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ON SKEW SYMMETRIC OPERATORS WITH EIGENVALUES

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Abstract. An operator T on a complex Hilbert space $\mathcal{H}$ is called skew symmetric if T can be represented as a skew symmetric matrix relative to some orthonormal basis for $\mathcal{H}$. In this paper, we study skew symmetric operators with eigenvalues. First, we provide an upper-triangular operator matrix representation for skew symmetric operators with nonzero eigenvalues. On the other hand, we give a description of certain skew symmetric triangular operators, which is based on the geometric relationship between eigenvectors.

1. Introduction

Throughout this paper, we denote by $\mathcal{H}$ a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Recall that a map C on $\mathcal{H}$ is called a *conjugation* if C is conjugate-linear, $C^{-1}=C$ and $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathcal{H}$. An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be *skew symmetric* if $CTC=-T^*$ for some conjugation C on $\mathcal{H}$. We remark that $T\in\mathcal{B}(\mathcal{H})$ is skew symmetric if and only if T admits a skew symmetric matrix representation with respect to some orthonormal basis (ONB, for short) of $\mathcal{H}$. Thus skew symmetric operators can be viewed as an infinite dimensional analogue of skew symmetric matrices.

Skew symmetric operators have been studied for many years in the finite dimensional setting. The most obvious examples of skew symmetric operators on finite dimensional spaces are those Jordan blocks with even ranks (see [14, Ex. 1.7]). Recently, there has been growing interest in skew symmetric operators in the infinite dimensional case, and some interesting results have been obtained [13, 14, 15, 16, 17, 18, 20]. In particular, skew symmetric normal operators, partial isometries, compact operators and weighted shifts are classified [13, 14, 18].

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The primary motivation for the study of skew symmetric operators lies in its connections to complex symmetric operators, which have received much attention in the last decade [4, 7, 8, 9, 10, 11, 12, 19]. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *complex symmetric* if $CTC = T^*$ for some conjugation C on $\mathcal{H}$. The following lemma, whose proof is omitted, summarizes some basic facts about complex symmetric operators and skew symmetric operators.

Lemma 1.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then*

(i) *there exist $A, B \in \mathcal{B}(\mathcal{H})$ such that $T = A + B$, $CAC = -A^*$ and $CBC = B^*$;*
(ii) *if $CTC = -T^*$, then T^{2n} is complex symmetric with respect to C for all $n \geq 1$;*
(iii) *if T is complex symmetric, then $T \oplus (-T)$ and $T^*T - TT^*$ are both skew symmetric.*

By Lemma 1.1, one can use complex symmetric operators to construct new skew symmetric operators. In particular, if T is complex symmetric, then $T^*T - TT^*$ is skew symmetric. In view of the description of skew symmetric normal operators [14, Thm. 1.10], this provides a new approach to describing complex symmetric operators. In a recent paper [12], one can see such an application to Toeplitz operators. The study of skew symmetric operators has applications to some special operators on function spaces [2, 3]. In particular, any commutator of two truncated Toplitz operators is skew symmetric.

Another motivation for the study of skew symmetric operators lies in the connection between skew symmetric operators and anti-automorphisms of singly generated C^*-algebras. Recall that an *anti-automorphism* of a C^*-algebra $\mathcal{A}$ is a vector space isomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathcal{A}$. It is proved that each C^*-algebra generated by a skew symmetric operator admits an involutory anti-automorphism on it (see [17, Cor. 3.2]).

The present aim of this paper is to explore the structure of skew symmetric operators with eigenvalues. For skew symmetric operators with nonzero eigenvalues, we give an upper-triangular operator matrix representation to describe their structure (see Theorem 2.5). An application to Foguel operators will be provided. On the other hand, we give a geometric description of certain skew symmetric triangular operators (see Theorems 3.4 and 3.7), which is based on the geometric relationship between eigenvectors.

2. Upper triangular representation

In this section, we shall provide an upper-triangular operator matrix representation for skew symmetric operators with nonzero eigenvalues and describe their structure. The main results of this section are Theorems 2.2 and 2.5.

We first make some preparation.

Definition 2.1 ([1], page 95)**.** Let $T \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called a *transpose* of T, if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

Note that if $T \in \mathcal{B}(\mathcal{H})$ is skew symmetric, then $-T = CT^*C$ for some conjugation C on $\mathcal{H}$; so $-T$ is a transpose of T. In general, an operator has more than one transpose [17, Ex. 2.2]. However, any two transposes of an operator are unitarily equivalent.

Recall that a map C on $\mathcal{H}$ is called an *antiunitary operator* if C is conjugate-linear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. Thus a conjugation is an involutory antiunitary operator.

If $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{N}$ is a subspace of $\mathcal{H}$, then $T_{\mathcal{N}}$ will denote the *compression* of T to $\mathcal{N}$, that is, the restriction of $P_{\mathcal{N}}T$ to $\mathcal{N}$, where $P_{\mathcal{N}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}$. It $\mathcal{N}$ is invariant under T, then $T_{\mathcal{N}} = T|_{\mathcal{N}}$. We write $\ker T$ for the kernel of T.

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\Gamma \subset \mathbb{C}$. Assume that*

$$\mathcal{M} = \bigvee_{\lambda\in\Gamma, n\geq 1} \ker(T-\lambda)^n, \quad \mathcal{N} = \bigvee_{\lambda\in\Gamma, n\geq 1} \ker\,(T+\lambda)^{*n},$$

where $\vee$ denotes closed linear span. If T is skew symmetric, then $T|_{\mathcal{M}} \cong (-T^t_{\mathcal{N}})$, where $\cong$ denotes unitary equivalence.

Proof. Assume that C is a conjugation on $\mathcal{H}$ and $CTC = -T^*$. Since $C(T-\lambda)^n = (-1)^n (T+\lambda)^{*n} C$ for $\lambda \in \mathbb{C}$, it follows that $C(\ker(T-\lambda)^n) = \ker\,(T+\lambda)^{*n}$. Thus

$$\begin{aligned} C(\mathcal{M}) &= C\Big(\bigvee_{\lambda\in\Gamma, n\geq 1} \ker(T-\lambda)^n\Big) \\ &= \bigvee_{\lambda\in\Gamma, n\geq 1} C\Big(\ker(T-\lambda)^n\Big) \\ &= \bigvee_{\lambda\in\Gamma, n\geq 1} \ker\,(T+\lambda)^{*n} = \mathcal{N}. \end{aligned}$$

It follows that $C(\mathcal{N}) = \mathcal{M}$.

