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ON A MULTI-PARAMETRIC GENERALIZATION OF THE UNIFORM ZERO-TWO LAW IN L^1-SPACES

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Abstract. Following an idea of Ornstein and Sucheston, Foguel proved the so-called uniform "zero-two" law: let $T : L^1(X, \mathcal{F}, \mu) \to L^1(X, \mathcal{F}, \mu)$ be a positive contraction. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

There are many papers devoted to generalizations of this law. In the present paper we provide a multi-parametric generalization of the uniform zero-two law for L^1-contractions.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space with a positive σ-additive measure μ. In what follows, for the sake of shortness, we denote by L^1 the usual $L^1(X, \mathcal{F}, \mu)$ space associated with $(X, \mathcal{F}, \mu)$. A linear operator $T : L^1 \to L^1$ is called a *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$.

Jamison and Orey [7] proved that if P is a Markov operator recurrent in the sense of Harris, with σ-finite invariant measure μ, then $\|P^n g\|_1 \to 0$ for every $g \in L^1$ with $\int g\, d\mu = 0$ if (and only if) the chain is aperiodic. Clearly, when the chain is not aperiodic, taking f with positive and negative parts supported in different sets of the cyclic decomposition, we have $\lim_{n\to\infty} \|P^n f\|_1 = 2\|f\|_1$.

Ornstein and Sucheston [13] obtained an analytic proof of the Jamison-Orey result, and in their work they proved the following theorem [13, Theorem 1.1].

Theorem 1.1. *Let $T : L^1 \to L^1$ be a positive contraction. Then either*

$$\sup_{\|f\|_1 \leq 1} \lim_{n\to\infty} \|T^{n+1} f - T^n f\| = 2, \tag{1.1}$$

or $\|T^{n+1} f - T^n f\| \to 0$ for every $f \in L^1$.

This result was later called a *strong zero-two law.* Consequently, [13, Theorem 1.3], if T is ergodic with $T^*\mathbf{1} = \mathbf{1}$ (e.g. T is ergodic and conservative),

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then either (1.1) holds, or $\|T^n g\|_1 \to 0$ for every $g \in L^1$ with $\int g \, d\mu = 0$. Some extensions of the strong zero-two law can be found in [2, 17, 21].

Interchanging "sup" and "lim" in the strong zero-two law we have the following *uniform zero-two law*, proved by Foguel [5] using ideas of [4] and [13].

Theorem 1.2. *Let $T : L^1 \to L^1$ be a positive contraction. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

A "zero-two" law for Markov processes was proved in [3], which allowed to study random walks on locally compact groups. Other extensions and generalizations of the formulated law have been investigated by many authors [4, 6, 10, 18, 20]. In all these investigations, the generalization was in direction replacement of the L^1-space by an abstract Banach lattice (see [8, 11, 15, 16, 18]). In [12] we have proposed another kind of generalization of the uniform zero-two law in L^1-spaces.

In this paper we continue the previous investigations and prove a multi-parametric generalization of the uniform "zero-two" law in L^1-space. Note that a different kind of generalization of the said law is given in [8, 15, 19].

2. Preliminaries

In this section, we provide necessary facts which will be used in the next section.

Let $T, S : L^1 \to L^1$ be two positive contractions. We write $T \leq S$ if $S - T$ is a positive operator. In this case we have

$$\|Sx - Tx\| = \|Sx\| - \|Tx\| \tag{2.1}$$

for every $x \geq 0$. Moreover, for a positive operator $T : L^1 \to L^1$ from $|Tf| \leq T(|f|)$ we obtain

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, x\geq 0} \|Tx\|. \tag{2.2}$$

In [12] we have proved the following:

Theorem 2.1 ([12]). *Let $T_1, T_2, S_1, S_2 : L^1 \to L^1$ be positive contractions such that $T_i \leq S_i$, $i = 1, 2$ and $S_1 S_2 = S_2 S_1$. If there is an $n_0 \in \mathbb{N}$ such that $\|S_1 S_2^{n_0} - T_1 T_2^{n_0}\| < 1$, then $\|S_1 S_2^n - T_1 T_2^n\| < 1$ for every $n \geq n_0$.*

From this theorem we immediately get a simple generalization of a result of [20, Theorem 1.1].

