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THE ANNIHILATING-IDEAL GRAPH OF A RING

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Abstract. Let S be a semigroup with 0 and R be a ring with 1. We extend the definition of the zero-divisor graphs of commutative semigroups to not necessarily commutative semigroups. We define an annihilating-ideal graph of a ring as a special type of zero-divisor graph of a semigroup. We introduce two ways to define the zero-divisor graphs of semigroups. The first definition gives a directed graph $\Gamma(S)$, and the other definition yields an undirected graph $\overline{\Gamma}(S)$. It is shown that $\Gamma(S)$ is not necessarily connected, but $\overline{\Gamma}(S)$ is always connected and $\mathrm{diam}(\overline{\Gamma}(S)) \leq 3$. For a ring R define a directed graph $\mathbb{APOG}(R)$ to be equal to $\Gamma(\mathbb{IPO}(R))$, where $\mathbb{IPO}(R)$ is a semigroup consisting of all products of two one-sided ideals of R, and define an undirected graph $\overline{\mathbb{APOG}}(R)$ to be equal to $\overline{\Gamma}(\mathbb{IPO}(R))$. We show that R is an Artinian (resp., Noetherian) ring if and only if $\mathbb{APOG}(R)$ has DCC (resp., ACC) on some special subset of its vertices. Also, it is shown that $\overline{\mathbb{APOG}}(R)$ is a complete graph if and only if either $(D(R))^2 = 0$, R is a direct product of two division rings, or R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathbb{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$. Finally, we investigate the diameter and the girth of square matrix rings over commutative rings $M_{n\times n}(R)$ where $n \geq 2$.

1. introduction

In [11], I. Beck associated to a commutative ring R its zero-divisor graph $G(R)$ whose vertices are all elements of R (including 0), and two distinct vertices a and b are adjacent if $ab = 0$. In [10], Anderson and Livingston introduced and studied the subgraph $\Gamma(R)$ (of $G(R)$) whose vertices are the nonzero zero-divisors of R. This graph turns out to best exhibit the properties of the set of zero-divisors of R, and the ideas and problems introduced

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in [10] were further studied in [4, 8, 9]. In [20], Redmond extended the definition of zero-divisor graph to non-commutative rings. Some fundamental results concerning zero-divisor graph for a non-commutative ring were given in [5, 6, 22]. For a commutative ring R with 1, denoted by $\mathbb{A}(R)$, the set of ideals with nonzero annihilator. The annihilating-ideal graph of R is an undirected graph $\mathbb{AG}(R)$ with vertices $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$, where distinct vertices I and J are adjacent if $IJ = (0)$. The concept of the annihilating-ideal graph of a commutative ring was introduced in [12, 13] were further studied in [1, 2, 3, 7]. For a ring R, let $D(R)$ be the set of one-sided zero-divisors of R and $\mathbb{IPO}(R) = \{A \subseteq R : A = IJ$ where I and J are left or right ideals of $R\}$. Let S be a semigroup with 0, and $D(S)$ be the set of one-sided zero-divisors of S. The zero-divisor graph of a commutative semigroup is an undirected graph with vertices $Z(S)^*$ (the set of non-zero zero-divisors) and two distinct vertices a and b are adjacent if $ab = 0$. The zero-divisor graph of a commutative semigroup was introduced in [15] and further studied in [14, 23, 24, 25].

Let Γ be a graph. For vertices x and y of Γ, let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such a path). The diameter of Γ is defined as $\operatorname{diam}(\Gamma) = sup\{d(x, y)|\ x$ and y are vertices of $\Gamma\}$. The girth of Γ, denoted by $\operatorname{gr}(\Gamma)$, is the length of a shortest cycle in Γ ($\operatorname{gr}(\Gamma) = \infty$ if Γ contains no cycles).

In Section 2, we introduce a directed graph $\Gamma(S)$ for a semigroup S with 0. We show that $\Gamma(S)$ is not necessarily connected. Then we find a necessarily and sufficient condition for $\Gamma(S)$ to be connected. After that we extend the annihilating-ideal graph to a (not necessarily commutative) ring. It is shown that $\mathbb{IPO}(R)$ is a semigroup. We associate to a ring R a directed graph (denote by $\mathbb{APOG}(R)$) the zero-divisor graph of $\mathbb{IPO}(R)$, i.e., $\mathbb{APOG}(R) = \Gamma(\mathbb{IPO}(R))$. Then we show that R is an Artinian (resp., Noetherian) ring if and only if $\mathbb{APOG}(R)$ has DCC (resp., ACC) on some subset of its vertices. In Section 3, we introduce an undirected graph $\overline{\Gamma}(S)$ for a semigroup S with 0. We show that $\overline{\Gamma}(S)$ is always connected and $\operatorname{diam}(\overline{\Gamma}(S)) \leq 3$. Moreover, if $\overline{\Gamma}(S)$ contains a cycle, then $\operatorname{gr}(\overline{\Gamma}(S)) \leq 4$. After that we define an undirected graph which extends the annihilating-ideal graph to a not necessarily commutative ring. We associate to a ring R an undirected graph (denoted by $\overline{\mathbb{APOG}}(R)$) the undirected zero-divisor graph of $\mathbb{IPO}(R)$, i.e., $\overline{\mathbb{APOG}}(R) = \overline{\Gamma}(\mathbb{IPO}(R))$. Finally, we characterize rings whose undirected annihilating-ideal graphs are complete graphs. In Section 4, we investigate the undirected annihilating-ideal graphs of matrix rings over commutative rings. It is shown that $\operatorname{diam}((\overline{\mathbb{APOG}}(M_n(R))) \geq 2$ where $n \geq 2$. Also, we show that $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R)) \geq \operatorname{diam}(\overline{\mathbb{APOG}}(R))$.

