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A GENERALIZED ADDITIVE-QUARTIC FUNCTIONAL EQUATION AND ITS STABILITY

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Abstract. We determine the general solution of the generalized additive-quartic functional equation $f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)-13\left[f(x+y)+f(x-y)\right]+24f(y)-12f(2y)=0$ without assuming any regularity conditions on the unknown function $f:\mathbb{R}\to\mathbb{R}$ and its stability is investigated.

1. Introduction

A function $A:\mathbb{R}\to\mathbb{R}$ is said to be *additive* [1], if $A(x+y)=A(x)+A(y)$ $(x,y\in\mathbb{R})$. For $n\in\mathbb{N}$, a function $A_n:\mathbb{R}^n\to\mathbb{R}$ that is additive in each of its variable is called n-additive. If

$$A_n(x_1,x_2,\ldots,x_n)=A_n(x_{\pi(1)},x_{\pi(2)},\ldots,x_{\pi(n)})$$

for every permutation $\{\pi(1),\pi(2),\ldots,\pi(n)\}$ of $\{1,2,\ldots,n\}$ where n is a positive number, then a function A_n is called *symmetric*. Denote the diagonal element $A_n(x,x,\ldots,x)$ by $A^n(x)$ if $A_n(x_1,x_2,\ldots,x_n)$ is n-additive symmetric function and denote the resulting function obtained by putting $x_1=x_2=\cdots=x_\ell=x$ and $x_{\ell+1}=x_{\ell+2}=\cdots=x_n=y$ in $A_n(x_1,x_2,\ldots,x_n)$ by $A^{\ell,n-\ell}(x,y)$. For $f:\mathbb{R}\to\mathbb{R}$, the difference operator Δ_h with $h\in\mathbb{R}$ is defined by

$$\Delta_h f(x)=f(x+h)-f(x).$$

Higher order differences are defined in the usual manner, namely,

$$\Delta_h^0 f(x)=f(x),\ \Delta_h^1 f(x)=\Delta_h f(x),\ \Delta_h^{n+1} f(x)=\Delta_h\circ\Delta_h^n f(x)\ (n\in\mathbb{N},h\in\mathbb{R}),$$

where $\Delta_h\circ\Delta_h^n$ denotes operator composition. The superposition of difference operators is defined by

$$\Delta_{h_1,\ldots,h_n} f=\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n} f,\ n\in\mathbb{N}. \tag{1.1}$$

For any given $n\in\mathbb{N}\cup\{0\}$, if f satisfies the functional equation

$$\Delta_h^{n+1} f(x)=0,\ x,h\in\mathbb{R}, \tag{1.2}$$

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then f is called a *polynomial function of order* n. In explicit form (1.2) can be written as

$$\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}f(x+kh)=0. \tag{1.3}$$

It is known (see [8]) that for functions defined over $\mathbb{R}$ the equation (1.3) is equivalent to the Fréchet functional equation

$$\Delta_{h_1,\ldots,h_{n+1}}f(x)=0, \tag{1.4}$$

where $x,h_1,\ldots,h_{n+1}\in\mathbb{R}$.

The following theorem is needed in our proof (see [3, pp. 71–77]).

Theorem 1.1. *The function $f:\mathbb{R}\to\mathbb{R}$ is a polynomial function of order n if and only if there exist k-additive symmetric functions $A_k:\mathbb{R}^k\to\mathbb{R}$ $(k=0,1,\ldots,n)$ such that the equation*

$$f(x)=\sum_{k=0}^{n}A^k(x)\ \ (x\in\mathbb{R})\ \textit{holds}, \tag{1.5}$$

where A^k are the diagonalizations of A_k $(k=0,1,\ldots,n)$.

In 2003, Chung and Sahoo [2] considered the functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4[f(x+y)+f(x-y)+6f(y)]\ (x,y\in\mathbb{R}). \tag{1.6}$$

It is easy to see that the function $f(x)=x^4$ is a solution of (1.6). The equation (1.6) is called a *quartic functional equation* and every solution of (1.6) is called a *quartic function*. Chung and Sahoo's results are:

Theorem 1.2. *If $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1.6)*, then f is a solution of the Fréchet functional equation $\Delta_{x_1,x_2,x_3,x_4,x_5}f(x_0)=0$ for all $x_0,x_1,x_2,x_3,x_4,x_5\in\mathbb{R}$.*

Theorem 1.3. *The function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1.6) *if and only if f is of the form $f(x)=A^4(x)$, where $A^4(x)$ is the diagonal of 4-additive symmetric function $A_4:\mathbb{R}^4\to\mathbb{R}$.*

Next, in 2004, Sahoo [10] solved the functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4\left[f(x+y)+f(x-y)\right]\ \ (x,y\in\mathbb{R}) \tag{1.7}$$

by finding its general solution to be of the form $f(x)=A^0+A^1(x)+A^2(x)+A^3(x)$, where $A^n(x)$ is the diagonal of n-additive symmetric function $A_n:\mathbb{R}^n\to\mathbb{R}$ $(n=1,2,3)$ and A^0 is an arbitrary constant. In the next year, he generalized (1.7) to

$$f_1(2x+y)+f_2(2x-y)=f_3(x+y)+f_4(x-y)+f_5(x), \tag{1.8}$$

and proved that the functions $f_1,f_2,f_3,f_4,f_5:\mathbb{R}\to\mathbb{R}$ satisfy the functional equation (1.8) for all $x,y\in\mathbb{R}$ if and only if

$$f_1(x)=A^3(x)+A^2(x)+A^1(x)+A^0+B^2(x)+B^1(x)+B^0,$$

$$f_2(x) = A^3(x) + A^2(x) + A^1(x) + A^0 - B^2(x) - B^1(x) - B^0,$$
$$f_3(x) = 2A^3(x) + A^2(x) + A^1(x) + \frac{1}{2}A^0 + C^1(x) + C^0 + 2B^2(x) + B^1(x) + B^0 + D^0,$$
$$f_4(x) = 2A^3(x) + A^2(x) + A^1(x) + \frac{1}{2}A^0 + C^1(x) + C^0 - 2B^2(x) - B^1(x) - B^0 - D^0,$$
$$f_5(x) = 12A^3(x) + 6A^2(x) + 2A^1(x) + A^0 - 2C^1(x) - 2C^0,$$

where A^0, B^0, C^0, D^0 are arbitrary constants, $A^n(x), B^n(x), C^n(x)$ are the diagonals of n-additive symmetric functions $A_n, B_n, C_n : \mathbb{R}^n \to \mathbb{R}$ $(n = 1, 2, 3)$, respectively.

