

## SHAPE OPERATOR AND GAUSS MAP OF POINTWISE 1-TYPE

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ABSTRACT. We examine the relationship of the shape operator of a surface of Euclidean 3-space with its Gauss map of pointwise 1-type. Surfaces with constant mean curvature and right circular cones with respect to some properties of the shape operator are characterized when their Gauss map is of pointwise 1-type.

### 1. Introduction

For last thirty years or so, the notion of finite type immersions has been widely applied in classifying and characterizing the submanifolds in Euclidean space since it was introduced in the late 1970's: Let  $x : M \rightarrow \mathbb{E}^m$  be an isometric immersion of a Riemannian manifold  $M$  into the Euclidean space of dimension  $m$ . If we identify  $x$  with the position vector at each point  $p$  of  $M$ , it is a vector valued function. Let  $\Delta$  be the Laplacian operator defined on  $M$ . If  $x$  is decomposed as  $x = x_0 + x_1 + x_2 + \cdots + x_k$ , it is said to be of finite type, where  $x_0$  is a constant vector and  $x_i$  non-constant maps satisfying  $\Delta x_i = \lambda_i x_i$  for some constant  $\lambda_i$ . In particular, if  $\lambda_i$ 's are different, we say that it is of  $k$ -type ([3, 4, 15]). It is well known that a submanifold of Euclidean space of 1-type is either a minimal submanifold of  $\mathbb{E}^m$  or a minimal submanifold of a hypersphere ([4]). It is still open whether a compact hypersurface of finite type in Euclidean space is of 1-type or not.

Also, we can apply the notion of finite type to the smooth maps defined on the submanifold  $M$  in Euclidean space naturally ([1, 2, 6]). A smooth map  $\phi$  on  $M$  of Euclidean space is said to be of finite type if  $\phi$  is expressed as a finite sum of eigenfunctions of the Laplacian  $\Delta$  of  $M$ , that is,  $\phi = \phi_0 + \sum_{i=1}^k \phi_i$ , where  $\phi_0$  is a constant function,  $\phi_i$  non-constant functions satisfying  $\Delta \phi_i = \lambda_i \phi_i$ ,  $\lambda_i \in \mathbb{R}$  ( $i = 1, \dots, k$ ). Among smooth maps defined on  $M$ , the Gauss map  $G$  draws a keen interest to differential geometry. It is identified with a unit normal vector field if  $M$  is a hypersurface of  $\mathbb{E}^m$ . Some surfaces such as helicoids and right

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cones have the interesting property. Their Gauss map  $G$  satisfies

$$(1.1) \quad \Delta G = f(G + C)$$

for some non-zero smooth function  $f$  and a constant vector  $C$ . Such a Gauss map is called of *pointwise 1-type* ([5, 7, 9, 12, 13, 14, 16]). In particular, if  $C = 0$ , it is said to be of pointwise 1-type of the first kind. If  $C \neq 0$ , then it is said to be of pointwise 1-type of the second kind. The typical example of a surface with pointwise 1-type Gauss map of the first kind is a helicoid and that of the second kind is a right circular cone ([5, 9]). We also call a submanifold of Euclidean space a *proper submanifold* regarding the Gauss map of pointwise 1-type if the function  $f$  defined in (1.1) is not constant.

However, in Section 4, we show that an isometric immersion  $x$  of a submanifold  $M$  into Euclidean space  $\mathbb{E}^m$  satisfying  $\Delta x = f(x + C)$  for a non-zero function  $f$  and a constant vector  $C$  is of 1-type, i.e., the function  $f$  is a constant.

In the present paper, we characterize surfaces of  $\mathbb{E}^3$  with the shape operator and pointwise 1-type Gauss map.

Throughout the paper all submanifolds are connected and maps are smooth unless otherwise stated.

