INSERTION-OF-FACTORS-PROPERTY
ON SKEW POLYNOMIAL RINGS

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ABSTRACT. In this paper, we investigate the insertion-of-factors-property (simply, IFP) on skew polynomial rings, introducing the concept of strongly σ-IFP for a ring endomorphism σ. A ring R is said to have *strongly σ-IFP* if the skew polynomial ring $R[x; \sigma]$ has IFP. We examine some characterizations and extensions of strongly σ-IFP rings in relation with several ring theoretic properties which have important roles in ring theory. We also extend many of related basic results to the wider classes, and so several known results follow as consequences of our results.

1. Introduction

Throughout this paper, all rings are associative with identity. We denote by $R[x]$ the polynomial ring with an indeterminate x over R. $\deg f(x)$ denotes the degree of $f(x) \in R[x]$. Let $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively.

Due to Bell [3], a ring R is called to satisfy the *insertion-of-factors-property* (simply, *an IFP ring*) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [22] and Shin [26] used the terms *semicommutative* and *SI* for the IFP, respectively. Commutative rings have clearly IFP, and any reduced ring (i.e., a ring without nonzero nilpotent elements) has IFP by a simple computation. A ring is called *Abelian* if every idempotent is central. IFP rings are Abelian by a simple computation.

Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia [25] called a ring R *Armendariz* if whenever the product of any two polynomials over a ring is zero, then so is the product of any pair of coefficients from the two polynomials. The class of IFP rings and the class of Armendariz rings do not imply each other by [25, Example 3.2] and [13, Example 14]. Note that the polynomial rings over IFP rings need not have IFP by [13, Example 2], but for an Armendariz ring R, R has IFP if and only if

$R[x]$ has IFP by [25, Proposition 4.6]. Based on this result, Kwak et al. [19] defined a *strongly IFP* ring: A ring R is called *strongly IFP* if $R[x]$ has IFP.

For an endomorphism σ of a ring R, a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) $R[x; \sigma]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \sigma(r)x$ for all $r \in R$. In $R[x; \sigma]$, consider $1x^n = x^n = x1x^{n-1} = \sigma(1)x^n$ for any $n \geq 1$ where 1 is the identity of R. So it is natural to set $\sigma(1) = 1$. The set $\{x_j\}_{j \geq 0}$ is easily seen to be a left Ore subset of $R[x; \sigma]$, so that one can localize $R[x; \sigma]$ and form the *skew Laurent polynomial* ring $R[x, x^{-1}; \sigma]$. Every element of $R[x, x^{-1}; \sigma]$ is a finite sum of elements of the form $x^{-j}rx^i$, where $r \in R$ and i, j are nonnegative integers.

According to Krempa [18], an endomorphism σ of a ring R is called *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$, and R is called a *σ-rigid* ring [8] if there exists a rigid endomorphism σ of R. Note that any rigid endomorphism of a ring is a monomorphism and σ-rigid rings are reduced rings by [8, Proposition 5], and $R[x; \sigma]$ is a reduced ring if and only if R is σ-rigid by [9, Proposition 3] and R is σ-rigid if and only if $R[x, x^{-1}; \sigma]$ is reduced by [23, Theorem 3].

Following Başer et al. [1, Definition 2.1], an endomorphism σ of a ring R is called to have *IFP* if whenever $ab = 0$ for $a, b \in R$, $aR\sigma(b) = 0$, and a ring R is called *σ-IFP* if there exists an endomorphism σ of R having IFP. A ring R is σ-IFP if and only if for $a, b \in R$, $ab = 0$ implies $aR\sigma^n(b) = 0$ for all $n \geq 1$ by [1, Remark 2.2]. It is proved that a ring R is σ-rigid if and only if R is reduced σ-IFP and σ is a monomorphism by [1, Theorem 2.4].