Corollary 2.2. *Let $Z, T, S : L^1 \to L^1$ be positive contractions such that $T \leq S$ and $ZS = SZ$. If there is an $n_0 \in \mathbb{N}$ such that $\|Z(S^{n_0} - T^{n_0})\| < 1$, then $\|Z(S^n - T^n)\| < 1$ for every $n \geq n_0$.*

Putting $Z = I$ we obtain the result of [20].

Corollary 2.3 ([20])**.** *Let $T, S : L^1 \to L^1$ be positive contractions such that $T \leq S$. If there is an $n_0 \in \mathbb{N}$ such that $\|S - T\| < 1$, then $\|S^n - T^n\| < 1$ for every $n \geq 1$.*

Let us provide an example of Z, S, T positive contractions for which the statement of Corollary 2.2 holds, but the condition of Corollary 2.3 is not satisfied.

Example. Consider $\mathbb{R}^2$ with the norm $\|\mathbf{x}\| = |x_1| + |x_2|$, where $\mathbf{x} = (x_1, x_2)$. An order in $\mathbb{R}^2$ is defined as usual, namely $\mathbf{x} \geq 0$ if and only if $x_1 \geq 0$, $x_2 \geq 0$. Now define mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$, respectively, by

$$S(x_1, x_2) = \left(\frac{1}{2}x_1 + \frac{1}{3}x_2, \frac{1}{2}x_1 + \frac{1}{3}x_2\right), \tag{2.3}$$

$$T(x_1, x_2) = \left(\frac{1}{4}x_2, 0\right). \tag{2.4}$$

It is clear that S and T are positive and $T \leq S$. Let us define $Z : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Z(x_1, x_2) = \left((1 - c)x_1 + \frac{2c}{3}x_2, cx_1 + \frac{3 - 4c}{3}x_2\right), \tag{2.5}$$

where $c \in (0, 3/4]$.

Then one can see that Z is positive and $ZS = SZ$. Moreover, one has

$$\begin{aligned} \|Z\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x} \geq 0}} \|Z\mathbf{x}\| &= \max_{\substack{x_1 + x_2 = 1 \\ x_1, x_2 \geq 0}} \left\{(1 - c)x_1 + \frac{2c}{3}x_2 + cx_1 + \frac{3 - 4c}{3}x_2\right\} \\ &= \max_{0 \leq x_1 \leq 1} \left\{\frac{2c}{3}x_1 + \frac{3 - 2c}{3}\right\} \\ &= 1. \end{aligned}$$

Similarly, we find that $\|S\| = 1$ and $\|T\| = 1/4$.

From (2.3), (2.4) one gets

$$\|S - T\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x} \geq 0}} \|(S - T)\mathbf{x}\| = \max_{0 \leq x_1 \leq 1} \left\{\frac{7x_1 + 5}{12}\right\} = 1, \tag{2.6}$$

$$\|S^2 - T^2\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x} \geq 0}} \|(S^2 - T^2)\mathbf{x}\| = \max_{0 \leq x_1 \leq 1} \left\{\frac{5x_1 + 10}{18}\right\} = \frac{15}{18}. \tag{2.7}$$

Similarly, from (2.5), (2.3), (2.4) we obtain

$$\|Z(S - T)\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x} \geq 0}} \|Z(S - T)\mathbf{x}\| = 1 - \frac{c}{3} < 1. \tag{2.8}$$

Consequently, we have positive contractions T and S with $S \geq T$ such that $\|S - T\| = 1$, $\|S^2 - T^2\| < 1$. This shows that the condition of Corollary 2.3 is not satisfied, but due to Corollary 2.2 we have $\|Z(S^n - T^n)\| < 1$ for all $n \geq 1$.

First note that for any $x, y \in L^1$ using the pointwise minimum one defines,

$$x \wedge y = \frac{1}{2}(x + y - |x - y|). \tag{2.9}$$

It is well-known (see [2, p. 11], [9, pp. 159–160]) that for any linear mapping S of L^1 one can define its modulus by

$$|S|x = \sup\{Sy : \ |y| \leq x\}, \quad x \in L^1, \ x \geq 0. \tag{2.10}$$

It is known that $|S|$ is linear, $\||S|\| = \|S\|$, and $|Sf| \leq |S|(|f|)$ (see [9, pp. 159–160]).