In 2010, Gordji [4] obtained the functional equation

$$f(2x + y) + f(2x - y) = 4\,[f(x + y) + f(x - y)] - \frac{3}{7}[f(2y) - 2f(y)] + 2f(2x) - 8f(x). \tag{1.9}$$

He proved that the function f satisfies (1.9) if and only if there exist a unique symmetric multiadditive function $B : X \times X \times X \times X \to Y$ and a unique additive function $A : X \to Y$ such that $f(x) = B(x, x, x, x) + A(x)$ for all $x \in X$.

In 2013, we (see [6]) considered the following functional equation

$$f(x + 3y) + f(x - 3y) + f(x + 2y) + f(x - 2y) + 22f(x) = 13\,[f(x + y) + f(x - y)] + 168f(y) \tag{1.10}$$

for all $x, y \in \mathbb{R}$ and we solved that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.10) if and only if it is of the form $f(x) = A^4(x)$, where $A^4(x)$ is the diagonal of 4-additive symmetric function $A_4 : \mathbb{R}^4 \to \mathbb{R}$.

Next, Recognizing the identity

$$(x + 3y)^4 + (x - 3y)^4 + (x + 2y)^4 + (x - 2y)^4 + 22x^4 - 13\left[(x + y)^4 + (x - y)^4\right] + 24y^4 - 12(2y)^4 = 0 \tag{1.11}$$

and

$$(x + 3y) + (x - 3y) + (x + 2y) + (x - 2y) + 22x - 13\,[(x + y) + (x - y)] + 24y - 12(2y) = 0, \tag{1.12}$$

which renders a solution $f(x) = x^4 + x$ to the functional equation

$$f(x + 3y) + f(x - 3y) + f(x + 2y) + f(x - 2y) + 22f(x) - 13\,[f(x + y) + f(x - y)] + 24f(y) - 12f(2y) = 0. \tag{1.13}$$

The aim of the present work is to find a general solution of the functional equation (1.13) without assuming any regularity condition and its stability. Our main result is:

Theorem 1.4. *The function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1.13) *for all $x,y\in\mathbb{R}$ if and only if it is of the form*

$$f(x)=A^4(x)+A(x)\ \ (x\in\mathbb{R}), \tag{1.14}$$

where $A^n(x)$ is the diagonal of a n-additive symmetric function $A_n:\mathbb{R}^n\to\mathbb{R}$ $(n=1,4)$.

2. Preliminary result

The following auxiliary lemma is shown in [6, Lemma 2.1].

Lemma 2.1. *If the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation*

$$\begin{aligned}&f(x+4y)-14f(x+2y)+35f(x+y)-35f(x)\\&+14f(x-y)-f(x-3y)=0\end{aligned} \tag{2.1}$$

for all $x,y\in\mathbb{R}$, then f is a solution of the Fréchet functional equation

$$\Delta_{x_1,x_2,x_3,x_4,x_5,}f(x_0)=0$$

for all $x_0,x_1,x_2,x_3,x_4,x_5\in\mathbb{R}$.

Theorem 2.2. *If $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1.13)*, then the following assertions hold.*

a) *If f is an even function, then f is a quartic function.*

b) *If f is an odd function, then f is an additive function.*

Proof. To prove a), letting $x=y=0$ in (1.13), we have $f(0)=0$. Replacing x by $x+y$ in (1.13), we have

$$\begin{aligned}&f(x+4y)+f(x-2y)+f(x+3y)+f(x-y)+22f(x+y)\\&-13\left[f(x+2y)+f(x)\right]+24f(y)-12f(2y)=0.\end{aligned} \tag{2.2}$$

Subtracting (1.13) from (2.2), we obtain

$$f(x+4y)-14f(x+2y)+35f(x+y)-35f(x)+14f(x-y)-f(x-3y)=0.$$

By Lemma 2.1, we have

$$\Delta_{x_1,\ldots,x_5}f(x_0)=0$$

for all $x_0,x_1,x_2,x_3,x_4,x_5\in\mathbb{R}$. Thus from Theorem 1.1 we have

$$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0\ \text{ for all } x\in\mathbb{R}, \tag{2.3}$$

where $A^n(x)$ is the diagonal of n-additive symmetric function $A_n:\mathbb{R}^n\to\mathbb{R}$ for $n=1,\ldots,4$ and A^0 is an arbitrary constant. Since $f(0)=0$, we have $A^0\equiv 0$ and f is even, then $A^3(x)$ and $A^1(x)$ must be vanish. Hence, from (2.3) we have

$$f(x)=A^4(x)+A^2(x). \tag{2.4}$$

Letting (2.4) into (1.13), we have

$$A^4(x+3y)+A^2(x+3y)+A^4(x-3y)+A^2(x-3y)$$

$+ A^4(x + 2y) + A^2(x + 2y) + A^4(x - 2y) + A^2(x - 2y) + 22[A^4(x) + A^2(x)]$
$- 13[A^4(x + y) + A^2(x + y)] - 13[A^4(x - y) + A^2(x - y)] + 24[A^4(y) + A^2(y)]$
$- 12[A^4(2y) + A^2(2y)] = 0.$

Thus, we obtain

$$A^4(x + 3y) + A^4(x - 3y) + A^2(x + 3y) + A^2(x - 3y)$$
$$+ A^4(x + 2y) + A^4(x - 2y) + A^2(x + 2y) + A^2(x - 2y) + 22[A^4(x) + A^2(x)]$$
$$- 13[A^4(x + y) + A^4(x - y)] - 13[A^2(x + y) + A^2(x - y)] + 24[A^4(y) + A^2(y)]$$
$$- 12[A^4(2y) + A^2(2y)] = 0.$$