On the other hand, a ring R is called *reversible* [4] if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Reversible rings have IFP by simple computation. The reversible ring property cannot go up to polynomial rings by [17, Example 2.1]. Recently, Kaynarca et al. called a ring R *strongly σ-reversible* [15, Definition 2.2] if the skew polynomial ring $R[x; \sigma]$ is reversible, and investigated some characterizations and extensions of strongly σ-reversible rings. (We change over from "strongly σ-skew reversible" in [15] to "strongly σ-reversible", so as to cohere with other related definitions.)

Motivated by the above, in this paper, we extend the insertion-of-factors-property of elements and polynomials to one of skew polynomials. We give a generalization of σ-rigid rings and an extension of strongly IFP rings, introducing the concept of *a strongly σ-IFP ring* for an endomorphism σ of a ring. Then the structure of strongly σ-IFP rings and their related properties including extensions are studied and the relations between strongly σ-IFP rings and generalized reduced rings is also investigated. Consequently, several known results are obtained as corollaries of our results.

From now on, σ denotes a nonzero and non identity endomorphism of each given ring, unless specified otherwise.

2. Strongly σ-IFP rings and related properties

Firstly we start by giving the following example.

Example 2.1. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R has IFP since R is reduced. Let $\sigma : R \to R$ be defined by $\sigma((a,b)) = (b,a)$. For $p(x) = (1,0)$, $q(x) = (0,1)x$ in $R[x;\sigma]$, $p(x)q(x) = 0$ but $p(x)xq(x) = (1,0)x^2 \neq 0$ for $x \in R[x;\sigma]$. Thus $R[x;\sigma]$ does not have IFP.

We can give the following definition.

Definition 2.2. For an endomorphism σ of a ring R with $\sigma(1) = 1$, R is called *strongly σ-IFP* if $R[x;\sigma]$ has IFP.

Every strongly σ-IFP ring has IFP (and so Abelian), and every subring S with $\sigma(S) \subseteq S$ of a strongly σ-IFP ring is also strongly σ-IFP. We will freely use these facts without reference. Every domain D with a monomorphism σ is clearly strongly σ-IFP, since D is σ-rigid. Any strongly σ-reversible rings are clearly strongly σ-IFP, but not conversely by the following example.

Example 2.3. Consider the polynomial ring $R = \mathbb{Z}[t]$. Define $\sigma : R \to R$ by $\sigma(f(t)) = f(0)$ where $f(t) \in R$. Then R is not strongly σ-reversible by [15, Example 2.6(2)].

Now, we will show that R is strongly σ-IFP. Suppose that $p(x)q(x) = 0$ for $p(x) = f_0(t) + f_1(t)x + \cdots + f_m(t)x^m$, $q(x) = g_0(t) + g_1(t)x + \cdots + g_n(t)x^n$ in $R[x;\sigma]$, where $f_i(t), g_j(t) \in R$ for all i, j. Without loss of generality, assume that $p(x) \neq 0$.

Case 1. Suppose that $f_0(t) \neq 0$. From $f_0(t)g_0(t) = 0$, we have $g_0(t) = 0$ since R is a domain. From $f_0(t)g_1(t) + f_1(t)\sigma(g_0(t)) = 0$ and $g_0(t) = 0$, we get $g_1(t) = 0$. Since $f_0(t)g_2(t) + f_1(t)\sigma(g_1(t)) + f_2(t)\sigma^2(g_0(t)) = 0$, we obtain $g_2(t) = 0$. Continuing this process, we have that $g_0(t) = g_1(t) = \cdots = g_n(t) = 0$, i.e., $q(x) = 0$. Thus $p(x)R[x;\sigma]q(x) = 0$.