Hence, similarly to (2.9) for given two linear mappings S, T of L^1 we define

$$(S \wedge T)x = \frac{1}{2}(Sx + Tx - |S - T|x), \quad x \in L^1. \tag{2.11}$$

It is immediate (using the linearity of the modulus) that $S \wedge T$ is linear, and easy to show that $S, T \geq S \wedge T$. One needs to show that if $R \leq S$ and $R \leq T$, then $R \leq S \wedge T$ (see [2, pp. 14–15]).

A linear operator $Z : L^1 \to L^1$ is called *a lattice homomorphism* whenever

$$Z(x \vee y) = Zx \vee Zy \tag{2.12}$$

holds for all $x, y \in L^1$. One can see that such an operator is positive. Note that such homomorphisms were studied in [14].

Recall that a net $\{x_\alpha\}$ in L^1 is *order convergent* to x, denoted $x_\alpha \to^o x$ whenever there exists another net $\{y_\alpha\}$ with the same index set satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$. An operator $T : L^1 \to L^1$ is said to be *order continuous*, if $x_\alpha \to^o 0$ implies $Tx_\alpha \to^o 0$.

Lemma 2.4 ([12]). *Let S, T be positive contractions of L^1, and Z be an order continuous lattice homomorphism of L^1. Then one has*

$$Z|S - T| = |Z(S - T)|. \tag{2.13}$$

Moreover, we have

$$Z(S \wedge T) = ZS \wedge ZT. \tag{2.14}$$

In what follows, an order continuous lattice homomorphism $Z : L^1 \to L^1$ with $\|Z\| \leq 1$, is called *a lattice contraction*.

3. A multi-parametric generalization of the zero-two law

In this section we prove a multi-parametric generalization of the uniform zero-two law for positive contractions on L^1.

Let us first introduce some notations. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $\mathbf{m} = (m_1, \ldots, m_d), \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ $(d \geq 1)$ we define in the usual way, $\mathbf{m} + \mathbf{n} = (m_1 + n_1, \ldots, m_d + n_d)$, $\ell\mathbf{n} = (\ell n_1, \ldots, \ell n_d)$, where $\ell \in \mathbb{N}_0$. We write $\mathbf{n} \leq \mathbf{k}$ if and only if $n_i \leq k_i$ $(i = 1, 2, \ldots, d)$. We denote $|\mathbf{n}| := n_1 + \cdots + n_d$.

Let us formulate our main result.

Theorem 3.1. *Let $Z : L^1 \to L^1$ be a lattice contraction. Assume that $T_k : L^1 \to L^1$ $(k = 1, \dots, d)$ are positive contractions such that such that $ZT_i = T_iZ$, $T_iT_j = T_jT_i$, for every $i, j \in \{1, \dots, d\}$. If for some $\mathbf{m} \in \mathbb{N}_0^d$, $\mathbf{k} \in \mathbb{N}_0^d$ one has $\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})\| < 2$, then for any $\varepsilon > 0$ there are $M \in \mathbb{N}$ and $\mathbf{n}_0 \in \mathbb{N}_0^d$ such that*

$$\|Z^M(\mathbf{T}^{\mathbf{n}+\mathbf{k}} - \mathbf{T}^{\mathbf{n}})\| < \varepsilon \quad \text{for all} \quad \mathbf{n} \geq \mathbf{n}_0.$$

Here $\mathbf{T}^{\mathbf{n}} := T_1^{n_1} \cdots T_d^{n_d}$, $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d$.

Proof. Due to the assumption one can find $\delta > 0$ such that $\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})\| = 2(1 - \delta)$. Let us first prove that

$$\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})\| < 1. \tag{3.1}$$

Assume that $\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})\| = 1$. Then (2.2) implies the existence $x \in L^1$ with $x \geq 0$, $\|x\| = 1$ such that

$$\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})x\| > 1 - \frac{\delta}{4},$$

which with (2.1) yields that $\|Z\mathbf{T}^{\mathbf{m}+\mathbf{k}}x\| > 1 - \delta/4$ and $\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})x\| < \delta/4$. From the commutativity of $\mathbf{T}$ and Z we get $\|Z\mathbf{T}^{\mathbf{m}}x\| > 1 - \delta/4$.