Note that $\mathcal{M}$ is invariant under T and $\mathcal{N}$ is invariant under T^*. Denote $A = T|_{\mathcal{M}}$ and $B = T_{\mathcal{N}}$. Thus $T^*|_{\mathcal{N}} = (T_{\mathcal{N}})^* = B^*$.

Define $D : \mathcal{M} \to \mathcal{N}$ as $Dx = Cx$ for $x \in \mathcal{M}$. Then D is an antiunitary operator and $D^{-1}y = Cy$ for all $y \in \mathcal{N}$.

Since $CT = -T^*C$, for given $x \in \mathcal{M}$, we have

$$DAx = CAx = CTx = -T^*Cx = -T^*Dx = -B^*Dx.$$

So $DA = -B^*D$. Choose a conjugation E on $\mathcal{N}$ and set $U = ED$. Then $U : \mathcal{M} \to \mathcal{N}$ is unitary and

$$UA = EDA = -EB^*D = -(EB^*E)(ED) = -(EB^*E)U.$$

That is, $A \cong (-B^t)$. $\square$

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal subset of $\mathcal{H}$. Assume that $Te_n \in \vee\{e_i : 1 \leq i \leq n\}$ and $\langle Te_n, e_n\rangle = \lambda_n$ for all $n \geq 1$,* *where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq -\lambda_j$ for all $i, j \geq 1$. If C is a conjugation on $\mathcal{H}$ and $CTC = -T^*$, then*

(i) $\langle TCe_i, Ce_i \rangle = -\lambda_i$ *for all $i \geq 1$, and*
(ii) $\langle Ce_i, e_j \rangle = 0$ *for all $i, j \geq 1$.*

Proof. For each $i \geq 1$, compute to see

$$\langle TCe_i, Ce_i \rangle = \langle Ce_i, T^*Ce_i \rangle = -\langle Ce_i, CTe_i \rangle = -\langle Te_i, e_i \rangle = -\lambda_i.$$

The rest is devoted to the proof of (ii). For $m_1, m_2 \in \mathbb{N}$, we say that (m_1, m_2) satisfies property (P) if

$$\langle Ce_i, e_j \rangle = 0, \quad \forall 1 \leq i \leq m_1, 1 \leq j \leq m_2.$$

Noting that $\langle Cx, y \rangle = \langle Cy, x \rangle$ for all x, y, one can see that (m_1, m_2) satisfies property (P) if and only (m_2, m_1) satisfies (P). Since

$$\langle Ce_i, Te_j \rangle = \langle CTe_j, e_i \rangle = -\langle T^*Ce_j, e_i \rangle = -\langle Ce_j, Te_i \rangle, \tag{2.1}$$

it follows readily that

$$\langle Ce_i, Te_i \rangle = 0, \qquad \forall i \geq 1. \tag{2.2}$$

We shall proceed by induction.

Claim 1. For $m \geq 1$, if $(1, m)$ satisfies (P), then $(1, m+1)$ satisfies (P).

Noting that $Te_{m+1} \in \vee\{e_i : 1 \leq i \leq m+1\}$ and $\langle Te_{m+1}, e_{m+1} \rangle = \lambda_{m+1}$, we may assume

$$Te_{m+1} = \lambda_{m+1}e_{m+1} + \sum_{i=1}^{m} \alpha_i e_i.$$

Since $\langle Ce_1, e_j \rangle = 0$ for all $1 \leq j \leq m$, it follows that $\langle Ce_1, Te_{m+1} \rangle = \langle Ce_1, \lambda_{m+1}e_{m+1} \rangle$. On the other hand, one can see from (2.1) that

$$\langle Ce_1, Te_{m+1} \rangle = -\langle Ce_{m+1}, Te_1 \rangle = -\langle Ce_{m+1}, \lambda_1 e_1 \rangle = -\langle Ce_1, \lambda_1 e_{m+1} \rangle.$$

Since $-\lambda_1 \neq \lambda_{m+1}$, we obtain $\langle Ce_1, e_{m+1} \rangle = 0$. This implies that $(1, m+1)$ satisfies (P).

Claim 2. For $n \geq 2$ and $m \geq 1$, if both (n, m) and $(n-1, m+1)$ satisfy (P), then $(n, m+1)$ satisfy (P).

Assume that

$$Te_{m+1} = \lambda_{m+1}e_{m+1} + \sum_{i=1}^{m} \alpha_i e_i, \quad Te_n = \lambda_n e_n + \sum_{j=1}^{n-1} \beta_j e_j.$$

Noting that (n, m) satisfies (P), we obtain $\langle Ce_n, Te_{m+1} \rangle = \langle Ce_n, \lambda_{m+1}e_{m+1} \rangle$. On the other hand, since $(m+1, n-1)$ satisfies (P), it follows from (2.1) that

$$\langle Ce_n, Te_{m+1} \rangle = -\langle Ce_{m+1}, Te_n \rangle = -\langle Ce_{m+1}, \lambda_n e_n \rangle = -\langle Ce_n, \lambda_n e_{m+1} \rangle.$$

Since $-\lambda_n \neq \lambda_{m+1}$, we obtain $\langle Ce_n, e_{m+1} \rangle = 0$. So $(n, m+1)$ satisfies (P).

Now we shall show that $\langle Ce_i, e_j \rangle = 0$ for all $i, j \geq 1$.

By (2.2), $\langle Ce_1, \lambda_1 e_1\rangle = \langle Ce_1, Te_1\rangle = 0$, that is, $(1,1)$ satisfies (P). In view of Claim 1, one can deduce recursively that $(1,n)$ satisfies (P) for all $n \geq 1$. Thus $(n,1)$ satisfies (P) for all $n \geq 1$.

Since $(2,1)$ and $(1,2)$ satisfy (P), it follows from Claim 2 that $(2,2)$ satisfies (P). On the other hand, noting that $(1,3)$ satisfies (P), it follows from Claim 2 again that $(2,3)$ satisfies (P). Recursively we can prove that $(2,n)$ satisfies (P) for all $n \geq 1$.

