+bm|^{p+q} + \Big|\frac{B}{A}\Big| m^{p+q}\Big)^n \|x\|^p\|y\|^q, \quad x, y \in X \setminus \{0\}
\end{aligned}$$

for every $n \in \mathbb{N}_0$. If $n = 0$, then (13) is simply (10). So, take $r \in \mathbb{N}_0$ and suppose that (13) holds for $n = r$. Then

$$\begin{aligned}
&\|\mathcal{T}^{r+1}f(ax+by) - A\mathcal{T}^{r+1}f(x) - B\mathcal{T}^{r+1}f(y)\|\\
&= \Big\|\frac{1}{A}\mathcal{T}^r f((a+bm)(ax+by)) - \frac{B}{A}\mathcal{T}^r f(m(ax+by))\\
&\quad - A\frac{1}{A}\mathcal{T}^r f((a+bm)x) + A\frac{B}{A}\mathcal{T}^r f(mx)\\
&\quad - B\frac{1}{A}\mathcal{T}^r f((a+bm)y) + B\frac{B}{A}\mathcal{T}^r f(my)\Big\|\\
&\le c\Big(\Big|\frac{1}{A}\Big||a+bm|^{p+q} + \Big|\frac{B}{A}\Big|m^{p+q}\Big)^r\Big|\frac{1}{A}\Big|\|(a+bm)x\|^p\|(a+bm)y\|^q\\
&\quad + c\Big(\Big|\frac{1}{A}\Big||a+bm|^{p+q} + \Big|\frac{B}{A}\Big|m^{p+q}\Big)^r\Big|\frac{B}{A}\Big|\|mx\|^p\|my\|^q\\
&= c\Big(\Big|\frac{1}{A}\Big||a+bm|^{p+q} + \Big|\frac{B}{A}\Big|m^{p+q}\Big)^{r+1}\|x\|^p\|y\|^q, \quad x,y \in X\setminus\{0\}.
\end{aligned}$$

Thus, by induction we have shown that (13) holds for every $n \in \mathbb{N}_0$.

Letting $n \to \infty$ in (13), we obtain that

$$F(ax+by) = AF(x) + BF(y), \quad x,y \in X\setminus\{0\}.$$

In this way, with Theorem 1.2, for every $m \ge m_0$ there exists a function F satisfying the linear equation (1) such that

$$\|f(x) - F(x)\| \le \frac{\frac{c}{|A|}m^q\|x\|^{p+q}}{1 - |\frac{1}{A}||a+bm|^{p+q} - |\frac{B}{A}|m^{p+q}}, \quad x \in X\setminus\{0\}.$$

It follows, with $m \to \infty$, that f is linear. □

In similar way we can prove the following theorem.

Theorem 2.2. *Let X, Y be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively, $a,b \in \mathbb{F}\setminus\{0\}$, $A,B \in \mathbb{K}\setminus\{0\}$, $c \ge 0$, $p,q \in \mathbb{R}$, $p+q > 0$ and $f\colon X \to Y$ satisfies (10). If $q > 0$ and $|a|^{p+q} \ne |A|$, or $p > 0$ and $|b|^{p+q} \ne |B|$, then f is linear.*

Proof. We present the proof only when $q > 0$ because the second case is similar. Let $q > 0$ and $\frac{|a|^{p+q}}{|A|} < 1$. Replacing y by $-\frac{a}{bm}x$, where $m \in \mathbb{N}$, in (10) we get

$$\Big\|f\Big(\Big(a - \frac{a}{m}\Big)x\Big) - Af(x) - Bf\Big(-\frac{a}{bm}x\Big)\Big\| \le c\|x\|^p\Big\|-\frac{a}{bm}x\Big\|^q, \quad x \in X\setminus\{0\},$$

thus

$$\begin{aligned}
&\Big\|\frac{1}{A}f\Big(\Big(a - \frac{a}{m}\Big)x\Big) - \frac{B}{A}f\Big(-\frac{a}{bm}x\Big) - f(x)\Big\|\\
&\le \frac{c}{|A|}\Big|\frac{a}{bm}\Big|^q\|x\|^{p+q}, \quad x \in X\setminus\{0\}.
\end{aligned} \tag{14}$$