2. Directed annihilating-ideal graph of a ring

Let S be a semigroup with 0 and $D(S)$ denote the set of one-sided zero-divisors of S. We associate to S a directed graph $\Gamma(S)$ with vertices set $D(S)^* =$

$D(S) \setminus \{0\}$ and $a \to b$ if $ab = 0$. In this section, we investigate the properties of $\Gamma(S)$ and we first show the following result.

Proposition 2.1. *Let R be a ring. Then $\mathbb{IPO}(R)$ is a semigroup.*

Proof. Let $A, B \in \mathbb{IPO}(R)$. Then there exist left or right ideals I_1, J_1, I_2, J_2 of R such that $A = I_1J_1$ and $B = I_2J_2$. We show that $AB = (I_1J_1)(I_2J_2) \in \mathbb{IPO}(R)$.

Case 1: J_1 is a left ideal. Then $AB = I_1(J_1I_2J_2) \in \mathbb{IPO}(R)$ (as $J_1I_2J_2$ is a left ideal of R).

Case 2: J_1 is a right ideal and either I_2 is a left ideal or J_2 is a right ideal. Then $AB = (I_1J_1)(I_2J_2) \in \mathbb{IPO}(R)$.

Case 3: J_1 is a right ideal, I_2 is a right ideal, and J_2 is a left ideal. Then $AB = (I_1J_1I_2)J_2 \in \mathbb{IPO}(R)$.

Thus $\mathbb{IPO}(R)$ is multiplicatively closed. Since the multiplication is associative, $\mathbb{IPO}(R)$ is a semigroup. □

It was shown in [15, Theorem 1.2] that the zero-divisor graph of a commutative semigroup S is connected and $\text{diam}(\Gamma(S)) \leq 3$. In the following example we show that $\Gamma(S)$ is not necessarily connected when S is a non-commutative semigroup.

Example 2.2. Let K be a field and $V = \oplus_{i=1}^{\infty} K$. Then $R = HOM_K(V, V)$, under the point-wise addition and the multiplication taken to be the composition of functions, is an infinite non-commutative ring with identity. Let $\pi_1 : V \to V$ be defined by $(a_1, a_2, \ldots) \mapsto (a_1, 0, \ldots)$ and $f : V \to V$ be defined by $(a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots)$. Then $\pi_1, f \in R$. Note that $(R\pi_1)(fR) = 0$, so $\Gamma(\mathbb{IPO}(R)) \neq \emptyset$. However, $\Gamma(\mathbb{IPO}(R))$ is not connected as there is no path leading from the vertex (fR) to any other vertex of $\Gamma(\mathbb{IPO}(R))$. This is because there exists $g : V \to V$ given by $(a_1, a_2, \ldots) \mapsto (a_2, a_3, \ldots)$ and $g \in R$ such that $gf = 1_R$. □

For a semigroup S, let

$$A^l(S) = \{a \in D(S)^* : \text{ there exists } b \in D(R)^* \text{ such that } ba = 0\}$$

and

$$A^r(S) = \{a \in D(S)^* : \text{ there exists } b \in D(R)^* \text{ such that } ab = 0\}.$$

Next we show that $\Gamma(S)$ is connected if and only if $A^l(S) = A^r(S)$. Moreover, if $\Gamma(S)$ is connected, then $\text{diam}(\Gamma(S)) \leq 3$.

Theorem 2.3. *Let S be a semigroup. Then $\Gamma(S)$ is connected if and only if $A^l(S) = A^r(S)$. Moreover, if $\Gamma(S)$ is connected, then diam$(\Gamma(S)) \leq 3$.*

Proof. Suppose that $A^l(S) = A^r(S)$. Let a and b be distinct vertices of $\Gamma(S)$. Then $a \neq 0$ and $b \neq 0$. We show that there is always a path with length at most 3 from a to b.

Case 1: $ab = 0$. Then $a \to b$ is a desired path.

Case 2: $ab \neq 0$. Then since $A^l(S) = A^r(S)$, there exists $c \in D(S) \setminus \{0\}$ such that $ac = 0$ and $d \in D(S) \setminus \{0\}$ such that $db = 0$.

Subcase 2.1: $c = d$. Then $a \to c \to b$ is a desired path.

Subcase 2.2: $c \neq d$. If $cd = 0$, then $a \to c \to d \to b$ is a desired path. If $cd \neq 0$, then $a \to cd \to b$ is a desired path.

Thus $\Gamma(S)$ is connected and $\mathrm{diam}(\Gamma(S)) \leq 3$.

Conversely, if $\Gamma(S)$ is connected, then it is easy to show that $A^l(S) = A^r(S)$. □

Now, we define a directed graph which extends the annihilating-ideal graph to an arbitrary ring. We associate to a ring R a directed graph (denoted by $\mathbb{APOG}(R)$) the zero-divisor graph of $\mathbb{IPO}(R)$, i.e., $\mathbb{APOG}(R) = \Gamma(\mathbb{IPO}(R))$.