Since $(2,3)$ and $(3,2)$ satisfy (P), it follows from Claim 2 that $(3,3)$ satisfies (P). On the other hand, noting that $(2,4)$ satisfies (P), it follows from Claim 2 again that $(3,4)$ satisfies (P). We can recursively prove that $(3,n)$ satisfies (P) for all $n \geq 1$.

For any $m \geq 1$, just we have done above, one can prove that (m,n) satisfies (P) for all $n \geq 1$. That is, $\langle Ce_i, e_j\rangle = 0$ for all $i, j \geq 1$. □

Remark 2.4. Let $A \in \mathcal{B}(\mathcal{H})$ and D be a conjugation on $\mathcal{H}$ satisfying $DAD = -A^*$. If $e \in \ker A$ and $e \neq 0$, then it is possible that $\langle De, e\rangle \neq 0$. Here is an example:

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & -N \end{bmatrix} \begin{matrix} \mathbb{C} \\ \mathcal{H}, \\ \mathcal{H} \end{matrix}$$

where N is an invertible normal operator on $\mathcal{H}$. By [14, Thm. 1.10], T is skew symmetric. Assume that C is a conjugation on $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}$ satisfying $CTC = -T^*$. Then $C(\ker T) = \ker T^*$. Since $\ker T = \ker T^* = \mathbb{C}$ is of dimension 1, it follows that $\langle Ce, e\rangle \neq 0$ for all nonzero $e \in \ker T$.

Given a conjugation C on $\mathcal{H}$, we denote by $S_C(\mathcal{H})$ the set of all skew symmetric operators on $\mathcal{H}$ with respect to C, that is,

$$S_C(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) : CXC = -X^*\}.$$

The following result shows that each skew symmetric operator with nonzero eigenvalues admits an upper-triangular operator matrix representation.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\Gamma \subset \mathbb{C}$ with $\Gamma \cap (-\Gamma) = \emptyset$. Denote*

$$\mathcal{M} = \bigvee_{\lambda\in\Gamma, n\geq 1} \ker(T-\lambda)^n, \quad \mathcal{N} = \bigvee_{\lambda\in\Gamma, n\geq 1} \ker{(T+\lambda)^{*n}}.$$

If T is skew symmetric, then $\mathcal{M}$ is orthogonal to $\mathcal{N}$ and

$$T = \begin{bmatrix} A & E & G \\ 0 & R & F \\ 0 & 0 & B \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{L}\,, \\ \mathcal{N} \end{matrix} \tag{2.3}$$

where

(i) $\mathcal{L} = (\mathcal{M} + \mathcal{N})^{\perp}$;
(ii) $A \cong (-B^t)$ *and R is skew symmetric*;

(iii) *all the following operators are skew symmetric*

$$\begin{bmatrix} A & G \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}$$

with respect to the same conjugation on $\mathcal{M}+\mathcal{N}$;

(iv) *all the following operators are skew symmetric*

$$\begin{bmatrix} 0 & E & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & G \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & B \end{bmatrix}$$

with respect to the same conjugation on $\mathcal{H}$.

Moreover, there exist a conjugation C *on* $\mathcal{M}$, $G_1 \in S_C(\mathcal{M})$ *and a conjugation* D *on* $\mathcal{L}$ *such that*

$$DRD = -R^*, \quad T \cong \begin{bmatrix} A & E & G_1 \\ 0 & R & -DE^*C \\ 0 & 0 & -CA^*C \end{bmatrix}.$$

Proof. It is obvious that $\mathcal{M}$ is invariant under T. Denote $A = T|_{\mathcal{M}}$. Then

$$\mathcal{M} = \bigvee_{\lambda\in\Gamma, n\geq 1} \ker(T-\lambda)^n = \bigvee_{\lambda\in\Gamma, n\geq 1} \ker(A-\lambda)^n.$$

Without loss of generality, we assume that $\dim \mathcal{M} = \infty$. Then there exists an ONB $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{M}$ with respect to which A admits the following upper-triangular matrix representation

$$A = \begin{array}{c} \begin{bmatrix} \lambda_1 & * & * & \cdots \\ & \lambda_2 & * & \cdots \\ & & \lambda_3 & \cdots \\ & & & \ddots \end{bmatrix} \end{array} \begin{array}{l} e_1 \\ e_2 \\ e_3, \\ \vdots \end{array}$$

where $\lambda_n \in \Gamma$ for all $n \geq 1$. Note that $Te_n = Ae_n \in \vee\{e_i : 1 \leq i \leq n\}$ for all $n \geq 1$.

Since T is skew symmetric, we assume that $\widehat{C}$ is a conjugation on $\mathcal{M}$ so that $\widehat{C}T\widehat{C} = -T^*$. It can be seen from the proof of Theorem 2.2 that $\widehat{C}(\mathcal{M}) = \mathcal{N}$ and $\widehat{C}(\mathcal{N}) = \mathcal{M}$. Thus $\{\widehat{C}e_i\}_{i=1}^{\infty}$ is an ONB of $\mathcal{N}$. Noting that $\lambda_i \neq -\lambda_j$ for all $i,j \geq 1$, it follows from Lemma 2.3 that $\langle \widehat{C}e_i, e_j\rangle = 0$ for all $i,j \geq 1$. So $\mathcal{M}$ is orthogonal to $\mathcal{N}$. Since $\mathcal{M}$ is invariant under T and $\mathcal{N}$ is invariant under T^*, we may assume that

$$T = \begin{bmatrix} A & E & G \\ 0 & R & F \\ 0 & 0 & B \end{bmatrix} \begin{array}{c} \mathcal{M} \\ \mathcal{L} \\ \mathcal{N} \end{array},$$

where $\mathcal{L} = (\mathcal{M}+\mathcal{N})^{\perp}$.

Noting that $\widehat{C}(\mathcal{M}) = \mathcal{N}$ and $\widehat{C}(\mathcal{N}) = \mathcal{M}$, we deduce that $\widehat{C}(\mathcal{L}) = \mathcal{L}$ and $\widehat{C}$ can be written as

$$\widehat{C} = \begin{bmatrix} 0 & 0 & C_3 \\ 0 & C_2 & 0 \\ C_1 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{L} \\ \mathcal{N} \end{matrix}.$$

It follows from $\widehat{C}^2 = I$ that C_2 is a conjugation on $\mathcal{L}$, $C_1 : \mathcal{M} \to \mathcal{N}$ is an antiunitary operator and $C_1^{-1} = C_3$.