Noting that

$$A^4(x + y) + A^4(x - y) = 2A^4(x) + 12A^{2,2}(x, y) + 2A^4(y),$$
$$A^2(x + y) + A^2(x - y) = 2A^2(x) + 2A^2(y),$$
$$A^{2,2}(x, 3y) = 9A^{2,2}(x, y),\ \ A^{2,2}(x, 2y) = 4A^{2,2}(x, y),$$
$$A^4(3y) = 81A^4(y),\ \ A^2(3y) = 9A^2(y),\ A^4(2y) = 16A^4(y),\ \ A^2(2y) = 4A^2(y),$$

then we get

$$2A^4(x) + 12A^{2,2}(x, 3y) + 2A^4(3y) + 2A^2(x) + 2A^2(3y)$$
$$+ 2A^4(x) + 12A^{2,2}(x, 2y) + 2A^4(2y) + 2A^2(x) + 2A^2(2y) + 22[A^4(x) + A^2(x)]$$
$$- 13[2A^4(x)+12A^{2,2}(x, y)+2A^4(y)]-13[2A^2(x)+2A^2(y)] + 24[A^4(y)+A^2(y)]$$
$$- 12[A^4(2y) + A^2(2y)] = 0.$$

That is,

$$2A^4(x) + 108A^{2,2}(x, y) + 162A^4(y) + 2A^2(x) + 18A^2(y)$$
$$+ 2A^4(x) + 48A^{2,2}(x, y) + 32A^4(y) + 2A^2(x) + 8A^2(y) + 22[A^4(x) + A^2(x)]$$
$$- 13[2A^4(x)+12A^{2,2}(x, y)+2A^4(y)]-13[2A^2(x)+2A^2(y)] + 24[A^4(y)+A^2(y)]$$
$$- 12[16A^4(y) + 4A^2(y)] = 0.$$

Thus, we obtain $A^2(y) = 0$ and from (2.4) we have $f(x) = A^4(x)$ for all $x \in \mathbb{R}$. Hence f is a quartic function.

To prove b), letting $x = y = 0$ in (1.13), we have $f(0) = 0$. Replacing x by $x + y$ in (1.13), we have

$$f(x + 4y) + f(x - 2y) + f(x + 3y) + f(x - y) + 22f(x + y)$$
$$- 13\,[f(x + 2y) + f(x)] + 24f(y) - 12f(2y) = 0. \tag{2.5}$$

Subtracting (1.13) from (2.5), we obtain

$$f(x + 4y) - 14f(x + 2y) + 35f(x + y) - 35f(x) + 14f(x - y) - f(x - 3y) = 0.$$

By Lemma 2.1, we have

$$\Delta_{x_1,\ldots,x_5} f(x_0) = 0$$

for all $x_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$. Thus from Theorem 1.1 we have

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0 \quad \text{for all } x \in \mathbb{R}, \tag{2.6}$$

where $A^n(x)$ is the diagonal of n-additive symmetric function $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, \ldots, 4$ and A^0 is an arbitrary constant. Since $f(0) = 0$, we have $A^0 \equiv 0$ and f is odd, then $A^4(x)$ and $A^2(x)$ must be vanish. Hence, from (2.6) we have

$$f(x) = A^3(x) + A^1(x). \tag{2.7}$$

Letting (2.7) into (1.13), we have

$$\begin{aligned}
&A^3(x+3y) + A^1(x+3y) + A^3(x-3y) + A^1(x-3y) \\
&+ A^3(x+2y) + A^1(x+2y) + A^3(x-2y) + A^1(x-2y) + 22[A^3(x) + A^1(x)] \\
&- 13[A^3(x+y) + A^1(x+y)] - 13[A^3(x-y) + A^1(x-y)] + 24[A^3(y) + A^1(y)] \\
&- 12[A^3(2y) + A^1(2y)] = 0.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&A^3(x+3y) + A^3(x-3y) + A^3(x+2y) + A^3(x-2y) + 22A^3(x) \\
&- 13[A^3(x+y) + A^3(x-y)] + 24A^3(y) - 12A^3(2y) \\
&+ A^1(x+3y) + A^1(x-3y) + A^1(x+2y) + A^1(x-2y) + 22A^1(x) \\
&- 13[A^1(x+y) + A^1(x-y)] + 24A^1(y) - 12A^1(2y) = 0.
\end{aligned}$$

Noting that

$$\begin{gathered}
A^3(x+y) + A^3(x-y) = 2A^3(x) + 6A^{1,2}(x,y), \\
A^1(x+y) = A^1(x) + A^1(y), \\
A^{1,2}(x,3y) = 9A^{1,2}(x,y), \ A^3(3y) = 27A^3(y), \\
A^{1,2}(x,2y) = 4A^{1,2}(x,y), \ A^3(2y) = 8A^3(y), \\
A^1(2y) = 2A^1(y), \ A^1(3y) = 3A^1(y) \text{ and } A^1(-y) = -A^1(y),
\end{gathered}$$

then we get

$$\begin{aligned}
&2A^3(x) + 54A^{1,2}(x,y) + 2A^3(x) + 24A^{1,2}(x,y) + 22A^3(x) \\
&- 13[2A^3(x) + 6A^{1,2}(x,y)] + 24A^3(y) - 96A^3(y) \\
&+ A^1(x) + A^1(3y) + A^1(x) - A^1(3y) + A^1(x) + A^1(2y) + A^1(x) - A^1(2y) \\
&+ 22A^1(x) - 13[A^1(x) + A^1(y) + A^1(x) - A^1(y)] + 24A^1(y) - 24A^1(y) = 0.
\end{aligned}$$

Thus, we obtain $A^3(y) = 0$ and from (2.7) we have $f(x) = A^1(x)$ for all $x \in \mathbb{R}$. Hence f is additive, which completes the proof. $\square$

3. Proof of Theorem 1.4

From

$$
\begin{aligned}
&f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)\\
&-13\,[f(x+y)+f(x-y)]+24f(y)-12f(2y)=0, \qquad (1.13)
\end{aligned}
$$

we define $f_e(x)=\frac{1}{2}[f(x)+f(-x)]$ and $f_o(x)=\frac{1}{2}[f(x)-f(-x)]$ are even and odd parts, respectively. Thus,

$$
\begin{aligned}
&f_e(x+3y)+f_e(x-3y)+f_e(x+2y)+f_e(x-2y)+22f_e(x)\\
&-13[f_e(x+y)+f_e(x-y)]+24f_e(y)-12f_e(2y)\\
=&\frac{1}{2}\,\{f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)\\
&-13[f(x+y)+f(x-y)]+24f(y)-12f(2y)\\
&+f(-(x+3y))+f(-(x-3y))+f(-(x+2y))+f(-(x-2y))\\
&+22f(-x)-13[f(-(x+y))+f(-(x-y))]\\
&+24f(-y)-12f(-2y)\}=0. \qquad (3.1)
\end{aligned}
$$

This implies that f_e satisfies (1.13). Since f_e is even, from Theorem 2.2, we have

$$f_e(x)=A^4(x), \qquad (3.2)$$

where $A^4(x)$ is the diagonal of 4-additive symmetric function $A_4:\mathbb{R}^4\to\mathbb{R}$.