Corollary 2.4. *Let R be a ring. Then $\mathbb{APOG}(R)$ is connected if and only if $A^l(\mathbb{IPO}(R)) = A^r(\mathbb{IPO}(R))$. Moreover, if $\mathbb{APOG}(R)$ is connected, then* $\mathrm{diam}(\mathbb{APOG}(R)) \leq 3$.

Proof. Since $\mathbb{APOG}(R)$ is equal to $\Gamma(\mathbb{IPO}(R))$, it follows from Theorem 2.3 that $\mathbb{APOG}(R)$ is a connected if and only if $A^l(\mathbb{IPO}(R)) = A^r(\mathbb{IPO}(R))$. Also, if $\mathbb{APOG}(R)$ is connected, then $\mathrm{diam}(\mathbb{APOG}(R)) \leq 3$. □

Recall that a Duo ring is a ring in which every one-sided ideal is a two-sided ideal.

Proposition 2.5. *Let R be an Artinian Duo ring. Then*

$$A^l(\mathbb{IPO}(R)) = A^r(\mathbb{IPO}(R)) = \mathbb{IPO}(R) \setminus \{0, R\}.$$

Moreover, $\mathbb{APOG}(R)$ is connected and $\mathrm{diam}(\mathbb{APOG}(R)) \leq 3$.

Proof. Let R be a Duo ring. Then by [17, Lemma 4.2], $R = (R_1, \mathfrak{m}_1) \times (R_2, \mathfrak{m}_2) \times \cdots \times (R_n, \mathfrak{m}_n)$, where each $R_i (1 \leq i \leq n)$ is an Artinian local ring with unique maximal ideal $\mathfrak{m}_i$. Let $A \in \mathbb{IPO}(R) \setminus \{0, R\}$. Then $A = (I_1 \times I_2 \times \cdots \times I_n)(J_1 \times J_2 \times \cdots \times J_n)$, where every $I_i (1 \leq i \leq n)$ is an one-sided ideal, so is every $J_j (1 \leq j \leq n)$. Since $A \neq R$, there exists I_i (or J_j) such that $I_i \neq R$ (or $J_j \neq R$). Without loss of generality we may assume that $I_i \neq R$. So $A = (I_1 \times I_2 \times \cdots \times I_n)(J_1 \times J_2 \times \cdots \times J_n) \subseteq (R_1 \times \cdots \times I_i \times \cdots \times R_n)(R_1 \times \cdots \times R_i \times \cdots \times R_n)$. Suppose k is the smallest positive integer such that ${I_i}^k = 0$. Thus $(0 \times \cdots \times I_i^{k-1} \times \cdots \times 0)((R_1 \times \cdots \times I_i \times \cdots \times R_n)(R_1 \times \cdots \times R_i \times \cdots \times R_n)) = 0$ and $((R_1 \times \cdots \times I_i \times \cdots \times R_n)(R_1 \times \cdots \times R_i \times \cdots \times R_n))(0 \times \cdots \times I_i^{k-1} \times \cdots \times 0) = 0$. Therefore $A \in A^l(\mathbb{IPO}(R))$ and $A \in A^r(\mathbb{IPO}(R))$. Thus $\mathbb{IPO}(R) \setminus \{0, R\} \subseteq A^r(\mathbb{IPO}(R))$ and $\mathbb{IPO}(R) \setminus \{0, R\} \subseteq A^l(\mathbb{IPO}(R))$. We conclude that $A^r(\mathbb{IPO}(R)) = \mathbb{IPO}(R) \setminus \{0, R\} = A^l(\mathbb{IPO}(R))$.

The second part follows from Theorem 2.3. □

It is well known that if $|D(R)| \geq 2$ is finite, then $|R|$ is finite. Let A, B be vertices of $\mathbb{APOG}(R)$. We use $A \rightleftharpoons B$ if $A \to B$ or $A \leftarrow B$. For any vertices C and D of $\mathbb{APOG}(R)$, let $\mathrm{ad}(C) = \{A$ is a vertex of $\mathbb{APOG}(R) : C = A$

or $C \rightleftharpoons A$ or there exists a vertex B of $\mathbb{APOG}(R)$ such that $C \rightleftharpoons B \rightleftharpoons A\}$ and $\mathrm{adu}(D) = \bigcup_{C \subseteq D} \mathrm{ad}(C)$. We know that $\mathrm{ad}(C) \subseteq D(R)$. The following proposition shows that if a principal left or right ideal I of R is a vertex of $\mathbb{APOG}(R)$ and all left and right ideals of $\mathrm{ad}(I)$ have finite cardinality, then R has finite cardinality.

Proposition 2.6. *Let R be a ring and I be a principal left or right ideal of R such that I is a vertex of $\mathbb{APOG}(R)$. If all left and right ideals of* $\mathrm{ad}(I)$ *have finite cardinality, then R has finite cardinality.*

Proof. Without loss of generality, we may assume that I is a left principal ideal. Thus $I = Rx$ for some non-zero $x \in R$. If $Ann_l(x) = 0$, then $|R| = |I| < \infty$. So we may always assume that $Ann_l(x) \neq 0$.