Case 2. Suppose that there is $f_k(t) \neq 0$ with $f_0(t) = \cdots = f_{k-1}(t) = 0$, where $0 < k \leq m$. Since $f_0(t)g_k(t) + f_1(t)\sigma(g_{k-1}(t)) + \cdots + f_{k-1}(t)\sigma^{k-1}(g_1(t)) + f_k(t)\sigma^k(g_0(t)) = 0$, we have $f_k(t)\sigma^k(g_0(t)) = 0$. Since R is a domain and $f_k(t) \neq 0$, $\sigma^k(g_0(t)) = 0$ and so $\sigma(g_0(t)) = 0$. From $f_0(t)g_{k+1}(t) + f_1(t)\sigma(g_k(t)) + \cdots + f_k(t)\sigma^k(g_1(t)) + f_{k+1}(t)\sigma^{k+1}(g_0(t)) = 0$, we have $f_k(t)\sigma^k(g_1(t)) = 0$ and so $\sigma^k(g_1(t)) = \sigma(g_1(t)) = 0$ by the same argument as above. Continuing this process, we obtain $\sigma(g_j(t)) = 0$ for all $0 \leq j \leq n$. This entails $xq(x) = 0$, and hence $p(x)(rx^s)q(x) = 0$ for any $r \in R$ and $s \geq 0$, noting that $p(x)rq(x) = rp(x)q(x) = 0$. This yields $p(x)R[x;\sigma]q(x) = 0$.

By the computations in Cases 1 and 2, we can conclude that R is strongly σ-IFP.

Note that the endomorphism σ of strongly σ-IFP rings need not be a monomorphism by Example 2.3.

Let R be a ring with an endomorphism σ and $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x; \sigma]$. Recall that a ring R is called σ-*skew Armendariz* [9, Definition] if whenever $p(x)q(x) = 0$, $a_i \sigma^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$, and R is called σ-*skew quasi-Armendariz* [10, Definition 2.1] if whenever $p(x)R[x; \sigma]q(x) = 0$, $a_i R \sigma^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Any σ-rigid ring is σ-skew Armendariz by [9, Corollary 4], and the concepts of σ-skew Armendariz rings and strongly σ-IFP rings do not imply each other, by help of [13, Example 14] and [25, Example 3.2]. Notice that any σ-skew Armendariz ring is clearly σ-skew quasi-Armendariz, when σ is an epimorphism.

Proposition 2.4. (1) *If R is a strongly σ-IFP ring, then R has both IFP and σ-IFP.*

(2) *Let R be a σ-skew Armendariz ring and σ an epimorphism. Then R has both IFP and σ-IFP if and only if R is strongly σ-IFP.*

(3) *Let R be a strongly σ-IFP ring and σ an epimorphism. Then R is σ-skew Armendariz if and only if R is σ-skew quasi-Armendariz.*

(4) *If R is a strongly σ-IFP ring, then $\sigma(e) = e$ for any $e^2 = e \in R$.*

Proof. (1) Assume that R is strongly σ-IFP. Let $ab = 0$ for $a, b \in R$. Since $aR[x; \sigma]b = 0$, $arb = 0$ and $asxb = 0$ for any $r, s \in R$. Hence $aRb = 0$ and $aR\sigma(b) = 0$, showing that R has both IFP and σ-IFP.

(2) It is enough to show the necessity by (1). Assume that R has IFP and σ-IFP. Let $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma]$. Then $a_i \sigma^i(b_j) = 0$ for all i, j, since R is σ-skew Armendariz. Hence $a_i R \sigma^{t+i}(b_j) = 0$ and so $(a_i x^i)(Rx^t)b_j = 0$ for all i, j and $t \geq 0$ by assumption. This implies $p(x)(cx^t)q(x) = 0$ for any $c \in R$ and $t \geq 0$. Therefore $p(x)R[x; \sigma]q(x) = 0$, proving that R is strongly σ-IFP.

(3) Suppose that R is σ-skew quasi-Armendariz and $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x; \sigma]$. Since R is strongly σ-IFP, $p(x)R[x; \sigma]q(x) = 0$, and so $a_i R \sigma^i(b_j) = 0$ for all i and j by hypothesis. Thus $a_i \sigma^i(b_j) = 0$, completing the proof.