For $x \in X\setminus\{0\}$ we define

$$\mathcal{T}_m\xi(x) := \frac{1}{A}\xi\Big(\Big(a - \frac{a}{m}\Big)x\Big) - \frac{B}{A}\xi\Big(-\frac{a}{bm}x\Big),$$

$$\varepsilon_m(x) := \frac{c}{|A|}\Big|\frac{a}{bm}\Big|^q \|x\|^{p+q},$$

$$\Lambda_m\eta(x) := \Big|\frac{1}{A}\Big|\eta\Big(\Big(a-\frac{a}{m}\Big)x\Big) + \Big|\frac{B}{A}\Big|\eta\Big(-\frac{a}{bm}x\Big),$$

and as in Theorem 2.1 we observe that (14) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X \setminus \{0\}$$

and Λ_m has the form described in (H3) with $k = 2$ and $f_1(x) = (a - \frac{a}{m})x$, $f_2(x) = -\frac{a}{bm}x$, $L_1(x) = \frac{1}{|A|}$, $L_2(x) = |\frac{B}{A}|$ for $x \in X \setminus \{0\}$. Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$

$$\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| \le \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|,$$

so (H2) is valid.

Next we can find $m_0 \in \mathbb{N}$, such that

$$\frac{|a|^{p+q}}{|A|}\Big|1-\frac{1}{m}\Big|^{p+q} + \Big|\frac{B}{A}\Big|\Big|\frac{b}{a}\Big|^{p+q}\Big(\frac{1}{m}\Big)^{p+q} < 1 \quad \text{for } m \in \mathbb{N}_{m_0}.$$

Therefore

$$\begin{aligned}\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x)\\ &= \frac{c}{|A|}\Big|\frac{a}{bm}\Big|^q\|x\|^{p+q}\sum_{n=0}^{\infty}\Big(\frac{|a|^{p+q}}{|A|}\Big|1-\frac{1}{m}\Big|^{p+q} + \Big|\frac{B}{A}\Big|\Big|\frac{b}{a}\Big|^{p+q}\Big(\frac{1}{m}\Big)^{p+q}\Big)^n\\ &= \frac{\frac{c}{|A|}|\frac{a}{bm}|^q\|x\|^{p+q}}{1-\frac{|a|^{p+q}}{|A|}|1-\frac{1}{m}|^{p+q} - |\frac{B}{A}||\frac{b}{a}|^{p+q}(\frac{1}{m})^{p+q}}, \quad m \in \mathbb{N}_{m_0},\ x \in X\setminus\{0\}.\end{aligned}$$

Hence, according to Theorem 1.1, for each $m \in \mathbb{N}_{m_0}$ there exists a unique solution $F_m \colon X \setminus \{0\} \to Y$ of the equation

$$F_m(x) = \frac{1}{A}F_m\Big(\Big(a-\frac{a}{m}\Big)x\Big) - \frac{B}{A}F_m\Big(-\frac{a}{bm}x\Big), \quad x \in X\setminus\{0\}$$

such that

$$\|f(x) - F_m(x)\| \le \frac{\frac{c}{|A|}|\frac{a}{bm}|^q\|x\|^{p+q}}{1-\frac{|a|^{p+q}}{|A|}|1-\frac{1}{m}|^{p+q} - |\frac{B}{A}||\frac{b}{a}|^{p+q}(\frac{1}{m})^{p+q}}, \quad x \in X\setminus\{0\}.$$

Moreover,

$$F_m(ax+by) = AF_m(x) + BF_m(y), \quad x, y \in X\setminus\{0\}.$$

In this way we obtain a sequence $(F_m)_{m\in\mathbb{N}_{m_0}}$ of linear functions such that

$$\|f(x) - F_m(x)\| \le \frac{\frac{c}{|A|}|\frac{a}{bm}|^q\|x\|^{p+q}}{1-\frac{|a|^{p+q}}{|A|}|1-\frac{1}{m}|^{p+q} - |\frac{B}{A}||\frac{b}{a}|^{p+q}(\frac{1}{m})^{p+q}}, \quad x \in X\setminus\{0\}.$$

So, with $m \to \infty$, f is linear on $X \setminus \{0\}$ and by Theorem 1.2 f is linear.

Let $q > 0$ and $\frac{|A|}{|a|^{p+q}} < 1$. Replacing x by $(\frac{1}{a} - \frac{1}{am})x$ and y by $\frac{1}{bm}x$, where $m \in \mathbb{N}$, in (10) we get

$$\begin{aligned}&\Big\| f\Big(a\Big(\frac{1}{a} - \frac{1}{am}\Big)x + b\frac{1}{bm}x\Big) - Af\Big(\Big(\frac{1}{a} - \frac{1}{am}\Big)x\Big) - Bf\Big(\frac{1}{bm}x\Big)\Big\| \\ &\quad \le c\Big\|\Big(\frac{1}{a} - \frac{1}{am}\Big)x\Big\|^p \Big\|\frac{1}{bm}x\Big\|^q, \quad x \in X \setminus \{0\}.\end{aligned}$$