## 2. Preliminaries

In the present section we recall definitions and some basic properties. Let  $x : M \rightarrow \mathbb{E}^m$  be an immersion from an  $n$ -dimensional connected Riemannian manifold  $M$  into an  $m$ -dimensional Euclidean space  $\mathbb{E}^m$ . We denote by  $g$  the metric tensor of  $\mathbb{E}^m$  as well as the induced metric on  $M$ . Let  $\tilde{\nabla}$  be the Levi-Civita connection of  $\mathbb{E}^m$  and  $\nabla$  the induced connection on  $M$ . Then the Gaussian and Weingarten formulas are given respectively by

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where  $X, Y$  are vector fields tangent to  $M$  and  $\xi$  normal to  $M$ . Here,  $h$  is the second fundamental form,  $D$  the linear connection induced in the normal bundle  $T^\perp M$ , called the normal connection and  $A_\xi$  the shape operator in the direction of  $\xi$  that is related with  $h$  by

$$g(h(X, Y), \xi) = g(A_\xi X, Y).$$

For an  $n$ -dimensional submanifold  $M$  in  $\mathbb{E}^m$ . The mean curvature vector  $\vec{H}$  is given by

$$\vec{H} = \frac{1}{n} \operatorname{tr} h.$$

A submanifold  $M$  is said to be minimal (respectively, totally geodesic) if  $\vec{H} \equiv 0$  (respectively,  $h \equiv 0$ ).

For a real valued function  $f$  on  $M$  the Laplacian of  $f$  is defined by

$$(2.3) \quad \Delta f = - \sum_i (e_i e_i f - \nabla_{e_i} e_i f),$$

or, locally, it is expressed as

$$(2.4) \quad \Delta f = - \frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{\mathcal{G}} g^{ij} \frac{\partial}{\partial x^j} \right) f,$$

where  $(g^{ij})$  are the components of the inverse matrix of the metric  $(g_{ji})$  of  $M$  and  $\mathcal{G}$  is the determinant of  $(g_{ji})$ .

Let us now define the Gauss map  $G$  of a submanifold  $M$  into  $G(n, m)$  in  $\wedge^n \mathbb{E}^m$ , where  $G(n, m)$  is the Grassmannian manifold consisting of all oriented  $n$ -planes through the origin of  $\mathbb{E}^m$  and  $\wedge^n \mathbb{E}^m$  is the vector space obtained by the exterior product of  $n$  vectors in  $\mathbb{E}^m$ . In a natural way, we can identify  $\wedge^n \mathbb{E}^m$  with some Euclidean space  $\mathbb{E}^N$ , where  $N = {}_m C_n$ . Let  $e_1, \dots, e_n, e_{n+1}, \dots, e_m$  be an adapted local orthonormal frame field in  $\mathbb{E}^m$  such that  $e_1, e_2, \dots, e_n$  are tangent to  $M$  and  $e_{n+1}, \dots, e_{n+2}, \dots, e_m$  normal to  $M$ . The map  $G : M \rightarrow G(n, m)$  defined by  $G(p) = (e_1 \wedge e_2 \wedge \dots \wedge e_n)(p)$  is called the Gauss map of  $M$  that is a smooth map which carries a point  $p$  in  $M$  into the oriented  $n$ -plane in  $\mathbb{E}^m$  obtained from the parallel translation of the tangent space of  $M$  at  $p$  in  $\mathbb{E}^m$  to the  $n$ -plane passing through the origin of  $\mathbb{E}^m$ . In particular, if  $M$  is a hypersurface of  $\mathbb{E}^m$ , the Gauss map  $G$  is obviously identified with a unit normal vector field on  $M$ . It is well known that mean curvature is constant if and only if the Gauss map is of pointwise 1-type of the first kind when a submanifold  $M$  is a hypersurface of Euclidean space ([5, 12]).