(4) Since $R[x; \sigma]$ has IFP, it is Abelian. Thus $ex = xe$ for any $e^2 = e \in R$, and so $\sigma(e) = e$. $\qquad\square$

Corollary 2.5. (1) *If R is a strongly σ-IFP ring, then R is Abelian.*

(2) *If R is an Armendariz ring, then R has IFP if and only if R is strongly IFP.*

(3) *If R is strongly IFP, then R is Armendariz if and only if R is quasi-Armendariz (i.e., whenever $f(x)R[x]g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, $a_i R b_j = 0$ for all i, j [7]).*

Proof. Each proof comes directly from Proposition 2.4(1,2,3). $\qquad\square$

The following illustrates Proposition 2.4(1,2) in detail.

Example 2.6. We refer to the argument and ring construction in [16, Example 2.8]. Consider the free algebra $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Define an automorphism α of A by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c,$$

respectively. Let B be the set of all polynomials with zero constant terms in A and consider the ideal I of A generated by

$$a_0b_0,\ a_0b_1 + a_1b_0,\ a_0b_2 + a_1b_1 + a_2b_0,\ a_1b_2 + a_2b_1,\ a_2b_2,\ a_0rb_0,\ a_2rb_2,$$

$$b_0a_0,\ b_0a_1 + b_1a_0,\ b_0a_2 + b_1a_1 + b_2a_0,\ b_1a_2 + b_2a_1,\ b_2a_2,\ b_0ra_0,\ b_2ra_2,$$

$$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2),\ (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2),$$

$$a_0a_0,\ a_2a_2,\ a_0ra_0,\ a_2ra_2,\ b_0b_0,\ b_2b_2,\ b_0rb_0,\ b_2rb_2,\ r_1r_2r_3r_4,$$

$$a_0a_1 + a_1a_0,\ a_0a_2 + a_1a_1 + a_2a_0,\ a_1a_2 + a_2a_1,$$

$$b_0b_1 + b_1b_0,\ b_0b_2 + b_1b_1 + b_2b_0,\ b_1b_2 + b_2b_1,$$

$$(a_0 + a_1 + a_2)r(a_0 + a_1 + a_2),\ (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2),$$

where $r, r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \subseteq I$. Set $R = A/I$. Since $\alpha(I) \subseteq I$, we can obtain an automorphism σ of R by defining $\sigma(s + I) = \alpha(s) + I$ for $s \in A$. Note that $\sigma^2 = 1_R$, where 1_R denotes the identity endomorphism of R. We identify every element of A with its image in R for simplicity. Then R has both IFP and σ-IFP by [16, Example 2.8].

However, R is neither strongly σ-IFP nor σ-skew Armendariz. For $a_0 + a_1x^2 + a_2x^4, b_0 + b_1x^2 + b_2x^4 \in R[x; \sigma]$, we have $(a_0 + a_1x^2 + a_2x^4)(b_0 + b_1x^2 + b_2x^4) = 0$. But

$$(a_0 + a_1x^2 + a_2x^4)cx(b_0 + b_1x^2 + b_2x^4)$$
$$= (a_0 + a_1x^2 + a_2x^4)c(\alpha(b_0) + \alpha(b_1)x^2 + \alpha(b_2)x^4)x$$
$$= (a_0 + a_1x^2 + a_2x^4)c(a_0 + a_1x^2 + a_2x^4)x \neq 0$$

since $a_0ca_1 + a_1ca_0 \neq 0$, showing that $R[x; \sigma]$ does not have IFP. Furthermore, we obtain $a_0b_1 \neq 0$ from $(a_0 + a_1x^2 + a_2x^4)(b_0 + b_1x^2 + b_2x^4) = 0$, entailing that R is not σ-skew Armendariz.

Let σ_γ be an endomorphism of a ring R_γ for each $\gamma \in \Gamma$. Then the map $\sigma : \prod_{\gamma \in \Gamma} R_\gamma \to \prod_{\gamma \in \Gamma} R_\gamma$ defined by $\sigma((a_\gamma)_\gamma) = (\sigma_\gamma(a_\gamma))_\gamma$ for $(a_\gamma)_\gamma \in \prod_{\gamma \in \Gamma} R_\gamma$ is an endomorphism of the product $\prod_{\gamma \in \Gamma} R_\gamma$.