Consider the functional equation

$$
\begin{aligned}
&f_o(x+3y)+f_o(x-3y)+f_o(x+2y)+f_o(x-2y)+22f_o(x)\\
&-13[f_o(x+y)+f_o(x-y)]+24f_o(y)-12f_o(2y)\\
=&\frac{1}{2}[f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)\\
&-13[f(x+y)+f(x-y)]+24f(y)-12f(2y)\\
&-[f(-(x+3y))+f(-(x-3y))+f(-(x+2y))+f(-(x-2y))\\
&+22f(-x)-13[f(-(x+y))+f(-(x-y))]]\\
&+24f(-y)-12f(-2y)]=0. \qquad (3.3)
\end{aligned}
$$

This shows that f_o satisfies (1.13). Since f_o is odd, from Theorem 2.2, we have

$$f_o(x)=A^1(x), \qquad (3.4)$$

where A_1 is additive. Hence, from (3.2) and (3.4), we obtain

$$f(x)=A^4(x)+A^1(x),$$

where $A^n(x)$ is the diagonal of n-additive symmetric function $A_n:\mathbb{R}^n\to\mathbb{R}$ for $n=1,4$.

Conversely, assume that $f(x)=A^4(x)+A^1(x)$.

Noting that

$$
\begin{aligned}
&A^4(x+y) + A^4(x-y) = 2A^4(x) + 12A^{2,2}(x,y) + 2A^4(y),\\
&\quad A^4(3y) = 81A^4(y),\ A^{2,2}(x,3y) = 9A^{2,2}(x,y),\ A^{2,2}(x,2y) = 4A^{2,2}(x,y),\\
&\quad A^4(2y) = 16A^4(y),\ A^1(x+y) = A^1(x) + A^1(y),\\
&\quad A^3(2y) = 8A^3(y),\ A^1(2y) = 2A^1(y),
\end{aligned}
$$

then we have

$$
\begin{aligned}
&[f(x+3y) + f(x-3y)] + [f(x+2y) + f(x-2y)] + 22f(x)\\
= {}& A^4(x+3y) + A^1(x+3y) + A^4(x-3y) + A^1(x-3y)\\
&+ A^4(x+2y) + A^1(x+2y) + A^4(x-2y) + A^1(x-2y)\\
&+ 22[A^4(x) + A^1(x)]\\
= {}& 2A^4(x) + 108A^{2,2}(x,y) + 162A^4(y) + 2A^4(x) + 48A^{2,2}(x,y)\\
&+ 32A^4(y) + 22A^4(x) + 26A^1(x)\\
= {}& 13[2A^4(x) + 12A^{2,2}(x,y) + 2A^4(y) + 2A^1(x)] - 24[A^4(y) + A^1(y)]\\
&+ 12[A^4(2y) + A^1(2y)]\\
= {}& 13[A^4(x+y) + A^1(x+y) + A^4(x-y) + A^1(x-y)] - 24[A^4(y) + A^1(y)]\\
&+ 12[A^4(2y) + A^1(2y)]\\
= {}& 13[f(x+y) + f(x-y)] - 24f(y) + 12f(2y).
\end{aligned}
$$

This completes the proof of the result.

4. Stability

In this section, we consider a *stability problem* which is proposed by Ulam [12] in 1940: Let f be a mapping from a group $(G_1,+)$ to a metric group $(G_2,+)$ with metric $d(\cdot,\cdot)$ such that

$$d(f(x+y), f(x)+f(y)) \le \epsilon.$$

Do there exist a group homomorphism $L : G_1 \to G_2$ and a constant $\delta_\epsilon > 0$ such that

$$d(f(x), L(y)) \le \delta_\epsilon$$

for all $x \in G_1$? This problem was solved one year later by Hyers [7] under the assumption that G_2 is a Banach space with $\|\cdot\|$. A generalized version of Hyers's result was proved by Th. M. Rassias [9] for linear mappings by considering an unbounded Cauchy difference in Banach spaces. In 1994, a generalized of Rassias's theorem was obtained by Gavruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 4, 5, 6, 7, 9, 10, 11, 12]).

The aim of this section is to investigate the stability of the generalized additive-quartic functional equation (1.13). Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then we define

$$\begin{aligned} D_f(x, y) := {} & f(x + 3y) + f(x - 3y) + f(x + 2y) + f(x - 2y) \\ & + 22f(x) - 13f(x + y) - 13f(x - y) + 24f(y) - 12f(2y) \end{aligned}$$

for all $x, y \in \mathbb{R}$.

Theorem 4.1. *Let $\phi : \mathbb{R}^2 \to [0, \infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{2^i} \text{ converges} \tag{4.1}$$

for all $x, y \in \mathbb{R}$. If the function $f : \mathbb{R} \to \mathbb{R}$ is an odd function which satisfies

$$|D_f(x, y)| \le \phi(x, y) \tag{4.2}$$

for all $x, y \in \mathbb{R}$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ which satisfies the equation (1.13) *and the inequality*

$$|f(y) - A(y)| \le \frac{1}{24} \sum_{i=0}^{\infty} \frac{\phi(0, 2^i y)}{2^i} \tag{4.3}$$

for all $y \in \mathbb{R}$, where the function A is defined by

$$A(y) = \lim_{n\to\infty} \frac{f(2^n y)}{2^n} \tag{4.4}$$

for all $y \in \mathbb{R}$.