For a submanifold of Euclidean space  $\mathbb{E}^m$ , considering the results of [17] concerning mean curvature vector, we have:

**Lemma 2.1** ([17]). *Let  $M$  be a submanifold of Euclidean space  $\mathbb{E}^m$ . Then, the mean curvature vector  $\vec{H}$  is parallel if and only if the Gauss map  $G$  is of pointwise 1-type of the first kind.*

In particular, if the surface  $M$  is a ruled surface, the authors et. al proved the following theorem:

**Theorem 2.2** ([9]). *A ruled surface in  $\mathbb{E}^3$  with pointwise 1-type Gauss map of the first kind is an open portion of either a right circular cylinder or a helicoid.*

Thus, we immediately have:

**Corollary 2.3.** *The helicoid is the only ruled surface in  $\mathbb{E}^3$  with proper pointwise 1-type Gauss map of the first kind.*

The authors proved:

**Theorem 2.4** ([8]). *Let  $M$  be a non-cylindrical ruled surface in  $\mathbb{E}^3$ . Then,  $M$  has pointwise 1-type Gauss map of the second kind if and only if  $M$  is a part of a right circular cone.*

U. Dursun ([13]) classified flat surfaces in  $\mathbb{E}^3$  with pointwise 1-type Gauss map of the second kind, which will be used in the later section.

**Theorem 2.5** ([13]). *Let  $M$  be a flat surface in the Euclidean space  $\mathbb{E}^3$ . Then  $M$  has pointwise 1-type Gauss map of the second kind if and only if  $M$  is an open part of one the following surfaces:*

- (1)  $M$  is a right circular cone.
- (2)  $M$  is a cylinder parameterized by

$$(2.5) \quad x(s, t) = \gamma(s) + t\beta,$$

where  $\gamma = \gamma(s)$  is a unit speed planar base curve with curvature  $k = k(s)$  satisfying the ordinary differential equation

$$(2.6) \quad \left(\frac{dk}{ds}\right)^2 = k^4(s)\{ak^2(s) + 2bk(s) - 1\}$$

for some real numbers  $a$  and  $b \neq 0$ , and  $\beta$  is the director vector  $(0, 0, 1)$ .

*Remark.* Suppose that a cylinder  $M$  is defined by a planar curve  $\gamma$  satisfying (2.6) with the director vector  $\beta = (0, 0, 1)$  as above. Then the shape operator  $A$  and the function  $f$  satisfy, respectively

$$(2.7) \quad \|A\|^2 = k^2 \quad \text{and} \quad f = bk^3.$$

A ruled surface  $M$  in  $\mathbb{E}^3$  can be divided into either cylindrical or non-cylindrical according the constancy of the director vector  $\beta$ . Combining Theorems 2.4 and 2.5, we have:

**Theorem 2.6.** *Let  $M$  be a ruled surface of  $\mathbb{E}^3$ . Then,  $M$  has pointwise 1-type Gauss map of the second kind if and only if  $M$  is an open part of one of the following:*

- (1)  $M$  is a right circular cone.
- (2)  $M$  is a cylinder parameterized by (2.5) satisfying (2.6).

*Remark.* The right circular cones in Theorem 2.7 includes planes.

### 3. Surfaces with pointwise 1-type Gauss map

In this section, we deal with a surface  $M$  in  $\mathbb{E}^3$  with pointwise 1-type Gauss map  $G$ . It is well known that:

**Lemma 3.1** ([1]). *Let  $M$  be a surface of Euclidean space  $\mathbb{E}^3$ . Then, the Gauss map  $G$  satisfies*

$$(3.1) \quad \Delta G = 2\nabla H + \|A\|^2 G,$$

where  $H$  denotes the mean curvature of  $M$  and  $\nabla H$  the gradient of  $H$ .

We now prove:

**Theorem 3.2.** *Let  $M$  be a surface of  $\mathbb{E}^3$  with pointwise 1-type Gauss map satisfying (1.1). Then, the following are equivalent:*

- (1) *The squared length of the Weingarten map is represented as a constant multiple of  $f$ , i.e.,  $\|A\|^2 = \alpha f$  for some  $\alpha \in \mathbb{R}$ .*
- (2)  *$M$  has constant mean curvature or  $M$  is part of a right circular cone.*

*Proof.* Suppose the squared length of the Weingarten map is represented by a constant multiple of  $f$  such as

$$(3.2) \quad \|A\|^2 = \alpha f$$

for some  $\alpha \in \mathbb{R}$ .