Whence

$$\begin{aligned}&\Big\| f(x) - Af\Big(\Big(\frac{1}{a} - \frac{1}{am}\Big)x\Big) - Bf\Big(\frac{1}{bm}x\Big)\Big\| \\ &\quad \le c\frac{1}{|a|^p}\frac{1}{|b|^q}\Big|1 - \frac{1}{m}\Big|^p \Big|\frac{1}{m}\Big|^q \|x\|^{p+q}, \quad x \in X \setminus \{0\}.\end{aligned}$$

For $x \in X \setminus \{0\}$ we define

$$\begin{aligned}\mathcal{T}_m\xi(x) &:= A\xi\Big(\Big(\frac{1}{a} - \frac{1}{am}\Big)x\Big) + B\xi\Big(\frac{1}{bm}x\Big), \\ \varepsilon_m(x) &:= c\frac{1}{|a|^p}\frac{1}{|b|^q}\Big|1 - \frac{1}{m}\Big|^p \Big|\frac{1}{m}\Big|^q \|x\|^{p+q}, \\ \Lambda_m\eta(x) &:= |A|\eta\Big(\Big(\frac{1}{a} - \frac{1}{am}\Big)x\Big) + |B|\eta\Big(\frac{1}{bm}x\Big),\end{aligned}$$

and as in Theorem 2.1 we observe that (14) takes form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X \setminus \{0\}$$

and Λ_m has the form described in (H3) with $k = 2$ and $f_1(x) = (\frac{1}{a} - \frac{1}{am})x$, $f_2(x) = \frac{1}{bm}x$, $L_1(x) = |A|$, $L_2(x) = |B|$ for $x \in X \setminus \{0\}$. Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$

$$\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| \le \sum_{i=1}^{2} L_i(x)\|(\xi - \mu)(f_i(x))\|,$$

so (H2) is valid.

Next we can find $m_0 \in \mathbb{N}$, such that

$$\frac{|A|}{|a|^{p+q}}\Big|1 - \frac{1}{m}\Big|^{p+q} + \frac{|B|}{|b|^{p+q}}\Big|\frac{1}{m}\Big|^{p+q} < 1 \quad \text{for } m \in \mathbb{N}_{m_0}.$$

Therefore

$$\begin{aligned}\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\ &= \varepsilon_m(x) \sum_{n=0}^{\infty} \Big(\frac{|A|}{|a|^{p+q}}\Big|1 - \frac{1}{m}\Big|^{p+q} + \frac{|B|}{|b|^{p+q}}\Big|\frac{1}{m}\Big|^{p+q}\Big)^n\end{aligned}$$

$$= \frac{c\frac{1}{|a|^p}\frac{1}{|b|^q}|1-\frac{1}{m}|^p|\frac{1}{m}|^q\|x\|^{p+q}}{1-\frac{|A|}{|a|^{p+q}}|1-\frac{1}{m}|^{p+q}-\frac{|B|}{|b|^{p+q}}|\frac{1}{m}|^{p+q}}, \quad m \in \mathbb{N}_{m_0},\ x \in X\setminus\{0\}.$$

Hence, according to Theorem 1.1, for each $m \in \mathbb{N}_{m_0}$ there exists a unique solution $F_m\colon X\setminus\{0\} \to Y$ of the equation

$$F_m(x) = AF_m\Big(\Big(\frac{1}{a}-\frac{1}{am}\Big)x\Big) + BF_m\Big(\frac{1}{bm}x\Big), \quad x \in X\setminus\{0\}$$

such that

$$\|f(x)-F_m(x)\| \le \frac{c\frac{1}{|a|^p}\frac{1}{|b|^q}|1-\frac{1}{m}|^p|\frac{1}{m}|^q\|x\|^{p+q}}{1-\frac{|A|}{|a|^{p+q}}|1-\frac{1}{m}|^{p+q}-\frac{|B|}{|b|^{p+q}}|\frac{1}{m}|^{p+q}}, \quad x \in X\setminus\{0\}. \tag{15}$$

Moreover,

$$F_m(ax+by) = AF_m(x) + BF_m(y), \quad x, y \in X.$$

In this way we obtain a sequence $(F_m)_{m\in\mathbb{N}_{m_0}}$ of linear functions such that (15) holds. It follows, with $m \to \infty$, that f is linear. □

Theorem 2.3. *Let X, Y be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\setminus\{0\}$, $A, B \in \mathbb{K}\setminus\{0\}$, $c \ge 0$, $p, q > 0$, and $f\colon X \to Y$ satisfies*

$$\|f(ax+by) - Af(x) - Bf(y)\| \le c\|x\|^p\|y\|^q, \quad x, y \in X. \tag{16}$$

If $|a|^{p+q} \ne |A|$ or $|b|^{p+q} \ne |B|$, then f is linear.