A ring R is called *local* if $R/J(R)$ is a division ring where $J(R)$ denotes the Jacobson radical of R, R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are Abelian and semilocal.

Proposition 2.7. (1) *Let R_γ be a ring with an endomorphism σ_γ for each $\gamma \in \Gamma$. The following are equivalent*:
   (i) *R_γ is strongly σ_γ-IFP for each $\gamma \in \Gamma$.*

(ii) *The direct product $\prod_{\gamma \in \Gamma} R_\gamma$ of R_γ, $(\gamma \in \Gamma)$ is strongly σ-IFP.*

(iii) *The direct sum of R_γ, $(\gamma \in \Gamma)$ is strongly σ-IFP.*

(2) *Let σ be an endomorphism of a ring R with $\sigma(e) = e$ for any $e^2 = e \in R$. Then we have the following.*

(i) *R is strongly σ-IFP and semiperfect if and only if R is a finite direct sum of local strongly σ_γ-IFP rings for $\gamma \in \Gamma$.*

(ii) *For a central idempotent e of R, eR and $(1-e)R$ are strongly σ-IFP if and only if R is strongly σ-IFP.*

Proof. (1) It is enough to show (i)$\Rightarrow$(ii). Assume (i) and let $p(x)q(x) = 0$ where $p(x) = (p_\gamma(x))_\gamma, q(x) = (q_\gamma(x))_\gamma \in \prod_{\gamma \in \Gamma} R_\gamma[x; \sigma]$. Then $p_\gamma(x)q_\gamma(x) = 0$ for each $\gamma \in \Gamma$, where $p_\gamma(x), q_\gamma(x) \in R_\gamma[x; \sigma_\gamma]$. By hypothesis, we have $p_\gamma(x)(r_\gamma x^s)q_\gamma(x) = 0$ for every $r_\gamma \in R_\gamma$, any $s \geq 0$ and each $\gamma \in \Gamma$ and this implies that $p(x)(rx^s)q(x) = 0$ for every $r = (r_\gamma)_\gamma \in \prod_{\gamma \in \Gamma} R_\gamma$ and $s \geq 0$. Therefore the direct product of R_γ is strongly σ-IFP.

(2)-(i) Suppose that R is strongly σ-IFP and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [20, Proposition 3.7.2], say $R = \sum_{i=1}^n e_i R$ such that each $e_i R e_i$ is a local ring. Since R is Abelian by Corollary 2.5(1). Thus each $e_i R$ is an ideal of R with $e_i R = e_i R e_i$. But each $e_i R$ is also a strongly σ-IFP ring as a subring since $\sigma(e_i R) \subseteq e_i R$ for each i by assumption.

Conversely assume that R is a finite direct sum of strongly σ_γ-IFP local rings. Then R is semiperfect since local rings are semiperfect, and moreover R is strongly σ-IFP by (1).

(2)-(ii) It comes from (1) with $R \cong eR \oplus (1-e)R$, since the class of strongly σ-IFP is closed under subrings. $\square$

Let R be a ring and σ be a monomorphism of R. Now we consider the Jordan's construction of an over-ring of R by σ (see [14] for more details). Let $A(R, \sigma)$ be the subset $\{x^{-i}rx^i \mid r \in R \text{ and } i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$. Note that for $j \geq 0$, $x^j r = \sigma^j(r)x^j$ implies $rx^{-j} = x^{-j}\sigma^j(r)$ for $r \in R$. This yields that for each $j \geq 0$ we have $x^{-i}rx^i = x^{-(i+j)}\sigma^j(r)x^{i+j}$. It follows that $A(R, \sigma)$ forms a subring of $R[x, x^{-1}; \sigma]$ with the following natural operations: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\sigma^j(r) + \sigma^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\sigma^j(r)\sigma^i(s)x^{i+j}$ for $r, s \in R$ and $i, j \geq 0$. Note that $A(R, \sigma)$ is an over-ring of R, and σ is actually an automorphism of $A(R, \sigma)$ with $\sigma(x^{-i}rx^i) = x^{-i}\sigma(r)x^i$. Jordan showed, with the use of left localization of the skew polynomial $R[x; \sigma]$ with respect to the set of powers of x, that for any pair (R, σ), such an extension $A(R, \sigma)$ always exists in [14]. This ring $A(R, \sigma)$ is usually said to be *Jordan extension* of R by σ.