**Case 1.** Suppose  $\alpha = 0$ . Then,  $M$  is totally geodesic and thus the mean curvature is obviously constant.

**Case 2.** Suppose  $\alpha = 1$ . If we combine (1.1) with (3.1), we have

$$2\nabla H = fC.$$

First, suppose the open subset  $U = \{p \in M \mid K(p) \neq 0\}$  is non-empty, where  $K$  is the Gaussian curvature of  $M$ . If there exists a point  $p \in U$  such that  $f(p) = 0$ , (3.2) yields  $A(p) = 0$  which contradicts  $K(p) \neq 0$ . Therefore,  $f(p) \neq 0$  for all  $p \in U$ . Thus, on  $U$ ,  $C = \frac{2}{f}\nabla H$ . So,  $\langle G, C \rangle = 0$  on  $U$ . It implies  $\langle AX, C \rangle = 0$  for any tangent vector field  $X$  defined on  $U$ . Therefore,  $AC = 0$  on  $U$  and thus  $C = 0$  since  $K$  never vanishes on  $U$ , or, equivalently, the mean curvature  $H$  is constant on a component  $U_1$  of  $U$ . Since  $C$  is constant vector,  $C = 0$  on  $M$  and hence  $G$  is of pointwise 1-type of the first kind. According to Lemma 2.1, the surface  $M$  has constant mean curvature.

We now suppose the surface  $M$  is flat, i.e., the Gaussian curvature  $K$  is vanishing. In case that the Gauss map  $G$  is of pointwise 1-type of the first kind, due to Lemma 2.1, we see that the mean curvature  $H$  is constant. In this case,  $M$  is part of a circular cylinder.

Next, suppose that the Gauss map is of pointwise 1-type of the second kind. Theorem 2.5 and Remark 2.6 imply  $M$  is either a right circular cone or a cylinder satisfying (2.6) and (2.7).

If  $M$  is part of a right circular cone, then the shape operator  $A$  does not satisfy  $\|A\|^2 = f$ . Therefore,  $M$  is part of a cylinder defined by (2.5) satisfying (2.6). By combining Remark 2.6 and the assumption  $\|A\|^2 = f$ , the curvature function  $k$  of the base curve must be constant and hence  $M$  is part of a plane or a circular cylinder. Since  $f$  is not vanishing,  $\|A\| \neq 0$ . Therefore,  $M$  is part of a circular cylinder and hence the mean curvature is constant. Hence,  $M$  has pointwise 1-type Gauss map of the first kind which is a contradiction.

**Case 3.** Suppose that  $\alpha \neq 0$  and  $\alpha \neq 1$ . (1.1) and (3.1) induce

$$(3.3) \quad f(1 + \langle G, C \rangle) = \|A\|^2,$$

$$(3.4) \quad 2\nabla H = fC^T,$$

where  $C^T$  denotes the tangential part of the constant vector  $C$ . Then,  $C^T$  becomes

$$(3.5) \quad C^T = C - (\alpha - 1)G$$

on the open subset  $V_0 = \{p \in M \mid f(p) \neq 0\}$ .

Suppose the open subset  $V_1 = \{p \in V_0 \mid K(p) \neq 0\}$  is non-empty. Let  $\mathbf{e}_1$  and  $\mathbf{e}_2$  be the principal directions associated with the principal curvatures  $\kappa_1$  and  $\kappa_2$ , respectively on  $V_1$ . On  $V_1$ , if we take the covariant differentiation to (3.5), we get

$$(3.6) \quad AX = \frac{1}{\alpha - 1} \nabla_X C^T$$

for a tangent vector field  $X$ . But, thanks to (3.5), we have  $h(X, C^T) = 0$  for all tangent vector field  $X$  on  $M$ . Since  $K \neq 0$  and  $\det A \neq 0$  on  $V_1$ ,  $C^T$  is zero on each component  $U_1$  of  $V_1$ . Then, the Gauss map  $G$  becomes a constant vector  $\frac{1}{\alpha - 1}C$  on  $U_1$  and so  $U_1$  is contained in a plane, which is a contradiction on  $V_1$ . Consequently,  $V_0$  is flat.