Proof. Of course this theorem follows from Theorem 2.2 but as p, q are positive we can set 0 in (16) and get an auxiliary equalities. In this way we obtain another proof which we present in the first case.

Assume that $|a|^{p+q} < |A|$. Setting $x = y = 0$ in (16) we get

$$f(0)(1-A-B) = 0. \tag{17}$$

With $y = 0$ in (16) we have

$$f(ax) = Af(x) + bf(0), \quad x \in X$$

thus

$$f(x) = Af\Big(\frac{x}{a}\Big) + Bf(0), \quad x \in X.$$

Using the last equality, (16) and (17) we get

$$\Big\|Af\Big(\frac{ax+by}{a}\Big) - AAf\Big(\frac{x}{a}\Big) - BAf\Big(\frac{y}{a}\Big)\Big\| \le c\|x\|^p\|y\|^q, \quad x, y \in X.$$

Replacing x by ax, y by ay and dividing the last inequality by $|A|$ we obtain

$$\|f(ax+by) - Af(x) - Bf(y)\| \le c\frac{|a|^{p+q}}{|A|}\|x\|^p\|y\|^q, \quad x, y \in X.$$

By induction it is easy to get

$$\|f(ax+by) - Af(x) - Bf(y)\| \le c\Big(\frac{|a|^{p+q}}{|A|}\Big)^n\|x\|^p\|y\|^q, \quad x, y \in X.$$

Whence, with $n \to \infty$, $f(ax+by) = Af(x) + Bf(y)$ for $x, y \in X$.

In the case $|A| < |a|^{p+q}$, we use the equation $f(x) = \frac{1}{A}f(ax) - \frac{b}{a}f(0)$ together with (16) and (17). □

The following examples show that the assumption in the above theorems are essential.

Example 2.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Then f satisfies

$$|f(x+y) - f(x) - f(y)| \le 2|x||y|, \quad x, y \in \mathbb{R},$$

but f does not satisfy the Cauchy equation.

Example 2.5. More generally a quadratic function $f(x) = x^2$, $x \in \mathbb{R}$ satisfies

$$|f(ax+by) - Af(x) - Bf(y)| \le 2|ab||x||y|, \quad x, y \in \mathbb{R},$$

where $A = a^2$, $B = b^2$, but f does not satisfy the linear equation (1).

Example 2.6. A function $f(x) = |x|$, $x \in \mathbb{R}$ satisfies

$$\left| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right| \le |x|^{\frac{1}{2}}|y|^{\frac{1}{2}}, \quad x, y \in \mathbb{R},$$

but f does not satisfy the Jensen equation.

It is known that for $p = q = 0$ we have the stability result and a function $f(x) = x + c$, $x \in \mathbb{R}$ satisfies

$$|f(x+y) - f(x) - f(y)| \le c, \quad x, y \in \mathbb{R}$$

but it is not linear.

To the end we show simple application of the above theorems.

Corollary 2.7. *Let X, Y be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K} \setminus \{0\}$, $c \ge 0$, $p, q \in \mathbb{R}$, $H\colon X^2 \to Y$, $H(w,z) \ne 0$ for some $z, w \in X$ and*

$$\|H(x,y)\| \le c\|x\|^p\|y\|^q, \quad x, y \in X \setminus \{0\}, \tag{18}$$

where $c \ge 0$, $p, q \in \mathbb{R}$. If one of the following conditions

(1) $p + q < 0$,
(2) $q > 0$ *and* $|a|^{p+q} \ne |A|$,
(3) $p > 0$ *and* $|b|^{p+q} \ne |B|$

holds, then the functional equation

$$h(ax+by) = Ah(x) + Bh(y) + H(x,y), \quad x, y \in X \tag{19}$$

has no solutions in the class of functions $h\colon X \to Y$.

Proof. Suppose that $h\colon X \to Y$ is a solution to (19). Then (10) holds, and consequently, according to the above theorems, h is linear, which means that $H(w,z) = 0$. This is a contradiction. □

Example 2.8. The functions $f\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ and $H\colon \mathbb{R}^2 \to \mathbb{R}$ given by $H(x, y) = 2xy$ satisfy the equation

$$f(x + y) = f(x) + f(y) + H(x, y), \quad x, y \in \mathbb{R}$$

and do not fulfill any condition (1)–(3) of Corollary 2.7.

Remark 2.9. We notice that our results correspond with the new results from hyperstability, for example in [4] was proved that linear equation is φ-hyperstabile with $\varphi(x, y) = c\|x\|^p\|y\|^q$, but there was considered only the case when $c, p, q \in [0, +\infty)$ (see Theorem 20).

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