Proof. Putting $x = 0$ in (4.2), we have

$$\begin{aligned} & |f(3y) + f(-3y) + f(2y) + f(-2y) + 22f(0) \\ & -13f(y) - 13f(-y) + 24f(y) - 12f(2y)| \le \phi(0, y). \end{aligned} \tag{4.5}$$

Since f is odd and $f(0) = 0$, we obtain

$$|24f(y) - 12f(2y)| \le \phi(0, y).$$

That is,

$$|12f(2y) - 24f(y)| \le \phi(0, y). \tag{4.6}$$

Dividing (4.6) by 24, we have

$$\left| \frac{f(2y)}{2} - f(y) \right| \le \frac{1}{24} \phi(0, y). \tag{4.7}$$

Replacing y by $2y$ in (4.7) and dividing this by 2, we obtain

$$\left| \frac{f(2^2 y)}{2^2} - \frac{f(2y)}{2} \right| \le \frac{1}{24} [\frac{\phi(0, 2y)}{2}]. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\left|\frac{f(2^2y)}{2^2} - f(y)\right| \le \frac{1}{24}[\frac{\phi(0,2y)}{2} + \phi(0,y)]. \tag{4.9}$$

Using the mathematical induction, we can extend (4.9) to

$$\left|\frac{f(2^ny)}{2^n} - f(y)\right| \le \frac{1}{24}\sum_{i=0}^{n-1}\frac{\phi(0,2^iy)}{2^i} \le \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^iy)}{2^i} \tag{4.10}$$

for all $y \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

Next, we will show that $\{\frac{f(2^ny)}{2^n}\}$ is a Cauchy sequence. For integers $m, n > 0$, we have

$$\begin{aligned}\left|\frac{f(2^{n+m}y)}{2^{n+m}} - \frac{f(2^my)}{2^m}\right| &= \frac{1}{2^m}\left|\frac{f(2^n2^my)}{2^n} - f(2^my)\right| \\ &\le \frac{1}{2^m}.\frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^i2^my)}{2^i} \\ &= \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^{i+m}y)}{2^{i+m}} \\ &= \frac{1}{24}\sum_{j=m}^{\infty}\frac{\phi(0,2^jy)}{2^j}. \end{aligned} \tag{4.11}$$

Thus this is the tail of the infinite series of (4.1), which converges (for any fixed y) to zero as $m \to \infty$. This implies that the sequence $\{\frac{f(2^ny)}{2^n}\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, there exists a function $A : \mathbb{R} \to \mathbb{R}$ such that

$$A(y) = \lim_{n\to\infty}\frac{f(2^ny)}{2^n}$$

for all $y \in \mathbb{R}$. By letting $n \to \infty$ in (4.10), we obtain

$$\left|\lim_{n\to\infty}\frac{f(2^ny)}{2^n} - f(y)\right| \le \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^iy)}{2^i}.$$

That is,

$$|A(y) - f(y)| \le \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^iy)}{2^i}.$$

This implies that

$$|f(y) - A(y)| \le \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^iy)}{2^i},$$

so we arrive at the formula (4.3) for all $y \in \mathbb{R}$.

To show that A satisfies the equation (1.13), consider

$$\begin{aligned}&A(x+3y) + A(x-3y) + A(x+2y) + A(x-2y) + 22A(x) \\ &- 13A(x+y) - 13A(x-y) + 24A(y) - 12A(2y)\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{2^n}[f(2^n(x+3y)) + f(2^n(x-3y)) + f(2^n(x+2y)) + f(2^n(x-2y)) + 22f(2^n x) - 13f(2^n(x+y)) - 13f(2^n(x-y)) + 24f(2^n y) - 12f(2^n 2y)]$$
$$= \lim_{n\to\infty} \frac{D_f(2^n x, 2^n y)}{2^n}.$$

Thus, by (4.2), we obtain

$$|D_A(x,y)| \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^n}. \tag{4.12}$$

Using (4.1), we have $D_A(x,y) = 0$. Hence A satisfies the equation (1.13). Since A is odd and by Theorem 2.2, we have A is additive.

To prove the uniqueness of A, suppose that there exists a function $S : \mathbb{R} \to \mathbb{R}$ which satisfies (1.13) and (4.3) with A replaced by S. Note that Theorem 1.4 gives us $A(2^n y) = 2^n A(y)$ and $S(2^n y) = 2^n S(y)$ for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
|A(y) - S(y)| &= \left|\frac{A(2^n y)}{2^n} - \frac{S(2^n y)}{2^n}\right| = \frac{1}{2^n}\,|A(2^n y) - S(2^n y)| \\
&= \frac{1}{2^n}\,|A(2^n y) - f(2^n y) + f(2^n y) - S(2^n y)| \\
&\le \frac{1}{2^n}\,|A(2^n y) - f(2^n y)| + \frac{1}{2^n}\,|f(2^n y) - S(2^n y)| \\
&\le \frac{1}{2^n}.\frac{1}{24}\sum_{i=0}^{\infty} \frac{\phi(0, 2^i 2^n y)}{2^i} + \frac{1}{2^n}.\frac{1}{24}\sum_{i=0}^{\infty} \frac{\phi(0, 2^i 2^n y)}{2^i} \\
&= \frac{1}{2^n}.\frac{2}{24}\sum_{i=0}^{\infty} \frac{\phi(0, 2^i 2^n y)}{2^i} \\
&= \frac{1}{12}\sum_{i=0}^{\infty} \frac{\phi(0, 2^{i+n} y)}{2^{i+n}} \\
&= \frac{1}{12}\sum_{j=n}^{\infty} \frac{\phi(0, 2^j y)}{2^j}.
\end{aligned}$$

Thus this is the tail of the infinite series of (4.1), which converges (for any fixed y) to zero as $n \to \infty$. Thus we immediately find the uniqueness of A. This completes the proof of the theorem. □

Theorem 4.2. *Let $\phi : \mathbb{R}^2 \to [0,\infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{16^i} \text{ converges} \tag{4.13}$$

for all $x, y \in \mathbb{R}$. If the function $f : \mathbb{R} \to \mathbb{R}$ is an even function which satisfies

$$|D_f(x,y)| \le \phi(x,y) \ (x, y \in \mathbb{R}), \tag{4.14}$$

and $f(0) = 0$, then there exists a unique function $Q : \mathbb{R} \to \mathbb{R}$ satisfying the equation (1.13) *and the inequality*

$$|f(y) - Q(y)| \le \frac{1}{960} \sum_{i=0}^{\infty} \frac{2\phi(3(2^i y),2^i y)+2\phi(2(2^i y),2^i y)+\phi(0,2(2^i y))+18\phi(2^i y,2^i y)+44\phi(0,2^i y)}{16^i} \tag{4.15}$$

for all $y \in \mathbb{R}$, where the function Q is defined by

$$Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \tag{4.16}$$

for all $y \in \mathbb{R}$.