Since $T\widehat{C} = -\widehat{C}T^*$, a direct matrical calculation shows that

$$AC_3 = -C_3B^*, \ BC_1 = -C_1A^*, \ RC_2 = -C_2R^*, \ GC_1 = -C_3G^*, \ FC_1 = -C_2E^*.$$

Set $D = C_2$. Then $DRD = -R^*$, which implies that R is skew symmetric. Also one can easily check that all the following operators are skew symmetric

$$\begin{bmatrix} 0 & E & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & G \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & B \end{bmatrix}$$

with respect to the same conjugation $\widehat{C}$. Define

$$\widehat{D} = \begin{bmatrix} 0 & C_3 \\ C_1 & 0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{N} \end{matrix}.$$

Then $\widehat{D}$ is a conjugation on $\mathcal{M} + \mathcal{N}$ and

$$\widehat{D} \begin{bmatrix} A & G \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & C_3B \\ C_1A & C_1G \end{bmatrix} = \begin{bmatrix} 0 & -A^*C_3 \\ -B^*C_1 & -G^*C_3 \end{bmatrix} = -\begin{bmatrix} A^* & 0 \\ G^* & B^* \end{bmatrix} \widehat{D}.$$

Similarly one can check that

$$\widehat{D} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = -\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \widehat{D}, \quad \widehat{D} \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ G^* & 0 \end{bmatrix} \widehat{D}.$$

Then the following operators are skew symmetric

$$\begin{bmatrix} A & G \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}$$

with respect to the same conjugation $\widehat{D}$.

Choose a conjugation C on $\mathcal{M}$ and set $U = C_1C$. Then $U : \mathcal{M} \to \mathcal{N}$ is unitary and $U^{-1} = CC_3$. Thus

$$BU = BC_1C = -C_1A^*C = -(C_1C)(CA^*C) = -U(CA^*C),$$

which implies $B \cong (-A^t)$ and, equivalently, $A \cong (-B^t)$.

Define $V : \mathcal{M} \oplus \mathcal{L} \oplus \mathcal{M} \to \mathcal{H}$ as $V(x, y, z) = x + y + Uz$. Then V is unitary and V can be written as

$$V = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & U \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{L} \\ \mathcal{M} \end{matrix}.$$

Then

$$TV = \begin{bmatrix} A & E & GU \\ 0 & R & FU \\ 0 & 0 & BU \end{bmatrix} = \begin{bmatrix} A & E & GU \\ 0 & R & -C_2E^*C \\ 0 & 0 & -U(CA^*C) \end{bmatrix} = V \begin{bmatrix} A & E & GU \\ 0 & R & -DE^*C \\ 0 & 0 & -CA^*C \end{bmatrix}.$$

Set $G_1 = GU$. Then $G_1 \in \mathcal{B}(\mathcal{M})$ and

$$CG_1C = CGUC = CGC_1 = -CC_3G^* = -U^*G^* = -(GU)^* = -G_1^*.$$

That is, $G_1 \in S_C(\mathcal{M})$. This completes the proof. □

Remark 2.6. One can see from the proof of Theorem 2.5 that A, B^* in (2.3) are both triangular, that is, A, B^* can be written as upper-triangular matrices with respect to suitably chosen orthonormal bases.

Let $S \in \mathcal{B}(\mathcal{H})$ be the unilateral shift defined by $Se_i = e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an ONB of $\mathcal{H}$. Assume that $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Define

$$R_T = \begin{bmatrix} S^{*n} & T \\ 0 & S^n \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Such an operator R_T is called a *Foguel operator* of order n. Since S^{*n} is a Cowen-Douglas operator with index n on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we have

$$\bigvee_{k\geq 1} \ker(S^{*n} - \frac{1}{2})^k = \mathcal{H} = \bigvee_{k\geq 1} \ker(S^{*n} + \frac{1}{2})^k.$$

Then one can check that

$$\bigvee_{k\geq 1} \ker(R_T - \frac{1}{2})^k = \mathcal{H}_1, \quad \bigvee_{k\geq 1} \ker(R_T^* + \frac{1}{2})^k = \mathcal{H}_2.$$

In view of Theorem 2.5, if R_T is skew symmetric, then there exist a conjugation C on $\mathcal{H}$ and $G \in S_C(\mathcal{H})$ such that

$$R_T \cong \begin{bmatrix} S^{*n} & G \\ 0 & -CS^nC \end{bmatrix}.$$

3. Skew symmetric triangular operators

This section is devoted to describing certain skew symmetric triangular operators. The main results of this section are Theorems 3.4 and 3.7. To proceed, we first introduce some notation and terminology.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *triangular* if

$$\bigvee_{\lambda\in\mathbb{C}, n\geq 1} \ker(T - \lambda)^n = \mathcal{H}.$$

We remark that T is triangular if and only if T admits an upper triangular matrix representation

$$T = \begin{bmatrix} \lambda_1 & * & * & \cdots \\ & \lambda_2 & * & \cdots \\ & & \lambda_3 & \cdots \\ & & & \ddots \end{bmatrix}$$

with respect to some orthonormal basis of $\mathcal{H}$, where each omitted entry is zero. The well-known Cowen-Douglas operators, which are closely related to complex geometry [5], are triangular.

When an operator T and its adjoint T^* are both triangular (in general, with respect to different orthonormal bases), T is called *bitriangular*. This class contains all algebraic operators, diagonal normal operators and block diagonal operators. Obviously, every operator on finite-dimensional Hilbert space is bitriangular. There exist triangular operators which are not bitriangular. The adjoint of the forward unilateral shift is such an example. However, each skew symmetric triangular operator must be bitriangular.