Due to Z being a lattice contraction (see (2.14)), one has

$$|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})| = Z\mathbf{T}^{\mathbf{m}+\mathbf{k}} + Z\mathbf{T}^{\mathbf{m}} - 2Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}}).$$

Hence, the last equality with (2.11) implies that

$$\begin{aligned} \big\||Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})|x\big\| &= \|Z\mathbf{T}^{\mathbf{m}+\mathbf{k}}x\| + \|Z\mathbf{T}^{\mathbf{m}}x\| - 2\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})x\| \\ &> 1 - \frac{\delta}{4} + 1 - \frac{\delta}{4} - 2 \cdot \frac{\delta}{4} \\ &= 2\left(1 - \frac{\delta}{2}\right). \end{aligned}$$

This with the equality

$$\big\||Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})|\big\| = \|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})\|,$$

contradicts to $\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - \mathbf{T}^{\mathbf{m}})\| = 2(1 - \delta/2)$.

Due to (see [19, p. 310]) for $\mathbf{T}^{\mathbf{k}}$ there is $\gamma > 0$ such that

$$\left\|\left(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\right)^{\ell} - \mathbf{T}^{\mathbf{k}}\left(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\right)^{\ell}\right\| \leq \frac{\gamma}{\sqrt{\ell}}. \tag{3.2}$$

Let $\varepsilon > 0$ and fix $\ell \in \mathbb{N}$ such that $\gamma/\sqrt{\ell} < \varepsilon/4$.

From (3.1) according to Corollary 2.2 we have

$$\big\|Z(\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} - (\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})^{\ell})\big\| < 1. \tag{3.3}$$

Hence,

$$\left\|Z\left(\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} - \left(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\right)^{\ell}(\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})^{\ell}\right)\right\|$$

$$= \left\| Z\left(\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} - \frac{1}{2^\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}\mathbf{T}^{i\mathbf{k}}(\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})^\ell \right) \right\|$$

$$\leq \frac{1}{2^\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}\left\| Z(\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} - \mathbf{T}^{i\mathbf{k}}(\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})^\ell)\right\|$$

$$\leq \frac{1}{2^\ell}\left\| Z(\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} - (\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})^\ell)\right\| + \frac{1}{2^\ell}\sum_{i=1}^{\ell}\binom{\ell}{i}$$

$$< \frac{1}{2^\ell} + \frac{1}{2^\ell}\sum_{i=1}^{\ell}\binom{\ell}{i} = 1. \tag{3.4}$$

Define

$$\mathbf{Q}_\ell := \mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} - \left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell (\mathbf{T}^{\mathbf{m}+\mathbf{k}} \wedge \mathbf{T}^{\mathbf{m}})^\ell$$

and put $\mathbf{V}_\ell^{(1)} = (\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} \wedge \mathbf{T}^{\mathbf{m}})^\ell$. Then $\|\mathbf{V}_\ell^{(1)}\| \leq 1$, since $\mathbf{V}_\ell^{(1)} \leq \mathbf{T}^{\mathbf{m}\ell}$. Moreover, one can see that

$$\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} = \left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell \mathbf{V}_\ell^{(1)} + \mathbf{Q}_\ell.$$

Now for every $d \in \mathbb{N}$, define

$$\mathbf{V}_\ell^{(d+1)} = \mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})}\mathbf{V}_\ell^{(d)} + \mathbf{V}_\ell^{(1)}\mathbf{Q}_\ell^d.$$

Then by induction one can establish (see [19]) that

$$\mathbf{T}^{d\ell(\mathbf{m}+\mathbf{k})} = \left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell \mathbf{V}_\ell^{(d)} + \mathbf{Q}_\ell^d \tag{3.5}$$

for every $d \in \mathbb{N}$.

Due to Proposition 2.1 [20] one has

$$\|\mathbf{V}_\ell^{(d)}\| \leq 2 \tag{3.6}$$

for all $d \in \mathbb{N}$.

It follows from (3.4) that $\|Z\mathbf{Q}_\ell\| < 1$, therefore there exists $M \in \mathbb{N}$ such that $\|(Z\mathbf{Q}_\ell)^M\| < \varepsilon/4$. So, commutativity Z and $\mathbf{T}$ implies that $Z\mathbf{Q}_\ell = \mathbf{Q}_\ell Z$, which yields that $\|Z^M\mathbf{Q}_\ell^M\| < \varepsilon/4$.