Proof. Putting $x = 3y$ in (4.14) and using $f(0) = 0$, we have

$$|f(6y) + f(5y) - 13f(4y) + 22f(3y) - 25f(2y) + 25f(y)| \le \phi(3y, y). \tag{4.17}$$

Putting $x = 2y$ in (4.14). Since $f(0) = 0$ and f is even, we have

$$|f(5y) + f(4y) - 13f(3y) + 10f(2y) + 12f(y)| \le \phi(2y, y). \tag{4.18}$$

From (4.17) and (4.18), we obtain

$$|f(6y) - 14f(4y) + 35f(3y) - 35f(2y) + 13f(y)| \le \phi(3y, y) + \phi(2y, y). \tag{4.19}$$

Putting $x = 0$ in (4.14). By using $f(0) = 0$ and f is even, we have

$$|2f(3y) - 10f(2y) - 2f(y)| \le \phi(0, y). \tag{4.20}$$

That is,

$$|f(3y) - 5f(2y) - f(y)| \le \frac{\phi(0, y)}{2}. \tag{4.21}$$

Replacing y by $2y$ in (4.21), we obtain

$$|f(6y) - 5f(4y) - f(2y)| \le \frac{\phi(0, 2y)}{2}. \tag{4.22}$$

From (4.19) and (4.22), we obtain

$$|-9f(4y) + 35f(3y) - 34f(2y) + 13f(y)| \le \phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2}. \tag{4.23}$$

Putting $x = y$ in (4.14). By using $f(0) = 0$ and f is even, we have

$$|f(4y) + f(3y) - 24f(2y) + 47f(y)| \le \phi(y, y). \tag{4.24}$$

Thus,

$$|9f(4y) + 9f(3y) - 216f(2y) + 423f(y)| \le 9\phi(y, y). \tag{4.25}$$

From (4.23) and (4.25), we obtain

$$|44f(3y) - 250f(2y) + 436f(y)| \le \phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2} + 9\phi(y, y). \tag{4.26}$$

From (4.21), we have

$$|44f(3y) - 220f(2y) - 44f(y)| \leq 22\phi(0, y). \tag{4.27}$$

From (4.26) and (4.27), we obtain

$$\begin{aligned} &|-30f(2y) + 480f(y)| \\ &\leq \phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2} + 9\phi(y, y) + 22\phi(0, y). \end{aligned} \tag{4.28}$$

Thus, we have

$$\begin{aligned} &|f(2y) - 16f(y)| \\ &\leq \frac{1}{30}[\phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2} + 9\phi(y, y) + 22\phi(0, y)]. \end{aligned} \tag{4.29}$$

Dividing by 16 in (4.29), we have

$$\begin{aligned} &\left|\frac{f(2y)}{16} - f(y)\right| \\ &\leq \frac{1}{960}[2\phi(3y, y) + 2\phi(2y, y) + \phi(0, 2y) + 18\phi(y, y) + 44\phi(0, y)]. \end{aligned} \tag{4.30}$$

Replacing y by $2y$ in (4.30) and dividing this by 16, we obtain

$$\begin{aligned} &\left|\frac{f(2^2y)}{(16)^2} - \frac{f(2y)}{16}\right| \\ &\leq \tfrac{1}{960}[\tfrac{2\phi(3(2y),2y)+2\phi(2(2y),2y)+\phi(0,2(2y))+18\phi(2y,2y)+44\phi(0,2y)}{16}]. \end{aligned} \tag{4.31}$$

From the equations (4.30) and (4.31), we have

$$\begin{aligned} &\left|\frac{f(2^2y)}{(16)^2} - f(y)\right| \\ &\leq \tfrac{1}{960}[\tfrac{2\phi(3(2y),2y)+2\phi(2(2y),2y)+\phi(0,2(2y))+18\phi(2y,2y)+44\phi(0,2y)}{16} \\ &\quad + 2\phi(3y, y) + 2\phi(2y, y) + \phi(0, 2y) + 18\phi(y, y) + 44\phi(0, y)]. \end{aligned} \tag{4.32}$$

Using the mathematical induction, we can extend (4.32) to

$$\begin{aligned} &\left|\frac{f(2^ny)}{16^n} - f(y)\right| \\ &\leq \frac{1}{960}\sum_{i=0}^{n-1} \tfrac{2\phi(3(2^iy),2^iy)+2\phi(2(2^iy),2^iy)+\phi(0,2(2^iy))+18\phi(2^iy,2^iy)+44\phi(0,2^iy)}{16^i} \\ &\leq \frac{1}{960}\sum_{i=0}^{\infty} \tfrac{2\phi(3(2^iy),2^iy)+2\phi(2(2^iy),2^iy)+\phi(0,2(2^iy))+18\phi(2^iy,2^iy)+44\phi(0,2^iy)}{16^i} \end{aligned} \tag{4.33}$$

for all $y \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

Next, we will show that $\{\frac{f(2^n y)}{16^n}\}$ is a Cauchy sequence. For integers $m, n > 0$, we have

$$\begin{aligned}
&\left|\frac{f(2^{n+m}y)}{16^{n+m}} - \frac{f(2^m y)}{16^m}\right| = \frac{1}{16^m}\left|\frac{1(2^n 2^m y)}{16^n} - f(2^m y)\right| \\
&\leq \frac{1}{16^m}\cdot\frac{1}{960}\sum_{i=0}^{\infty}\frac{1}{16^i}[2\phi(3(2^i 2^m y), 2^i 2^m y) + 2\phi(2(2^i 2^m y), 2^i 2^m y) \\
&\quad + \phi(0, 2(2^i 2^m y)) + 18\phi(2^i 2^m y, 2^i 2^m y) + 44\phi(0, 2^i 2^m y)]. \\
&= \frac{1}{960}\sum_{i=0}^{\infty}\frac{1}{16^{i+m}}[2\phi(3(2^{i+m} y), 2^{i+m} y) + 2\phi(2(2^{i+m} y), 2^{i+m} y) \\
&\quad + \phi(0, 2(2^{i+m} y)) + 18\phi(2^{i+m} y, 2^{i+m} y) + 44\phi(0, 2^{i+m} y)]. \qquad (4.34)
\end{aligned}$$