Lemma 3.1. *If $T \in \mathcal{B}(\mathcal{H})$ is skew symmetric and triangular, then T is bitriangular.*

Proof. Since T is skew symmetric, there is a conjugation C on $\mathcal{H}$ such that $T^*C = -CT$. Hence $(-1)^n(T^* + \overline{\lambda})^n C = C(T-\lambda)^n$ and $C(\ker(T-\lambda)^n) = \ker(T^* + \overline{\lambda})^n$ for all $\lambda \in \mathbb{C}$ and $n \geq 1$.

Note that

$$\bigvee_{\lambda \in \mathbb{C}, n \geq 1} \ker(T-\lambda)^n = \mathcal{H}.$$

Since C is a conjugation, it follows that

$$\mathcal{H} = C(\mathcal{H}) = \bigvee_{\lambda \in \mathbb{C}, n \geq 1} C(\ker(T-\lambda)^n) = \bigvee_{\lambda \in \mathbb{C}, n \geq 1} \ker(T^* + \overline{\lambda})^n.$$

Hence T^* is triangular and T is bitriangular. □

Remark 3.2. Let $T \in \mathcal{B}(\mathcal{H})$ be skew symmetric. From the proof of Lemma 3.1, one can see that $\lambda \in \sigma_p(T)$ if and only if $-\overline{\lambda} \in \sigma_p(T^*)$, where $\sigma_p(\cdot)$ denotes point spectrum. In particular, $\dim \ker(T-\lambda) = \dim \ker(T+\lambda)^*$.

Lemma 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Assume that $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and $u \in \ker(T-\lambda_1), v \in \ker(T-\lambda_2)^*$. Then $\langle u, v\rangle = 0$.*

Proof. Compute to see

$$\lambda_1\langle u, v\rangle = \langle Tu, v\rangle = \langle u, T^*v\rangle = \lambda_2\langle u, v\rangle.$$

Since $\lambda_1 \neq \lambda_2$, it follows that $\langle u, v\rangle = 0$. □

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\{\lambda_i : i \in \mathbb{N}\}$ are distinct eigenvalues of T and $u_i \in \ker(T - \lambda_i)$ is a unit vector for $i \in \mathbb{N}$. If $\vee\{u_i : i \in \mathbb{N}\} = \mathcal{H}$, then T is skew symmetric if and only if there exist unit vectors $\{v_i : i \in \mathbb{N}\}$ with $v_i \in \ker(T + \lambda_i)^*$ for $i \in \mathbb{N}$ such that $\vee\{v_j : j \in \mathbb{N}\} = \mathcal{H}$, $\langle u_i, u_j\rangle = \langle v_j, v_i\rangle$ and $\langle u_i, v_j\rangle = \langle u_j, v_i\rangle$ for any $i, j \in \mathbb{N}$.*

Proof. "$\Longrightarrow$". Assume that C is a conjugation on $\mathcal{H}$ satisfying $CTC = -T^*$. For each $i \geq 1$, set $v_i = Cu_i$. Note that

$$T^* v_i = T^* C u_i = -CTu_i = -\overline{\lambda_i} C u_i = -\overline{\lambda_i} v_i.$$

It follows that each v_i is a normalized eigenvector of T^* corresponding to $-\overline{\lambda_i}$. Moreover, we have

$$\vee\{v_i : i \geq 1\} = \vee\{Cu_i : i \geq 1\} = C(\vee\{u_i : i \geq 1\}) = C(\mathcal{H}) = \mathcal{H}.$$

For $i, j \geq 1$, since C is a conjugation, it follows that

$$\langle v_j, v_i\rangle = \langle Cu_j, Cu_i\rangle = \langle u_i, u_j\rangle$$

and

$$\langle u_i, v_j\rangle = \langle Cv_j, Cu_i\rangle = \langle u_j, v_i\rangle.$$

This proves the necessity.

"$\Longleftarrow$". Assume that v_i is a normalized eigenvector of T^* corresponding to $-\overline{\lambda_i}$ for $i \geq 1$, $\vee\{v_i : i \geq 1\} = \mathcal{H}$ and

$$\langle u_i, u_j\rangle = \langle v_j, v_i\rangle, \quad \langle u_i, v_j\rangle = \langle u_j, v_i\rangle, \quad \forall i, j \geq 1.$$

We shall construct a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$.

Denote by $\mathcal{H}_0$ the set of all finite linear combinations of u_i's, and by $\mathcal{H}_1$ the set of all finite linear combinations of v_i's. By the hypothesis, $\mathcal{H}_i$ is a dense linear manifold of $\mathcal{H}$, $i = 1, 2$.

For each $x \in \mathcal{H}_0$ with $x = \sum_{i=1}^n \alpha_i u_i$, define $Cx = \sum_{i=1}^n \overline{\alpha_i} v_i$. If $y \in \mathcal{H}_0$ and $y = \sum_{j=1}^n \beta_j u_j$, one can check that

$$\begin{aligned}
\langle Cx, Cy\rangle &= \Big\langle \sum_{i=1}^n \overline{\alpha_i} v_i, \sum_{j=1}^n \overline{\beta_j} v_j \Big\rangle \\
&= \sum_{i,j=1}^n \overline{\alpha_i}\beta_j \langle v_i, v_j\rangle \\
&= \sum_{i,j=1}^n \overline{\alpha_i}\beta_j \langle u_j, u_i\rangle \\
&= \Big\langle \sum_{j=1}^n \beta_j u_j, \sum_{i=1}^n \alpha_i u_i \Big\rangle \\
&= \langle y, x\rangle.
\end{aligned}$$

It follows that the map $C : \mathcal{H}_0 \to \mathcal{H}_1$ is conjugate-linear, isometric and hence well defined. Moreover, C admits a continuous extension to $\mathcal{H}$, denoted by C

again. It is obvious that C is surjective and hence invertible. In particular, we have

$$\langle Cx, Cy\rangle = \langle y, x\rangle, \quad \forall x, y \in \mathcal{H}.$$

We claim that C is a conjugation. Now it suffices to prove that C is involutive, that is, $C^2 = I$. Since $\vee\{u_i : i \geq 1\} = \mathcal{H}$, we need only check that $C^2 u_i = u_i$ for each $i \geq 1$.