Case 1: $I = Ann_r(x)$ and $Ann_r(x)Ann_l(x) = 0$. Then

$$I \to Ann_l(x)$$

and so $Ann_l(x) \in \mathrm{ad}(I)$. Therefore, $Ann_l(x)$ is finite. Since $I \cong R/Ann_l(x)$, $|R| = |I||Ann_l(x)| < \infty$.

Case 2: $I \neq Ann_r(x)$ and $Ann_r(x)Ann_l(x) = 0$. If $Ann_r(x) \neq 0$, then

$$I \to Ann_r(x) \to Ann_l(x)$$

and so $Ann_l(x) \in \mathrm{ad}(I)$. Therefore, $Ann_l(x)$ is finite. Since $I \cong R/Ann_l(x)$, $|R| = |I||Ann_l(x)| < \infty$. If $Ann_r(x) = 0$, then since Rx is a vertex of $\mathbb{APOG}(R)$, there exists a (nonzero right ideal) J such that $JRx = 0$ (replace J by JR if necessary). Since $Ann_r(x) = 0$, we have xJ is a nonzero right ideal and so

$$Ann_l(x) \to xJ \to I.$$

Thus $Ann_l(x) \in \mathrm{ad}(I)$, so $Ann_l(x)$ is finite. Again, we have $|R| = |I||Ann_l(x)| < \infty$.

Case 3: $I \neq Ann_r(x)$ and $Ann_r(x)Ann_l(x) \neq 0$. Then

$$Ann_r(x) \leftarrow I \to Ann_r(x)Ann_l(x) \to (xR)$$

and so $(xR), Ann_r(x) \in \mathrm{ad}(I)$. Therefore, (xR) and $Ann_r(x)$ are finite. Since $(xR) \cong R/Ann_r(x)$, $|R| = |(xR)||Ann_r(x)| < \infty$. This completes the proof. □

Here is our main result in this section.

Theorem 2.7. *Let R be a ring such that $\mathbb{APOG}(R) \neq \emptyset$. Then R is Artinian* (*resp., Noetherian*) *if and only if for a left or right ideal I in the vertex set of* $\mathbb{APOG}(R)$*,* $\mathrm{adu}(I)$ *has DCC* (*resp., ACC*) *on both its left and right ideals.*

Proof. If R is Artinian, then $\mathbb{IPO}(R)$ has DCC on both its left ideals and right ideals. Thus for every left or right ideal of the vertex set of $\mathbb{APOG}(R)$, $\mathrm{adu}(I)$ has DCC on both its left and right ideals as $\mathrm{adu}(I) \subseteq \mathbb{IPO}(R)$.

Conversely, without loss of generality let I be a left ideal of vertex set of $\mathbb{APOG}(R)$ such that $\mathrm{adu}(I)$ has DCC on its left and right ideals. Assume that $x \in I$. We have the following cases:

Case 1: $xRx \neq \{0\}$, $Ann_l(x) \neq 0$, and $Ann_r(x) \neq 0$. Then

$$(xR) \leftarrow Ann_l(x) \leftarrow xRx \rightarrow Ann_r(x) \leftarrow (Rx).$$

Therefore $(xR), Ann_r(x), Ann_l(x), (Rx) \in \mathrm{ad}(xRx)$. Since $\mathrm{ad}(xRx) \subseteq \mathrm{adu}(I)$ and $\mathrm{adu}(I)$ has DCC on its left and right ideals, we conclude that (Rx) and $Ann_l(x)$ are left Artinian R-modules, and (xR) and $Ann_r(x)$ are right Artinian R-modules. Since $(Rx) \cong R/Ann_l(x)$ and $(xR) \cong R/Ann_r(x)$, by [18, (1.20)] we conclude that R is Artinian.

Case 2: $xRx = \{0\}$, $Ann_l(x) \neq 0$, and $Ann_r(x) \neq 0$. Then

$$Ann_l(x) \rightarrow (xR) \rightarrow (Rx) \rightarrow Ann_r(x).$$

Since $\mathrm{ad}(Rx) \subseteq \mathrm{adu}(I)$ and $\mathrm{adu}(I)$ has DCC on its left and right ideals, we conclude that (Rx) and $Ann_l(x)$ are left Artinian R-modules, and (xR) and $Ann_r(x)$ are right Artinian R-modules. Since $(Rx) \cong R/Ann_l(x)$ and $(xR) \cong R/Ann_r(x)$, by [18, (1.20)] we conclude that R is Artinian.

Case 3: $Ann_l(x) = \{0\}$. Then $Rx \cong R$. Therefore, R is a left Artinian module. Since Rx is a vertex of $\mathbb{APOG}(R)$, we have $Ann_r(x) \neq \{0\}$. So there exists $y \in D(R) \setminus \{0\}$ such that $xy = 0$.

Subcase 3.1: $yRy \neq \{0\}$. If $Ann_r(y) = \{0\}$, then since

$$Rx \rightarrow yR,$$

we have $yR \in \mathrm{adu}(I)$, so yR is a Artinian right R-module. Note that $yR \cong R$. Therefore, R is a right Artinian module. If $Ann_r(y) \neq \{0\}$, then

$$Ann_r(y) \leftarrow yRy \leftarrow yRx \rightarrow yR.$$

Therefore $(yR), Ann_r(y) \in \mathrm{ad}(yRx) \subseteq \mathrm{adu}(I)$. Since $\mathrm{adu}(I)$ has DCC on its right ideals, we conclude that (yR) and $Ann_r(y)$ are right Artinian R-modules. Note that $(yR) \cong R/Ann_r(y)$, by [18, (1.20)] we conclude that R is a right Artinian module.