Since the right-hand side of (4.34) converges (for any fixed y) to 0 as $n \to \infty$, then the sequence $\{\frac{f(2^n y)}{16^n}\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, there exists a function $Q : \mathbb{R} \to \mathbb{R}$ such that

$$Q(y) = \lim_{n\to\infty}\frac{f(2^n y)}{16^n}$$

for all $y \in \mathbb{R}$. From (4.33), we obtain

$$\begin{aligned}
&\left|\lim_{n\to\infty}\frac{f(2^n y)}{16^n} - f(y)\right| \\
&\leq \tfrac{1}{960}\sum_{i=0}^{\infty}\tfrac{2\phi(3(2^i y),2^i y)+2\phi(2(2^i y),2^i y)+\phi(0,2(2^i y))+18\phi(2^i y,2^i y)+44\phi(0,2^i y)}{16^i}. \qquad (4.35)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|f(y) - Q(y)| \\
&\leq \tfrac{1}{960}\sum_{i=0}^{\infty}\tfrac{2\phi(3(2^i y),2^i y)+2\phi(2(2^i y),2^i y)+\phi(0,2(2^i y))+18\phi(2^i y,2^i y)+44\phi(0,2^i y)}{16^i}. \qquad (4.36)
\end{aligned}$$

Next, we claim that Q satisfies the equation (1.13). Consider

$$\begin{aligned}
&D_Q(x, y) \\
&= Q(x+3y) + Q(x-3y) + Q(x+2y) + Q(x-2y) + 22Q(x) \\
&\quad - 13Q(x+y) - 13Q(x-y) + 24Q(y) - 12Q(2y) \\
&= \lim_{n\to\infty}\frac{1}{16^n}[f(2^n(x+3y)) + f(2^n(x-3y)) + f(2^n(x+2y)) + f(2^n(x-2y)) \\
&\quad + 22f(2^n x) - 13f(2^n(x+y)) - 13f(2^n(x-y)) + 24f(2^n y) - 12f(2^n y)] \\
&= \lim_{n\to\infty}\frac{D_f(2^n x, 2^n y)}{16^n}.
\end{aligned}$$

Thus, we obtain

$$|D_Q(x,y)| \le \lim_{n\to\infty} \frac{1}{16^n} |D_f(2^n x, 2^n y)| \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{16^n} = 0$$

for all $x, y \in \mathbb{R}$. This means that Q satisfies (1.13).

To prove the uniqueness of Q, suppose that there exists a function $S : \mathbb{R} \to \mathbb{R}$ satisfying (1.13) and (4.15). Note that Theorem 1.4 gives us $Q(2^n y) = 16^n Q(y)$ and $S(2^n y) = 16^n S(y)$ for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
|Q(y) - S(y)| &= \left| \frac{Q(2^n y)}{16^n} - \frac{S(2^n y)}{16^n} \right| \\
&= \frac{1}{16^n} |Q(2^n y) - S(2^n y)| \\
&= \frac{1}{16^n} |Q(2^n y) - f(2^n y) + f(2^n y) - S(2^n y)| \\
&\le \frac{1}{16^n} |Q(2^n y) - f(2^n y)| + \frac{1}{16^n} |f(2^n y) - S(2^n y)| \\
&\le \frac{2}{16^n} \cdot \frac{1}{960} \sum_{i=0}^{\infty} \frac{1}{16^i} [2\phi(3(2^i 2^n y), 2^i 2^n y) + 2\phi(2(2^i 2^n y), 2^i 2^n y) \\
&\quad + \phi(0, 2(2^i 2^n y)) + 18\phi(2^i 2^n y, 2^i 2^n y) + 44\phi(0, 2^i 2^n y)] \\
&\le \frac{2}{960} \sum_{i=0}^{\infty} \frac{1}{16^{i+n}} [2\phi(3(2^{i+n} y), 2^{i+n} y) + 2\phi(2(2^{i+n} y), 2^{i+n} y) \\
&\quad + \phi(0, 2(2^{i+n} y)) + 18\phi(2^{i+n} y, 2^{i+n} y) + 44\phi(0, 2^{i+n} y)] \qquad (4.37)
\end{aligned}$$

for all $y \in \mathbb{R}$. Since the right-hand side of (4.37) converges to 0 as $n \to \infty$, this completes the proof. □

Theorem 4.3. *Let $\phi : \mathbb{R}^2 \to [0, \infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{2^i} \text{ converges} \qquad (4.38)$$

for all $x, y \in \mathbb{R}$. If the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|D_f(x,y)| \le \phi(x,y) \qquad (4.39)$$

for all $x, y \in \mathbb{R}$ and $f(0) = 0$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ and a unique quartic function $Q : \mathbb{R} \to \mathbb{R}$ which satisfies the equation (1.13) *and the inequality*

$$\begin{aligned}
&|f(y) - A(y) - Q(y)| \\
&\le \frac{1}{48} \sum_{i=0}^{\infty} [\frac{\phi(0, 2^i y) + \phi(0, -2^i y)}{2^i} + \frac{1}{40 \times 16^i} [2\phi(3(2^i y), 2^i y) \\
&\quad + 2\phi(3(-2^i y), -2^i y) + 2\phi(2(2^i y), 2^i y) + 2\phi(2(-2^i y), -2^i y) \\
&\quad + \phi(0, 2(2^i y)) + \phi(0, 2(-2^i y)) + 18\phi(2^i y, 2^i y) + 18\phi(-2^i y, -2^i y)
\end{aligned}$$

$$+ 44\phi(0, 2^i y) + 44\phi(0, -2^i y)]] \tag{4.40}$$

for all $y \in \mathbb{R}$.

Proof. Since

$$\begin{aligned}
& f_o(x+3y) + f_o(x-3y) + f_o(x+2y) + f_o(x-2y) + 22f_o(x) \\
& - 13[f_o(x+y) + f_o(x-y)] + 24f_o(y) - 12f_o(2y) \\
= & \frac{1}{2}[f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) \\
& - 13[f(x+y) + f(x-y)] + 24f(y) - 12f(2y) \\
& - [f(-(x+3y)) + f(-(x-3y)) + f(-(x+2y)) + f(-(x-2y)) \\
& + 22f(-x) - 13[f(-(x+y)) + f(-(x-y))]] \\
& + 24f(-y) - 12f(-2y)].
\end{aligned} \tag{4.41}$$

Thus, we obtain

$$\begin{aligned}
|D_{f_o}(x,y)| &= \frac{1}{2}[|D_f(x,y) - D_f(-x,-y)|] \\
&\le \frac{1}{2}[|D_f(x,y)| + |D_f(-x,-y)|] \\
&\le \frac{1}{2}[\phi(x,y) + \phi(-x,-y)].
\end{aligned} \tag{4.42}$$