Now fix an $i \geq 1$. Since $\vee\{v_j : j \geq 1\} = \mathcal{H}$ and $u_i \neq 0$, there exists some $\tau_i \geq 1$ such that $\langle u_i, v_{\tau_i}\rangle \neq 0$. By Lemma 3.3, it follows that $\lambda_i = -\lambda_{\tau_i}$. Since λ_j's are pairwise distinct, such τ_i is unique. Thus $u_i \in \{v_j : j \neq \tau_i\}^\perp$. Since $\vee\{v_j : j \geq 1\} = \mathcal{H}$, it follows that $\{v_j : j \neq \tau_i\}^\perp = \vee\{u_i\}$. For $j \geq 1$ with $j \neq \tau_i$, we have

$$\langle Cv_i, v_j\rangle = \langle Cv_i, Cu_j\rangle = \langle u_j, v_i\rangle = \langle u_i, v_j\rangle = 0.$$

So $Cv_i \in \vee\{u_i\}$. Since τ is isometric, we obtain $Cv_i = \alpha u_i$ for some unimodular constant α. So

$$\langle u_{\tau_i}, v_i\rangle = \langle Cv_i, Cu_{\tau_i}\rangle = \langle \alpha u_i, v_{\tau_i}\rangle = \alpha\langle u_{\tau_i}, v_i\rangle.$$

Noting that $\langle u_i, v_{\tau_i}\rangle \neq 0$, we have $\alpha = 1$ and hence $C^2 u_i = Cv_i = u_i$. Thus we have proved that C is a conjugation.

For each $i \geq 1$, compute to see that

$$CTu_i = C(\lambda_i u_i) = \overline{\lambda_i} Cu_i = \overline{\lambda_i} v_i = -T^* v_i = -T^* C u_i,$$

which implies that $CT = -T^*C$. Hence T is skew symmetric. □

From the proof for the sufficiency of Theorem 3.4, one can see the following result.

Corollary 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\{\lambda_i : i \in \mathbb{N}\}$ are distinct eigenvalues of T and $u_i \in \ker(T - \lambda_i)$ is a unit vector for $i \geq 1$. If $\vee\{u_i : i \geq 1\} = \mathcal{H}$ and T is skew symmetric, then $\{\lambda_i : i \in \mathbb{N}\} = \{-\lambda_i : i \in \mathbb{N}\}$.*

For a general skew symmetric operator T, $\lambda \in \sigma_p(T)$ does not imply $-\lambda \in \sigma_p(T)$. Here is an example.

Example 3.6. Let $\{e_i\}_{i=1}^\infty$ be an ONB of $\mathcal{H}$ and S be the operator on $\mathcal{H}$ defined as

$$Se_i = e_{i+1}, \qquad \forall i \geq 1.$$

For $x \in \mathcal{H}$ with $x = \sum_{i=1}^\infty \alpha_i e_i$, define $Cx = \sum_{i=1}^\infty \overline{\alpha_i} e_i$. Then C is a conjugation on $\mathcal{H}$ and it is easy to check $CSC = S$. Set

$$T = \begin{bmatrix} I - S & 0 \\ 0 & S^* - I \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}, \qquad D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Then D is a conjugation on $\mathcal{H} \oplus \mathcal{H}$ and one can see $DTD = -T^*$. So T is skew symmetric. Note that $\sigma_p(T) = \{z \in \mathbb{C} : |z + 1| < 1\}$.

For certain irreducible triangular operators, the following result provides a geometric characterization of skew symmetry.

Theorem 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be irreducible. Suppose that $\{\lambda_i : i \in \mathbb{N}\}$ are distinct eigenvalues of T, u_i is a normalized eigenvector of T corresponding to λ_i and v_i is a normalized eigenvector of T^* corresponding to $-\overline{\lambda_i}$ for $i \in \mathbb{N}$. If*

$$\dim\ker(T-\lambda_i) = 1 = \dim\ker(T+\lambda_i)^*, \quad \forall i \geq 1,$$

and $\vee\{u_i : i \geq 1\} = \mathcal{H} = \vee\{v_i : i \geq 1\}$, then the following are equivalent:

(i) *T is skew symmetric*;

(ii) *there exist unimodular constants $\{\alpha_i : i \geq 1\}$ such that*

$$\alpha_i\langle u_i, u_j\rangle = \alpha_j\langle v_j, v_i\rangle, \quad \alpha_i\langle u_i, v_j\rangle = \alpha_j\langle u_j, v_i\rangle, \quad \forall i,j \geq 1;$$

(iii) *the conditions*

$$\begin{aligned}&\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle\cdots\langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, u_{i_1}\rangle\\ &= \langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle\cdots\langle v_{i_n}, v_{i_{n-1}}\rangle\langle v_{i_1}, v_{i_n}\rangle\end{aligned}$$

and

$$\begin{aligned}&\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle\cdots\langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, v_{i_1}\rangle\\ &= \langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle\cdots\langle v_{i_n}, v_{i_{n-1}}\rangle\langle u_{i_1}, v_{i_n}\rangle\end{aligned}$$

hold for any $n \in \mathbb{N}$ and any n-tuple $(i_1, i_2, \ldots, i_n)$ in $\mathbb{N}$.

Proof. "(ii)$\Longrightarrow$(i)". Set $w_i = \alpha_i v_i$ for $i \geq 1$. It is easy to see that

$$\langle u_i, u_j\rangle = \langle w_j, w_i\rangle, \quad \langle u_i, w_j\rangle = \langle u_j, w_i\rangle, \quad i,j \geq 1.$$

Note that $w_i \in \ker(T+\lambda_i)^*$ and $\|w_i\| = 1$ for $i \geq 1$. In view of Theorem 3.4, T is skew symmetric.

"(i)$\Longrightarrow$(iii)". Assume that C is a conjugation on $\mathcal{H}$ satisfying $CTC = -T^*$.

For each $i \geq 1$, we note that $T^*Cu_i = -CTu_i = -\overline{\lambda_i}Cu_i$. Thus $Cu_i \in \ker(T+\lambda_i)^*$. Note that $\ker(T+\lambda_i)^* = \vee\{v_i\}$. Thus there exist unimodular α_i such that $Cu_i = \alpha_i v_i$ for $i \geq 1$.