Subcase 3.2: $yRy = \{0\}$. Then

$$yR \leftarrow yRx \leftarrow Ry \rightarrow Ann_r(y).$$

Since $(yR), Ann_r(y) \in \mathrm{ad}(yRx) \subseteq \mathrm{adu}(I)$, we conclude that (yR) and $Ann_r(y)$ are right Artinian R-modules. Note that $(yR) \cong R/Ann_r(y)$, by [18, (1.20)] we conclude that R is a right Artinian module.

Case 4: $Ann_r(x) = \{0\}$. Then $xRx \neq \{0\}$ and since Rx is a vertex of $\mathbb{APOG}(R)$, we have $Ann_l(x) \neq \{0\}$. Therefore,

$$(xR) \leftarrow Ann_l(x) \rightarrow xRx.$$

We conclude that $xR, Ann_l(x) \in \mathrm{ad}(xRx) \subseteq \mathrm{adu}(I)$. Since $xR, Rx, Ann_l(x) \in \mathrm{adu}(I)$, we have Rx and $Ann_l(x)$ are left Artinian modules and xR is a right

Artinian module. Note that $(Rx) \cong R/Ann_l(x)$ and $(xR) \cong R/Ann_r(x)$. Again by [18, (1.20)] we conclude that R is Artinian. $\square$

Corollary 2.8. *Let R be a ring such that $\mathbb{APOG}(R) \neq \emptyset$. Then R is Artinian (resp., Noetherian) if and only if $\mathbb{APOG}(R)$ has DCC (resp., ACC) on left and right ideals of its vertex set.*

Proof. Since vertex set of $\mathbb{APOG}(R)$ is a subset of $\mathbb{IPO}(R)$, As in the proof of Theorem 2.7, if R is Artinian (resp., Noetherian), then $\mathbb{APOG}(R)$ has DCC (resp., ACC) on left and right ideals of its vertex set.

Conversely, since for a left or right ideal I of the vertex set of $\mathbb{APOG}(R)$, adu(I) is a subset of the vertex set of $\mathbb{APOG}(R)$, it follows from Theorem 2.7 that R is Artinian. $\square$

A directed graph Γ is called a tournament if for every two distinct vertices x and y of Γ exactly one of xy and yx is an edge of Γ. In other words, a tournament is a complete graph with exactly one direction assigned to each edge.

Proposition 2.9. *Let R be a ring such that $A^2 \neq \{0\}$ for every non-zero $A \in \mathbb{IPO}(R)$ and $A^l(\mathbb{IPO}(R)) \cap A^r(\mathbb{IPO}(R)) \neq \emptyset$. Then $\mathbb{APOG}(R)$ is not a tournament.*

Proof. Assume $\mathbb{APOG}(R)$ is a tournament. Since $A^l(\mathbb{IPO}(R)) \cap A^r(\mathbb{IPO}(R)) \neq \emptyset$, there exists $B \in A^l(\mathbb{IPO}(R)) \cap A^r(\mathbb{IPO}(R))$, that is, there exist distinct non-zero $A, C \in \mathbb{IPO}(R)$ such that $A \to B \to C$ is a path in $\mathbb{APOG}(R)$. If $CA \neq \{0\}$, then $B(CA) = (BC)A = \{0\}$ and $(CA)B = C(AB) = \{0\}$, which is a contradiction. So $CA = \{0\}$ and therefore $AC \neq \{0\}$ since $\mathbb{APOG}(R)$ is a tournament. Also, $AC \neq A$ (otherwise $A^2 = (ACAC) = A(CA)C = \{0\}$) and similarly, $AC \neq C$. Let $a, a_1 \in A$ and $c, c_1 \in C$. Then we have $B \to C \to ((a - a_1c)R)$ and $(R(c - ac_1)) \to A \to B$. As the above $((a - a_1c)R)B = \{0\}$ and $B(R(c - ac_1)) = \{0\}$. Let $b \in B$ be an arbitrary element. Then $-acb = a_1b - acb \in ((a - a_1c)R)B = \{0\}$ and $bac = bc_1 - bac \in B(R(c - ac_1)) = \{0\}$. Therefore, $ACB = \{0\}$ and $BAC = \{0\}$. Thus both $AC \to B$ and $B \to AC$ are edges of $\mathbb{APOG}(R)$. This is a contradiction, hence, $\mathbb{APOG}(R)$ cannot be a tournament. $\square$

3. Undirected annihilating-ideal graph of a ring

Let S be a semigroup with 0 and recall that $D(S)$ denotes the set of one-sided zero-divisors of S. We associate to S an undirected graph $\overline{\Gamma}(S)$ with vertices set $D(S)^* = D(S) \setminus \{0\}$ and two distinct vertices a and b are adjacent if $ab = 0$ or $ba = 0$. Similarly, we associate to a ring R an undirected graph (denoted by $\overline{\mathbb{APOG}}(R)$) the undirected zero-divisor graph of $\mathbb{IPO}(R)$, i.e., $\overline{\mathbb{APOG}}(R) = \overline{\Gamma}(\mathbb{IPO}(R))$. The only difference between $\mathbb{APOG}(R)$ and $\overline{\mathbb{APOG}}(R)$ is that the former is a directed graph and the latter is undirected (that is, these graphs share the same vertices and the same edges if directions on the edges are ignored). If R is a commutative ring, this definition agrees with the previous definition of the annihilating-ideal graph. In this section we study the properties of $\overline{\Gamma}(S)$. We first show that $\overline{\Gamma}(S)$ is always connected with diameter at most 3.