Proposition 2.8. *Let R be a ring and σ a monomorphism of R. Then R is a strongly σ-IFP ring if and only if so is $A(R, \sigma)$.*

Proof. It is enough to show the necessity. Let $A = A(R, \sigma) = \{t^{-i}rt^i \mid r \in R$ and $i \geq 0\}$, using t in place of x in the argument above. Suppose that R is

strongly σ-IFP and let $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{n} b_j x^j \in A[x;\sigma]$, where $a_i = t^{-v(i)} a_i' t^{v(i)}$ and $b_j = t^{-u(j)} b_j' t^{u(j)}$ for $a_i', b_j' \in R$ and $v(i), u(j) \geq 0$. We use the property that $x^{-i} r x^i = x^{-(i+j)} \sigma^j(r) x^{i+j}$ for each $j \geq 0$ and that σ is an automorphism of A. Then we can express all a_i, b_j by

$$a_i = t^{-s} \alpha_i t^s \text{ and } b_j = t^{-s} \beta_j t^s$$

for some $s \geq 0$ and $\alpha_i, \beta_j \in R$. This yields that

$$0 = \sum_{k=0}^{m+n} \left(\sum_{k=i+j} a_i \sigma^i(b_j) \right) x^k = \sum_{k=0}^{m+n} \left(\sum_{k=i+j} t^{-s} \alpha_i t^s \sigma^i(t^{-s} \beta_j t^s) \right) x^k$$

$$= \sum_{k=0}^{m+n} \left(\sum_{k=i+j} t^{-s} \alpha_i t^s t^{-s} \sigma^i(\beta_j) t^s \right) x^k = t^{-s} \left(\sum_{k=0}^{m+n} \left(\sum_{k=i+j} \alpha_i \sigma^i(\beta_j) \right) x^k \right) t^s.$$

This entails $\sum_{k=0}^{m+n} \left(\sum_{k=i+j} \alpha_i \sigma^i(\beta_j) \right) x^k = 0$. Next let

$$p'(x) = \sum_{i=0}^{m} \alpha_i x^i \text{ and } q'(x) = \sum_{j=0}^{n} \beta_j x^j \in R[x;\sigma].$$

Then we get $p'(x)q'(x) = 0$ by the result above, noting that $p(x) = t^{-s} p'(x) t^s$ and $q(x) = t^{-s} q'(x) t^s$. Since R is strongly σ-IFP, $p'(x)(rx^l)q'(x) = 0$ for all $r \in R$ and $l \geq 0$, entailing that

$$(*) \qquad t^{-u} p'(x)(rx^l) q'(x) t^u = 0 \text{ for any } u \geq 0.$$

Let $a = t^{-c} r t^c \in A$ with $r \in R$. Here we can let $c = s$, after resetting c or s in the expression of a_i, b_j if necessary. Now we have

$$\begin{aligned}
p(x)(ax^l)q(x) &= (t^{-s} p'(x) t^s)(ax^l)(t^{-s} q'(x) t^s) \\
&= (t^{-s} p'(x) t^s)(t^{-s} r t^s x^l)(t^{-s} q'(x) t^s) \\
&= t^{-s} p'(x) r t^s t^{-s} x^l q'(x) t^s \\
&= t^{-s} p'(x) r x^l q'(x) t^s \\
&= 0
\end{aligned}$$

by the equality $(*)$. This implies that $p(x)A[x;\sigma]q(x) = 0$, and therefore A is strongly σ-IFP. $\qquad \square$