For $i,j \geq 1$, since C is a conjugation, it follows that

$$\langle u_i, u_j\rangle = \langle Cu_j, Cu_i\rangle = \langle \alpha_j v_j, \alpha_i v_i\rangle = \overline{\alpha_i}\alpha_j\langle v_j, v_i\rangle$$

and

$$\langle u_i, v_j\rangle = \langle Cv_j, Cu_i\rangle = \langle \alpha_j u_j, \alpha_i v_i\rangle = \alpha_j\overline{\alpha_i}\langle u_j, v_i\rangle.$$

The desired equalities follow readily.

"(iii)$\Longrightarrow$(ii)". By the hypothesis, one can easily check that

$$|\langle u_i, u_j\rangle| = |\langle v_j, v_i\rangle|, \quad |\langle u_i, v_j\rangle| = |\langle u_j, v_i\rangle|, \quad \forall i,j \geq 1. \tag{3.1}$$

For $i,j \in \mathbb{N}$, we define $i \sim j$ if there exist $i_1, i_2, \ldots, i_n \in \mathbb{N}$ such that

$$\langle u_i, u_{i_1}\rangle\langle u_{i_1}, u_{i_2}\rangle\cdots\langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, u_j\rangle \neq 0.$$

One can verify that $\sim$ is an equivalence relation on $\mathbb{N}$.

Denote $\Lambda = \{j \in \mathbb{N} : 1 \sim j\}$. Then $\langle u_i, u_j\rangle = 0$ for all $i \in \Lambda$ and $j \in \mathbb{N}\setminus\Lambda$. It follows that $\mathcal{M} = \vee\{u_i : i \in \Lambda\}$ is orthogonal to $\mathcal{N} = \vee\{u_i : i \in \mathbb{N}\setminus\Lambda\}$. We claim that $\Lambda = \mathbb{N}$. If not, then $\mathcal{M}, \mathcal{N}$ are nonzero subspaces of $\mathcal{H}$. Noting that each u_i is an eigenvector of T, it follows that $\mathcal{M}$ and $\mathcal{N}$ are both invariant under T. Since $\vee\{u_i : i \in \mathbb{N}\} = \mathcal{H}$, we deduce that T is reducible, a contradiction.

For convenience, we denote

$$\Phi(i_1, i_2, \ldots, i_k) = \frac{\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle \cdots \langle u_{i_{k-1}}, u_{i_k}\rangle}{\langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle \cdots \langle v_{i_k}, v_{i_{k-1}}\rangle}$$

for $k \geq 2$ and k-tuple $(i_1, i_2, \ldots, i_k)$ in $\mathbb{N}$. By the hypothesis, if $\langle u_{i_l}, u_{i_{l+1}}\rangle \neq 0$ for all $1 \leq l \leq k-1$ and $\langle u_{i_k}, u_{i_1}\rangle \neq 0$, then $|\Phi(i_1, i_2, \ldots, i_k)| = 1 = \Phi(i_1, i_2, \ldots, i_k, i_1)$ and $\overline{\Phi(i_1, i_2, \ldots, i_k)} = \Phi(i_k, i_{k-1}, \ldots, i_1)$. If, in addition, $\langle u_{i_k}, v_{i_1}\rangle \neq 0$, then

$$\Phi(i_1, i_2, \ldots, i_k)\langle u_{i_k}, v_{i_1}\rangle = \langle u_{i_1}, v_{i_k}\rangle. \tag{3.2}$$

For each $i \in \mathbb{N}$, we can find $i_1, i_2, \ldots, i_n \in \mathbb{N}$ such that

$$\langle u_1, u_{i_1}\rangle\langle u_{i_1}, u_{i_2}\rangle \cdots \langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, u_i\rangle \neq 0.$$

Set $\alpha_i = \Phi(1, i_1, i_2, \ldots, i_n, i)$.

Assume that $(j_1, j_2, \ldots, j_k)$ is a k-tuple in $\mathbb{N}$ satisfying

$$\langle u_1, u_{j_1}\rangle\langle u_{j_1}, u_{j_2}\rangle \cdots \langle u_{j_{k-1}}, u_{j_k}\rangle\langle u_{j_k}, u_i\rangle \neq 0.$$

Since

$$\begin{aligned}
&\Phi(1, i_1, i_2, \ldots, i_n, i)\overline{\Phi(1, j_1, j_2, \ldots, j_k, i)} \\
&= \Phi(1, i_1, i_2, \ldots, i_n, i)\Phi(i, j_k, j_{k-1}, \ldots, j_1, 1) \\
&= \Phi(1, i_1, i_2, \ldots, i_n, i, j_k, j_{k-1}, \ldots, j_1, 1) = 1.
\end{aligned}$$

Thus $\Phi(1, i_1, i_2, \ldots, i_n, i) = \Phi(1, j_1, j_2, \ldots, j_k, i)$. It shows that the definition of α_i does not depend on the choice of $(i_1, i_2, \ldots, i_n)$.

Now it remains to check that

$$\alpha_i\langle u_i, u_j\rangle = \alpha_j\langle v_j, v_i\rangle, \quad \alpha_i\langle u_i, v_j\rangle = \alpha_j\langle u_j, v_i\rangle, \quad \forall i, j \geq 1.$$

Assume that $\alpha_i = \Phi(1, i_1, i_2, \ldots, i_n, i)$ and $\alpha_j = \Phi(1, j_1, j_2, \ldots, j_k, j)$. By (3.1), $\langle u_i, u_j\rangle = 0$ if and only if $\langle v_j, v_i\rangle = 0$. Thus we may assume that $\langle u_i, u_j\rangle\langle v_i, v_j\rangle \neq 0$. So

$$\begin{aligned}
\alpha_i\langle u_i, u_j\rangle &= \Phi(1, i_1, i_2, \ldots, i_n, i)\langle u_i, u_j\rangle \\
&= \Phi(1, i_1, i_2, \ldots, i_n, i, j)\langle v_j, v_i\rangle = \alpha_j\langle v_j, v_i\rangle.
\end{aligned}$$

By (3.1), $\langle u_i, v_j\rangle = 0$ if and only if $\langle u_j, v_i\rangle = 0$. We may assume that $\langle u_i, v_j\rangle\langle u_j, v_i\rangle \neq 0$. Then, in view of (3.2), we obtain

$$\begin{aligned}
\overline{\alpha_j}\alpha_i\langle u_i, v_j\rangle &= \Phi(j, j_k, j_{k-1}, \ldots, j_1, 1)\Phi(1, i_1, i_2, \ldots, i_n, i)\langle u_i, v_j\rangle \\
&= \Phi(j, j_k, j_{k-1}, \ldots, j_1, 1, i_1, i_2, \ldots, i_n, i)\langle u_i, v_j\rangle \\
&= \langle u_j, v_i\rangle.
\end{aligned}$$

This completes the proof. □

Now we give an example of skew symmetric operators satisfying the conditions in Theorem 2.5.