We have

$$|D_{f_o}(x,y)| \le \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \tag{4.43}$$

for all $x, y \in \mathbb{R}$. Since f_o is odd, then by Theorem 4.1, there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_o(y) - A(y)| \le \frac{1}{24} \sum_{i=0}^{\infty} \frac{\phi(0, 2^i y) + \phi(0, -2^i y)}{2 \times 2^i} \tag{4.44}$$

for all $y \in \mathbb{R}$. Since

$$\begin{aligned}
& f_e(x+3y) + f_e(x-3y) + f_e(x+2y) + f_e(x-2y) + 22f_e(x) \\
& - 13[f_e(x+y) + f_e(x-y)] + 24f_e(y) - 12f_e(2y) \\
= & \frac{1}{2}[f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) \\
& - 13[f(x+y) + f(x-y)] + 24f(y) - 12f(2y) \\
& + f(-(x+3y)) + f(-(x-3y)) + f(-(x+2y)) + f(-(x-2y)) \\
& + 22f(-x) - 13[f(-(x+y)) + f(-(x-y))] \\
& + 24f(-y) - 12f(-2y)] = 0.
\end{aligned} \tag{4.45}$$

Thus, we obtain

$$|D_{f_e}(x,y)| = \frac{1}{2}[|D_f(x,y) + D_f(-x,-y)|]$$

$$\leq \frac{1}{2}[|D_f(x,y)| + |D_f(-x,-y)|]$$
$$\leq \frac{1}{2}[\phi(x,y) + \phi(-x,-y)]. \tag{4.46}$$

We have

$$|D_{f_e}(x,y)| \leq \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \tag{4.47}$$

for all $x, y \in \mathbb{R}$. Since f_e is even and $f(0) = 0$, then by Theorem 4.2, there exists a unique quartic function $Q : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{aligned} &|f_e(y) - Q(y)| \\ &\leq \frac{1}{960}\sum_{i=0}^{\infty}\frac{1}{2\times 16^i}[2\phi(3(2^iy),2^iy) + 2\phi(2(2^iy),2^iy) + \phi(0,2(2^iy)) \\ &\quad + 18\phi(2^iy,2^iy)+44\phi(0,2^iy)+2\phi(3(-2^iy),-2^iy)+2\phi(2(-2^iy),-2^iy) \\ &\quad + \phi(0,2(-2^iy))+18\phi(-2^iy,-2^iy)+44\phi(0,-2^iy)] \end{aligned} \tag{4.48}$$

for all $y \in \mathbb{R}$. Combining (4.44) and (4.48), we obtain

$$\begin{aligned} &|f_o(y) - A(y)| + |f_e(y) - Q(y)| \\ &\leq \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^iy) + \phi(0,-2^iy)}{2\times 2^i} \\ &\quad + \frac{1}{960}\sum_{i=0}^{\infty}\frac{1}{2\times 16^i}[2\phi(3(2^iy),2^iy)+2\phi(2(2^iy),2^iy) \\ &\quad +\phi(0,2(2^iy))+18\phi(2^iy,2^iy)+44\phi(0,2^iy)+2\phi(3(-2^iy),-2^iy) \\ &\quad +2\phi(2(-2^iy),-2^iy)+\phi(0,2(-2^iy))+18\phi(-2^iy,-2^iy)+44\phi(0,-2^iy)]. \end{aligned} \tag{4.49}$$

Thus, we have

$$\begin{aligned} &|f_o(y) - A(y) + f_e(y) - Q(y)| \\ &\leq \frac{1}{48}\sum_{i=0}^{\infty}[\frac{\phi(0,2^iy) + \phi(0,-2^iy)}{2^i} + \frac{1}{40}.\frac{1}{16^i}[2\phi(3(2^iy),2^iy) \\ &\quad +2\phi(2(2^iy),2^iy) + \phi(0,2(2^iy)) + 18\phi(2^iy,2^iy) + 44\phi(0,2^iy) \\ &\quad +2\phi(3(-2^iy),-2^iy) + 2\phi(2(-2^iy),-2^iy) + \phi(0,2(-2^iy)) \\ &\quad +18\phi(-2^iy,-2^iy) + 44\phi(0,-2^iy)]]. \end{aligned} \tag{4.50}$$

That is,

$$\begin{aligned} &|f(y) - A(y) - Q(y)| \\ &\leq \frac{1}{48}\sum_{i=0}^{\infty}[\frac{\phi(0,2^iy) + \phi(0,-2^iy)}{2^i} + \frac{1}{40\times 16^i}[2\phi(3(2^iy),2^iy) \\ &\quad + 2\phi(3(-2^iy),-2^iy) + 2\phi(2(2^iy),2^iy) + 2\phi(2(-2^iy),-2^iy) \end{aligned}$$

$$+\phi(0,2(2^i y))+\phi(0,2(-2^i y))+18\phi(2^i y,2^i y)+18\phi(-2^i y,-2^i y)$$
$$+44\phi(0,2^i y)+44\phi(0,-2^i y)]]. \tag{4.51}$$

This completes the proof of the theorem. □

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References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[2] J. K. Chung and P. K. Sahoo, *On the general solution of a quartic functional equation*, Bull. Korean Math. Soc. **40** (2003), no. 4, 565–576.

[3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.

[4] M. Eshaghi Gordji, *Stability of a functional equation deriving from quartic and additive functions*, Bull. Korean Math. Soc. **47** (2010), no. 3, 491–502.

[5] P. Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436.

[6] C. Hengkrawit and A. Thanyacharoen, *A general solution of a generalized quartic functional equation and its stability*, Int. J. Pure Appl. Math. **80** (2013), no. 4, 691–706.

[7] D. H. Hyers, *On the stability of the linear functional equations*, Proc. Natl. Acad. Sci. USA **27** (1941), no. 4, 222–224.

[8] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe-Uniwersylet Ślaski, Warszawa-Kraków-Katowice, 1985.

[9] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.

[10] P. K. Sahoo, *On a functional equation characterizing polynomials of degree three*, Bull. Inst. Math. Acad. Sin. (N.S.) **32** (2004), no. 1, 35–44.

[11] ______, *A generalized cubic functional equation*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 5, 1159–1166.

[12] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1964.

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