Theorem 3.1. *Let S be a semigroup. Then $\overline{\Gamma}(S)$ is a connected graph and* $\mathrm{diam}(\overline{\Gamma}(S)) \leq 3$.

Proof. Let a and b be distinct vertices of $\overline{\Gamma}(S)$. If $ab = 0$ or $ba = 0$, then $a - b$ is a path. Next assume that $ab \neq 0$ and $ba \neq 0$.

Case 1: $a^2 = 0$ and $b^2 = 0$. Then $a - ab - b$ is a path.

Case 2: $a^2 = 0$ and $b^2 \neq 0$. Then there is a some $c \in D(S) \setminus \{a, b, 0\}$ such that either $cb = 0$ or $bc = 0$. If either $ac = 0$ or $ca = 0$, then $a - c - b$ is a path. If $ac \neq 0$ and $ca \neq 0$, then $a - ca - b$ is a path if $bc = 0$ and $a - ac - b$ is a path if $cb = 0$.

Case 3: $a^2 \neq 0$ and $b^2 = 0$. We can use an argument similar to that of the above case to obtain a path.

Case 4: $a^2 \neq 0$ and $b^2 \neq 0$. Then there exist $c, d \in D(S) \setminus \{a, b, 0\}$ such that either $ca = 0$ or $ac = 0$ and either $db = 0$ or $bd = 0$. If $bc = 0$ or $cb = 0$, then $a - c - b$ is a path. Similarly, if $ad = 0$ or $da = 0$, $a - d - b$ is a path. So we may assume that $c \neq d$. If $cd = 0$ or $dc = 0$, then $a - c - d - b$ is a path. Thus we may further assume that $cd \neq 0, dc \neq 0$, $bc \neq 0, cb \neq 0$, $ad \neq 0$ and $da \neq 0$. We divide the proof into 4 subcases.

Subcase 4.1: $ac = 0$ and $db = 0$. Then $a - cd - b$ is a path.

Subcase 4.2: $ac = 0$ and $bd = 0$. Then $a - cb - d - b$ is a path.

Subcase 4.3: $ca = 0$ and $bd = 0$. Then $a - dc - b$ is a path.

Subcase 4.4: $ca = 0$ and $db = 0$. $a - bc - d - b$ is a path.

Thus $\overline{\Gamma}(S))$ is connected and $\mathrm{diam}(\overline{\Gamma}(S)) \leq 3$. □

In [10], Anderson and Livingston proved that if $\Gamma(R)$ (the zero-divisor graph of a commutative ring R) contains a cycle, then $\mathrm{gr}(\Gamma(R)) \leq 7$. They also proved that $\mathrm{gr}(\Gamma(R)) \leq 4$ when R is Artinian and conjectured that this is the case for all commutative rings R. Their conjecture was proved independently by Mulay [19] and DeMeyer and Schneider [16]. Also, in [20], Redmond proved that if $\overline{\Gamma}(R)$ (the undirected zero-divisor graph of a non-commutative ring) contains a cycle, then $\mathrm{gr}(\overline{\Gamma}(R)) \leq 4$. The following is our first main result in this section which shows that for a (not necessarily commutative) semigroup S, if $\overline{\Gamma}(S)$ contains a cycle, then $gr(\overline{\Gamma}(S)) \leq 4$.

Theorem 3.2. *Let S be a semigroup. If $\overline{\Gamma}(S)$ contains a cycle, then $gr(\overline{\Gamma}(S)) \leq$ 4.*

Proof. Let $a_1 - a_2 - \cdots - a_{n-1} - a_n - a_1$ be a cycle of shortest length in $\overline{\Gamma}(S)$. Assume that $gr(\overline{\Gamma}(S)) > 4$, i.e., assume $n \geq 5$. Note that $a_2 a_{n-1} \neq 0$ and $a_{n-1} a_2 \neq 0$ (as $n \geq 5$). If $a_2 a_{n-1} \notin \{a_1, a_n\}$, then $a_1 - a_2 a_{n-1} - a_n - a_1$ is a cycle of length 3, yielding a contradiction. Also, if $a_{n-1} a_2 \notin \{a_1, a_n\}$, then

$a_1 - a_{n-1}a_2 - a_n - a_1$ is a cycle of length 3, yielding a contradiction. We have the following cases:

Case 1 : $a_2a_{n-1} = a_1$ and $a_{n-1}a_2 = a_n$. If $a_2a_3 = 0$, then $a_na_3 = (a_{n-1}a_2)a_3 = 0$. Therefore, $a_1 - a_2 - a_3 - a_n - a_1$ is a cycle of length 4, yielding a contradiction. So, $a_3a_2 = 0$. Thus, $a_3a_1 = a_3(a_2a_{n-1}) = 0$. Therefore, $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$ is a cycle of length $n-1$, yielding a contradiction.