Proposition 2.9. *Let $\alpha : R \to S$ be a ring isomorphism. Then R is a strongly σ-IFP if and only if S is a strongly $\alpha \sigma \alpha^{-1}$-IFP ring.*

Proof. Note that $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$ if and only if $p'(x) = \sum_{i=0}^{m} a_i' x^i$, $q'(x) = \sum_{j=0}^{n} b_j' x^j \in S[x; \alpha \sigma \alpha^{-1}]$, letting $\alpha(a_i) = a_i'$, $\alpha(b_j) = b_j' \in S$ for all i and j, and also that $r \in R$ if and only if $r' \in S$, letting $\alpha(r) = r'$, since α is bijective.

For any $r \in R$ and $s \geq 0$,

$$p(x)(rx^s)q(x) = 0 \text{ in } R[x; \sigma]$$

$$\Leftrightarrow \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i \sigma^i(r\sigma^s(b_j)) \right) x^{k+s} = 0$$

$$\Leftrightarrow \sum_{k=0}^{m+n} \left(\sum_{i+j=k} \alpha(a_i \sigma^i(r\sigma^s(b_j))) \right) x^{k+s} = 0, \text{ since } \alpha \text{ is bijective}$$

$$\Leftrightarrow \sum_{k=0}^{m+n} \left(\sum_{i+j=k} \alpha(a_i)\alpha(\sigma^i(r\sigma^s(b_j))) \right) x^{k+s} = 0$$

$$\Leftrightarrow \sum_{k=0}^{m+n} \left(\sum_{i+j=k} \alpha(a_i)(\alpha\sigma\alpha^{-1})^i \left(\alpha(r)(\alpha\sigma\alpha^{-1})^s\alpha(b_j) \right) \right) x^{k+s} = 0,$$

since $(\alpha\sigma\alpha^{-1})^t = \alpha\sigma^t\alpha^{-1}$ for any positive integer t

$$\Leftrightarrow \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i'(\alpha\sigma\alpha^{-1})^i \left(r'(\alpha\sigma\alpha^{-1})^s(b_j') \right) \right) x^{k+s} = 0$$

$$\Leftrightarrow p'(x)(r'x^s)q'(x) = 0 \text{ in } S[x; \alpha\sigma\alpha^{-1}] \text{ for any } r' \in S \text{ and } s \geq 0. \qquad \square$$

Theorem 2.10. *A ring R is σ-rigid if and only if R is a semiprime strongly σ-IFP ring and σ is a monomorphism of R.*

Proof. If R is σ-rigid (i.e., $R[x; \sigma]$ is reduced), then R is clearly strongly σ-IFP, and moreover R is reduced (so semiprime) and σ is a monomorphism by [8, Proposition 5].

Now, suppose that R is semiprime strongly σ-IFP and σ is a monomorphism of R. Let $\sigma(a)a = 0$ for $a \in R$. Since R is σ-IFP, $\sigma(a)R\sigma(a) = 0$. Then $\sigma(a) = 0$ and so $a = 0$ by hypothesis. By [12, Proposition 2.4], R is σ-rigid. $\qquad \square$

The following example shows that the conditions "R is a semiprime ring" and "σ is a monomorphism" in Theorem 2.10 cannot be dropped, respectively.

Example 2.11. (1) Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $\sigma : R \to R$ be an automorphism defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Clearly, R is not semiprime and so R is not σ-rigid. By [15, Propostion 2.13], R is strongly σ-reversible and hence R is strongly σ-IFP.

(2) Recall the strongly σ-IFP ring $R = \mathbb{Z}[t]$ and the endomorphism σ of R which is not a monomorphism, as in Example 2.3. Then R is a domain and so semiprime. But R is not σ-rigid, since $x\sigma(x) = 0$ for $0 \neq x \in R$.