Put $\mathbf{n}_0 = M\ell(\mathbf{m}+\mathbf{k})$, then from (3.5) with (3.2), (3.6) we obtain

$$\|Z^M(\mathbf{T}^{\mathbf{n}_0+\mathbf{k}} - T^{\mathbf{n}_0})\| = \left\| Z^M\left(\mathbf{T}^{\mathbf{k}}\left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell - \left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell\right)\mathbf{V}_\ell^{(d)} \right.$$

$$\left. + Z^M(\mathbf{T}^{\mathbf{k}}\mathbf{Q}_\ell^M - \mathbf{Q}_\ell^M)\right\|$$

$$\leq \left\| \left(\mathbf{T}^{\mathbf{k}}\left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell - \left(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\right)^\ell\right)\mathbf{V}_\ell^{(M)} \right\|$$

$$+\|Z^M \mathbf{Q}_\ell^M(\mathbf{T}^{\mathbf{k}}-1)\|$$
$$\leq 2\cdot\frac{k\gamma}{\sqrt{\ell}}+2\cdot\frac{\varepsilon}{4}<\varepsilon.$$

Take any $\mathbf{n} \geq \mathbf{n}_0$, then from the last inequality one finds

$$\begin{aligned}\|Z^M(\mathbf{T}^{\mathbf{n}+\mathbf{k}}-\mathbf{T}^{\mathbf{n}})\| &= \|\mathbf{T}^{\mathbf{n}-\mathbf{n}_0}Z^M(\mathbf{T}^{\mathbf{n}_0+\mathbf{k}}-\mathbf{T}^{\mathbf{n}_0})\| \\ &\leq \|Z^M(\mathbf{T}^{\mathbf{n}_0+\mathbf{k}}-\mathbf{T}^{\mathbf{n}_0})\|<\varepsilon\end{aligned}$$

which completes the proof. □

Remark 3.2. The proved theorem is a multi-parametric generalization of the main result of [12]. Hence, it generalizes all main results of [3, 4, 6, 13, 20].

We remark that in the spacial case $Z = I$ Theorem 3.1 yields a multi-dimensional extension of Foguel's result [5]. Namely, we have:

Theorem 3.3. *Let $T_1, \dots, T_d$ be commuting positive contractions of L^1. If there exist $\mathbf{m}$ and $\mathbf{k}$ in $\mathbb{N}_0^d$ such that $\|\mathbf{T}^{\mathbf{m}+\mathbf{k}}-\mathbf{T}^{\mathbf{m}}\| < 2$, then for any $\varepsilon > 0$ there exists $\mathbf{n}_0 \in \mathbb{N}_0^d$ such that*

$$\|\mathbf{T}^{\mathbf{n}+\mathbf{k}}-\mathbf{T}^{\mathbf{n}}\|<\varepsilon \quad \text{for all} \ \ \mathbf{n}\geq\mathbf{n}_0.$$

Corollary 3.4. *Let $T, S : L^1 \to L^1$ be two commuting positive contractions. If for some $k, m_0 \in \mathbb{N}$ one has $\|T^{m_0+k}S^{m_0} - T^{m_0}S^{m_0}\| < 2$, then*

$$\lim_{n,m\to\infty}\|T^{n+k}S^m - T^nS^m\| = 0.$$

The proof immediately follows from Theorem 3.3 if one takes $\mathbf{m} = (m_0, m_0)$ and $\mathbf{k} = (k, 0)$.

Remark 3.5. It is clear that if for commuting T and S contractions of L^1 one has $\lim_n \|T^n - T^{n+k_1}\| = 0$ and $\lim_n \|S^n - S^{n+k_2}\| = 0$ for some $k_1, k_2 \in \mathbb{N}$, then

$$\lim_{\min(n,m)\to\infty}\|T^nS^m - T^{n+k_1}S^{m+k_2}\| = 0.$$

Remark 3.6. Since the dual of L^1 is L^∞ then due to the duality theory the proved Theorem 3.1 holds true if we replace L^1-space with L^∞.

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