Case 2 : $a_2a_{n-1} = a_1$ and $a_{n-1}a_2 = a_1$. If $a_2a_3 = 0$, then $a_1a_3 = (a_{n-1}a_2)a_3 = 0$. Therefore, $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$ is a cycle of length $n-1$, yielding a contradiction. So, $a_3a_2 = 0$. Thus, $a_3a_1 = a_3(a_2a_{n-1}) = 0$. Therefore, $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$ is a cycle of length $n-1$, yielding a contradiction.

Case 3 : $a_2a_{n-1} = a_n$ and $a_{n-1}a_2 = a_1$. If $a_2a_3 = 0$, then $a_1a_3 = (a_{n-1}a_2)a_3 = 0$. Therefore, $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$ is a cycle of length $n-1$, yielding a contradiction. So, $a_3a_2 = 0$. Thus, $a_3a_n = a_3(a_2a_{n-1}) = 0$. Therefore, $a_1 - a_2 - a_3 - a_n - a_1$ is a cycle of length 4, yielding a contradiction.

Case 4 : $a_2a_{n-1} = a_n$ and $a_{n-1}a_2 = a_n$. If $a_2a_3 = 0$, then $a_na_3 = (a_{n-1}a_2)a_3 = 0$. If $a_3a_2 = 0$, then $a_3a_n = a_3(a_2a_{n-1}) = 0$. Therefore, $a_1 - a_2 - a_3 - a_n - a_1$ is a cycle of length 4, yielding a contradiction.

Since in all cases we have found contradictions, we conclude that if $\overline{\Gamma}(S)$ contains a cycle, then $gr(\overline{\Gamma}(S)) \leq 4$. □

Corollary 3.3. *Let R be a ring. Then $\overline{\mathbb{APOG}}(R)$ is a connected graph and* $\operatorname{diam}(\overline{\mathbb{APOG}}(R)) \leq 3$. *Moreover, If $\overline{\mathbb{APOG}}(R)$ contains a cycle, then*

$$\operatorname{gr}(\overline{\mathbb{APOG}}(R)) \leq 4.$$

Proof. Note that $\overline{\mathbb{APOG}}(R)$ is equal to $\overline{\Gamma}(\mathbb{IPO}(R))$. So by Theorem 3.1, $\overline{\mathbb{APOG}}(R)$ is a connected graph and $\operatorname{diam}(\overline{\mathbb{APOG}}(R)) \leq 3$. Also, by Theorem 3.2, if $\overline{\mathbb{APOG}}(R)$ contains a cycle, then $\operatorname{gr}(\overline{\mathbb{APOG}}(R)) \leq 4$. □

For a not necessarily commutative ring R, we define a simple undirected graph $\overline{\Gamma}(R)$ with vertex set $D(R)^*$ (the set of all non-zero zero-divisors of R) in which two distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$ (see [20]). The Jacobson radical of R, denoted by $J(R)$, is equal to the intersection of all maximal right ideals of R. It is well-known that $J(R)$ is also equal to the intersection of all maximal left ideals of R. In our second main theorem in this section we characterize rings whose undirected annihilating-ideal graphs are complete graphs.

Theorem 3.4. *Let R be a ring. Then $\overline{\mathbb{APOG}}(R)$ is a complete graph if and only if either $(D(R))^2 = 0$, or R is a direct product of two division rings, or R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathbb{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$.*

Proof. Assume that $\overline{\mathbb{APOG}}(R)$ is a complete graph. If $\overline{\Gamma}(R)$ is a complete graph, then by [5, Theorem 5], either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $D(R)^2 = \{0\}$. So the forward direction holds. Next assume that $\overline{\Gamma}(R)$ is not a complete graph. So there exist different vertices x and y of $\overline{\Gamma}(R)$ such that x and y are not adjacent. We have the following cases:

Case 1: $x \in A^r(R)$. Without loss of generality assume that $y \in A^r(R)$. If $Rx \neq Ry$, then since $\mathbb{APOG}(R)$ is a complete graph, we have Rx is adjacent to Ry in $\overline{\mathbb{APOG}}(R)$, so x and y are adjacent in $\overline{\Gamma}(R)$, yielding a contradiction. Thus $Rx = Ry$. Since $x \in A^r(R)$, there exists non-zero element $z \in D(R)$ such that $xz = 0$. If $Rx \subseteq zR$, then $(Rx)^2 = \{0\}$. So $(Rx)(Ry) = \{0\}$, and x and y are adjacent in $\overline{\Gamma}(R)$, yielding a contradiction. Therefore, $Rx \nsubseteq zR$. If there exists a left or right ideal I of R expect zR such that $I \nsubseteq Rx$, then there exists nonzero element $s \in I \setminus Rx$. Then $(Rs + Rx)(zR) = \{0\}$. Since $\overline{\mathbb{APOG}}(R)$ is a complete graph Rx is adjacent to $(Rs + Rx) = \{0\}$. Thus $(Rx)^2 = \{0\}$, and so x and y are adjacent in $\overline{\Gamma}(R)$, yielding a contradiction. Therefore, $\{zR, Rx\}$ is the set of nonzero proper left or right ideals of R. Thus by Corollary 2.8, R is an Artinian ring. We have the following subcases:

Subcase 1: $zR \nsubseteq Rx$. Then zR and Rx are maximal ideals. If zR or Rx is not a two-sided ideal, then $zR = J(R) = Rx$, yielding a contradiction. Therefore, Rx and zR are two-sided ideals. Also, Rx and zR are minimal ideals and so $Rx \cap zR = \{0\}$. Thus by Brauer's Lemma (see [18, 10.22]), $(Rx)^2 = 0$ or $Rx = Re$, where e is a idempotent in R. If $(Rx)^2 = \{0\}$, then x is adjacent to y in $\overline{\Gamma}(R)$, yielding a contradiction. So $Rx = Re$, where e is an idempotent in R. Therefore, $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Since $\{zR, Rx\}$ is the set of nonzero proper left or right ideals of R and $Rx \cap zR = \{0\}$, we conclude that $Re = Rx = eR$ and $(1-e)R = zR = R(1-e)$. Therefore, $(1-e)Re = (1-e)eR = \{0\}$ and $eR(1-e) = e(1-e)R = \{0\}$. So $R = eRe \oplus (1-e)R(1-e)$. Since R is an Artinian ring with two nonzero left or right ideals, we conclude that eRe and $(1-e)R(1-e)$ are division rings.

Subcase 2: $zR \subseteq Rx$. Then $Rx = D(R)$. If $(Rx)^2 = \{0\}$, then x is adjacent to y in $\overline{\Gamma}(R)$, yielding a contradiction. If $D(R)^2 \neq 0$, then $D(R)^2 = zR$. Therefore, R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathbb{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$.

In summary, we obtain that either R is a direct product of two division rings, or R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathbb{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$. Thus the forward direction holds.

Case 2: $x \in A^l(R)$. Similar to Case 1, we conclude that either R is a direct product of two division rings, or R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathbb{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$. So the forward direction holds.

The converse is obvious. □

4. Undirected annihilating-ideal graphs for matrix rings over commutative rings

In this section we investigate the undirected annihilating-ideal graphs of matrix rings over commutative rings. By Theorem 3.3, $\mathrm{diam}(\overline{\mathbb{APOG}}(R)) \leq 3$ for any ring R. In Proposition 4.1 we show that $\mathrm{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq 2$

where $n \geq 2$. A natural question is whether or not $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq \operatorname{diam}(\overline{\mathbb{APOG}}(R))$. We show that the answer to this question is affirmative.

Proposition 4.1. *Let R be a commutative ring. Then*

$$\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq 2, \text{ where } n \geq 2.$$

Proof. Let

$$A = (M_n(R)\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}) \text{ and } B = (\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M_n(R)).$$

Since

$$A(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M_n(R)) = 0 \text{ and } (M_n(R)\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix})B = 0,$$

we conclude that A and B are vertices in $(\overline{\mathbb{APOG}}(M_n(R)))$. Note that

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2 \neq 0 \text{ and } \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in A \cap B,$$

so $AB \neq 0$. Therefore, $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq 2$. □

Theorem 4.2. *Let R be a commutative ring. Then* $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq \operatorname{diam}(\mathbb{AG}(R)) = \operatorname{diam}(\overline{\mathbb{APOG}}(R))$.

Proof. By [12, Theorem 2.1], $\operatorname{diam}(\mathbb{AG}(R)) \leq 3$.

Case 1: $diam(\mathbb{AG}(R)) \leq 2$. By Proposition 4.1, $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq 2$. Thus $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq \operatorname{diam}(\mathbb{AG}(R))$.

Case 2: $\operatorname{diam}(\mathbb{AG}(R)) = 3$. Then there exist vertices I, J, K, and L of $\mathbb{AG}(R)$ such that $I - K - L - J$ is a shortest path between I and J. So $d(I, J) = 3$. Since I and J are vertices of $\mathbb{AG}(R)$, $M_n(I)$ and $M_n(J)$ are vertices of $\overline{\mathbb{APOG}}(M_n(R))$. Suppose that $\operatorname{diam}(\overline{\mathbb{APOG}}(M_n(R))) = 2$. So we can assume that there exists $\alpha = [a_{ij}] \in M_n(R)$ such that $M_n(I)\alpha = \alpha M_n(J) = 0$. Without loss of generality, we may assume that $a_{11} \neq 0$. For every $a \in I$,

$$\begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} A = 0,$$

so $aa_{11} = 0$. Therefore $I(a_{11}R) = 0$. For every $b \in J$,

$$A \begin{bmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = 0.$$

Therefore $(a_{11}R)J = 0$. Thus $I - (a_{11}R) - J$ is a path of length 2 in $\mathbb{AG}(R)$, and so $d(I, J) \leq 2$, yielding a contradiction. Therefore, $\mathrm{diam}(\overline{\mathbb{APOG}}(M_n(R))) = 3$ and we are done. $\square$

It was shown in Corollary 3.3 that $\mathrm{gr}(\overline{\mathbb{APOG}}(R)) \leq 4$. We now show that $\mathrm{gr}(\overline{\mathbb{APOG}}(M_n(R))) = 3$ where $n \geq 2$.

Proposition 4.3. *Let R be a commutative ring. Then* $\mathrm{gr}(\overline{\mathbb{APOG}}(M_n(R))) = 3$ *where $n \geq 2$.*

Proof. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $(AM_n(R)A) - (BM_n(R)B) - (CM_n(R)C)$ is a cycle in $\overline{\mathbb{APOG}}(M_n(R))$, so $\mathrm{gr}(\overline{\mathbb{APOG}}(M_n(R))) = 3$. $\square$

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