If  $C = 0$ , then  $G$  is of pointwise 1-type of the first kind and thus  $M$  has constant mean curvature.

Suppose  $C \neq 0$ , that is, the Gauss map is of pointwise 1-type of the second kind. Then, due to Theorem 2.5, we see that  $V_0$  is part of either a right circular cone or a cylinder defined by (2.5) satisfying (2.6).

Suppose  $V_0$  is part of a cylinder defined by (2.5) satisfying (2.6). By Remark 2.6 and (3.2), we see that the curvature  $k$  of the plane curve  $\gamma$  is a non-zero constant. Thus, each component  $W_1$  of  $V_0$  is contained in a circular cylinder and hence the constant vector  $C$  is zero on  $W_1$  which contradicts  $C \neq 0$ . Consequently,  $W_1$  is contained in a right circular cone. If we choose a surface patch  $x(u, v) = (v \cos u, v \sin u, av)$ , ( $v > 0$ ) of a right circular cone for a constant  $a > 0$ , we easily have the mean curvature  $H = \frac{a}{2v\sqrt{1+a^2}} > 0$ .

Suppose  $M - V_0$  has a non-empty interior. Then,  $M - V_0$  is contained in a plane. Therefore,  $M - V_0$  is either empty or the whole surface  $M$ . Consequently, we can conclude that the surface  $M$  has constant mean curvature or  $M$  is part of a right circular cone.

The converse is obvious. □

As we see in the proof of Theorem 3.2, if the function  $f$  on a surface  $M$  of  $\mathbb{E}^3$  with pointwise 1-type Gauss map admitting (1.1) satisfies  $f = \|A\|^2$ , the Gauss map  $G$  is of the first kind, i.e.,  $C = 0$ . Consequently, we have a characterization of right circular cones including planes as follows:

**Theorem 3.3** (Characterization). *Let  $M$  be a surface of  $\mathbb{E}^3$ . Then,  $M$  is part of a right circular cone if and only if its Gauss map  $G$  of  $M$  satisfies*

$$\Delta G = \alpha \|A\|^2 (G + C)$$

for some non-zero real number  $\alpha \neq 1$  and a non-zero constant vector  $C$ .

#### 4. Submanifolds of $\mathbb{E}^m$ satisfying $\Delta x = f(x + C)$

Let  $M$  be an  $n$ -dimensional submanifold of  $\mathbb{E}^m$  satisfying

$$(4.1) \quad \Delta x = f(x + C)$$

for some function  $f$  and a constant vector  $C$ . By the well known result  $\Delta x = -n\vec{H}$ , we have

$$(4.2) \quad -n\vec{H} = f(x + C).$$

Differentiating covariantly (4.2) with respect to a tangent vector field  $X$  to  $M$  and using Weingarten formula (2.2), we get

$$(4.3) \quad -n(-A_{\vec{H}}X + D_X\vec{H}) = (Xf)(x + C) + fX.$$

Suppose that  $f \neq 0$ , that is,  $M$  is not minimal in  $\mathbb{E}^m$ . Then, there exists a non-empty open subset  $U = \{p \in M \mid f(p) \neq 0\}$ . By using (4.2), if we compare the tangential part and the normal part of (4.3), we get

$$(4.4) \quad A_{\vec{H}}X = \frac{f}{n}X,$$

$$(4.5) \quad \frac{Xf}{f}\vec{H} = D_X\vec{H}$$

on  $U$ . From (4.4) we see that

$$(4.6) \quad f = n\|\vec{H}\|^2.$$

Therefore,  $A_{\vec{H}}X = \|\vec{H}\|^2 X$  for every tangent vector field  $X$ , that is,  $M$  is pseudo-umbilical on  $U$ .