Example 3.8. Let $\{e_i\}_{i=1}^{\infty}$ be an ONB of $\mathcal{H}$. Set

$$A = \sum_{n=1}^{\infty} \frac{e_n \otimes e_n}{n}, \qquad E = e_1 \otimes e_2 - e_2 \otimes e_1,$$

where $e_1 \otimes e_2(x) = \langle x, e_2 \rangle e_1$ for $x \in \mathcal{H}$. Then A is a diagonal operator, $A = A^*$, $\sigma(A) = \{\frac{1}{n} : n \geq 1\} \cup \{0\}$ and

$$\bigvee_{n \geq 1} \ker(A - \frac{1}{n}) = \mathcal{H}.$$

Also we note that rank $(A - \lambda) = \infty$ for all $\lambda \in \mathbb{C}$ and A is not an algebraic operator of degree 2. Then, by [6, Main Theorem], we can choose an irreducible $B \in \mathcal{B}(\mathcal{H})$ which is similar to A. It follows readily that B^* is also similar to A and

$$\bigvee_{n \geq 1} \ker(B - \frac{1}{n}) = \mathcal{H} = \bigvee_{n \geq 1} \ker(B^* - \frac{1}{n}). \tag{3.3}$$

For $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, define $Cx = \sum_{i=1}^{\infty} \overline{\alpha_i} e_i$. Then C is a conjugation on $\mathcal{H}$ and one can check that $CEC = -E^*$. Define

$$T = \begin{bmatrix} B & E \\ 0 & -CB^*C \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. In the remaining, we shall prove that T is skew symmetric and satisfies all the conditions stated in Theorem 3.7.

First we claim that T is skew symmetric. Set

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Then D is a conjugation on $\mathcal{H}_1 \oplus \mathcal{H}_2$. Compute to see

$$DTD = \begin{bmatrix} -B^* & 0 \\ CEC & CBC \end{bmatrix} = \begin{bmatrix} -B^* & 0 \\ -E^* & CBC \end{bmatrix} = -T^*.$$

So T is skew symmetric.

Next we shall check that

$$\bigvee \left\{ \ker(T - \frac{1}{n}), \ker(T + \frac{1}{n}) : n \geq 1 \right\} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

We note that $\sigma(B) = \sigma(A) = \{\frac{1}{n} : n \geq 1\} \cup \{0\}$ and $\sigma(-CB^*C) = -\sigma(B)$.

For each $n \geq 1$, since $-CB^*C - \frac{1}{n}$ is invertible, we have $\ker(T - \frac{1}{n}) = \ker(B - \frac{1}{n})$ and hence

$$\dim \ker(T - \frac{1}{n}) = \dim \ker(B - \frac{1}{n}) = \dim \ker(A - \frac{1}{n}) = 1.$$

In view of (3.3), we obtain

$$\bigvee_{n \geq 1} \ker(T - \frac{1}{n}) = \bigvee_{n \geq 1} \ker(B - \frac{1}{n}) = \mathcal{H}_1. \tag{3.4}$$

On the other hand, since $B + \frac{1}{n}$ is invertible, each $x \in \ker(T + \frac{1}{n})$ has the form

$$\begin{pmatrix} -(B+\frac{1}{n})^{-1}Ey \\ y \end{pmatrix}, \tag{3.5}$$

where $y \in \ker(-CB^*C + \frac{1}{n})$. Note that $\ker(-CB^*C + \frac{1}{n}) = C(\ker(B^* - \frac{1}{n}))$. Then

$$\begin{aligned} \dim\ker(T + \frac{1}{n}) &= \dim\ker(-CB^*C + \frac{1}{n}) = \dim\ker(B^* - \frac{1}{n}) \\ &= \dim\ker(A^* - \frac{1}{n}) = \dim\ker(A - \frac{1}{n}) = 1. \end{aligned}$$

Furthermore, in view of (3.4) and (3.5), we have

$$\bigvee\Big\{\ker(T - \frac{1}{n}), \ker(T + \frac{1}{n}) : n \geq 1\Big\} = \mathcal{H}_1 \oplus \bigvee_{n\geq 1} \ker(-CB^*C + \frac{1}{n}).$$

Note that

$$\begin{aligned} \mathcal{H} = C(\mathcal{H}) &= C(\vee_{n\geq 1}\ker(B^* - \frac{1}{n})) \\ &= \bigvee_{n\geq 1} C(\ker(B^* - \frac{1}{n})) \\ &= \bigvee_{n\geq 1} \ker(CB^*C - \frac{1}{n}) \\ &= \bigvee_{n\geq 1} \ker(-CB^*C + \frac{1}{n}). \end{aligned}$$

This implies that

$$\bigvee\Big\{\ker(T - \frac{1}{n}), \ker(T + \frac{1}{n}) : n \geq 1\Big\} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Now it remains to prove that T is irreducible. In view of (3.4), $\mathcal{H}_1$ is hyperinvariant under T. Then each orthogonal projection Q on $\mathcal{H}_1 \oplus \mathcal{H}_2$ commuting with T admits the form

$$Q = \begin{bmatrix} Q_1 & Q_{1,2} \\ 0 & Q_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Since $Q = Q^*$, we obtain $Q_{1,2} = 0$. Thus Q_1, Q_2 are orthogonal projections commuting with B and CB^*C respectively. Note that B and CB^*C are irreducible. Hence $Q_i = 0$ or I, $i = 1, 2$. From $TQ = QT$, one can see $Q_1E = EQ_2$. Since $E \neq 0$, we obtain either $Q_1 = Q_2 = 0$ or $Q_1 = Q_2 = I$. This implies that T is irreducible. By Theorem 3.4, T satisfies all conditions stated in Theorem 3.7.

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