Next, taking the inner product to (4.5) with  $\vec{H}$  and making use of (4.6), we have

$$Xf = 0$$

for every tangent vector field  $X$  on  $U$ . Thus, the function  $f$  is constant on each component of  $U$ . Since  $f$  is continuous on  $M$ ,  $f$  is a constant on  $M$ , that is,  $M$  is of usual 1-type.

If we make use of the classification theorem of 1-type submanifold of  $\mathbb{E}^m$  in [3], we have:

**Theorem 4.1.** *Let  $M$  be an  $n$ -dimensional submanifold of  $m$ -dimensional Euclidean space  $\mathbb{E}^m$ . Then, the following are equivalent:*

- (1) *The immersion  $x$  of  $M$  into  $\mathbb{E}^m$  satisfies (4.1).*
- (2)  *$M$  is a minimal submanifold of  $\mathbb{E}^m$  or a minimal submanifold of a hypersphere in  $\mathbb{E}^m$ .*

### 5. Examples

In this section, we consider a few examples of surfaces of  $\mathbb{E}^3$  which have pointwise 1-type Gauss map and compare the squared length of the Weingarten maps and the function  $f$  defined in (1.1).

**Example 5.1.** Let  $M$  be a circular cylinder parameterized by

$$x(u, v) = (\cos u, \sin u, v), \quad 0 < u < 2\pi, \quad v \in \mathbb{R}.$$

Its Gauss map  $G$  is given by

$$G = (\cos u, \sin u, 0).$$

Then,  $\Delta G = G$  and the Weingarten map  $A$  satisfies  $\|A\|^2 = 1$ .

**Example 5.2.** Consider a right circular cone  $C_a$  which is parameterized by

$$x(u, v) = (v \cos u, v \sin u, av), \quad a > 0.$$

Then the Gauss map  $G$  and its Laplacian  $\Delta G$  are respectively given by

$$G = \frac{1}{\sqrt{1+a^2}}(a \cos u, a \sin u, -1)$$

and

$$\Delta G = \frac{1}{v^2} \left( G + \left( 0, 0, \frac{1}{\sqrt{1+a^2}} \right) \right),$$

from which, it is easy to compute  $\|A\|^2 = \frac{a^2}{1+a^2} \frac{1}{v^2}$ .

**Example 5.3.** Let  $M$  be a catenoid parameterized by

$$x(u, v) = (\sqrt{1+v^2} \cos u, \sqrt{1+v^2} \sin u, \sinh^{-1} v).$$

Then, its Gauss map  $G$  is given by

$$G = \frac{1}{\sqrt{1+v^2}}(\cos u, \sin u, -v)$$

and, the Laplacian  $\Delta G$  satisfies

$$\Delta G = \frac{2}{(1+v^2)^2} G.$$

In this case, the Weingarten map  $A$  is given by

$$A = \begin{pmatrix} \frac{1}{\sqrt{1+v^2}} & 0 \\ 0 & -\frac{1}{\sqrt{1+v^2}} \end{pmatrix}.$$

Thus, we have

$$\|A\|^2 = \frac{2}{1+v^2}.$$



**Example 5.4.** Let  $f$  be a function of variable  $s$  satisfying

$$\sin^{-1}\left(\frac{c^2 f^{-1/3} - 1}{c_1^2 + c_2^2}\right) - \sqrt{c_1^2 + c_2^2 - (c^2 f^{-1/3} - 1)^2} = \pm c^3(s + 1),$$

where  $c_1$  and  $c_2$  are some constants with  $(c_1, c_2) \neq (0, 0)$  and  $c$  is a non-zero real number. Let

$$x(s, t) = \left( \int \cos \theta(s) ds, \int \sin \theta(s) ds, t \right)$$

where  $\theta'(s) = c\sqrt[3]{f}$ . Then,  $x$  defines a cylinder of an infinite type (For details, see [11]). This cylinder satisfies  $\Delta G = f(G + (c_1, c_2, 0))$ , that is, the Gauss map  $G$  is of pointwise 1-type of the second kind. But, in this case,  $\|A\|^2 = (\theta'(s))^2 = c^2\sqrt[3]{f^2}$ .

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