Jordan's construction also yields $R[x, x^{-1}; \sigma] \cong A(R, \sigma)[x, x^{-1}; \sigma]$, by way of an isomorphism defined by $x^{-i}rx^j \mapsto \sigma^i(r)x^{j-i}$, noting $\sigma^i(r) = \sigma^i(r)x^{-u}x^u = x^{-u}\sigma^{u+i}(r)x^u$ for any $u \geq 0$ and $r \in R$. So Proposition 2.8 provides the following.

Corollary 2.12. *Let R be a semiprime ring and σ a monomorphism of R. Then the following are equivalent*:
  (1) *R is strongly σ-reversible.*
  (2) *R is strongly σ-IFP.*
  (3) *R is σ-IFP.*
  (4) *R is σ-rigid.*
  (5) *$R[x, x^{-1}; \sigma]$ is a reduced ring.*
  (6) *$A(R, \sigma)$ is strongly σ-reversible.*
  (7) *$A(R, \sigma)$ is strongly σ-IFP.*
  (8) *$A(R, \sigma)$ is σ-IFP.*
  (9) *$A(R, \sigma)$ is σ-rigid.*

Proof. (4)$\Rightarrow$(1)$\Rightarrow$(2)$\Rightarrow$(3) and (9)$\Rightarrow$(6)$\Rightarrow$(7)$\Rightarrow$(8) are obvious. (3)$\Rightarrow$(4) and (8)$\Rightarrow$(9) follow from the proof of Theorem 2.10, and (2)$\Leftrightarrow$(7) is shown by Proposition 2.8. We see that R is σ-rigid if and only if $R[x, x^{-1}; \sigma]$ is reduced, in [23, Theorem 3]. Since $R[x, x^{-1}; \sigma]$ is isomorphic to $A(R, \sigma)[x, x^{-1}; \sigma]$, we can obtain the equivalence of (4) and (5). $\square$

McCoy [21] showed that if two polynomials annihilate each other over a commutative ring, then each polynomial has a nonzero annihilator in the base ring. Nielsen [24] and Rege-Chhawchharia [25] called a noncommutative ring R *right McCoy* (resp., left McCoy) if whenever any nonzero polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ (resp., $cg(x) = 0$) for some nonzero $c \in R$, and a ring R is called McCoy if it is both left and right McCoy. Reversible rings are McCoy by [24, Theorem 2], but there exists an IFP ring which is not McCoy [24, Section 3]. The McCoy condition on a ring is also extended on the skew polynomial ring. A ring R is called *σ-skew Mc-Coy* [2, Defiition 1] if for any nonzero polynomials $p(x)$ and $q(x) \in R[x; \sigma]$, $p(x)q(x) = 0$ implies $p(x)c = 0$ for some nonzero $c \in R$.

Proposition 2.13. *If R is a strongly σ-IFP ring, then R is a σ-skew McCoy ring.*

Proof. Suppose that R is a strongly σ-IFP ring. Let $p(x)q(x) = 0$ for any nonzero skew polynomials $p(x), q(x) \in R[x; \sigma]$. Then $p(x)R[x; \sigma]q(x) = 0$. By [11, Theorem 1], there exists $0 \neq c \in R$ such that $p(x)R[x; \sigma]c = 0$ and so $p(x)c = 0$, showing that R is σ-skew McCoy. $\square$

Corollary 2.14 ([19, Proposition 3.19]). *If R is a strongly IFP ring, then R is a McCoy ring.*

3. Examples of strongly σ-IFP rings

Let R be an algebra over a commutative ring S. Following [5], the *Dorroh extension* of R by S is the Abelian group $D = R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$, where $r_i \in R$ and $s_i \in S$. For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

Theorem 3.1. *Let R be an algebra over a commutative domain S and σ an endomorphism of R. Then R is strongly σ-IFP if and only if the Dorroh extension D of R by S is strongly $\bar{\sigma}$-IFP.*

Proof. Recall that $\sigma(1) = 1$ and note that $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$. It is sufficient to show that the Dorroh extension D is strongly $\bar{\sigma}$-IFP when R is strongly σ-IFP. Assume that R is strongly σ-IFP. Let $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m}(a_i, b_i)x^i$, $q(x) = \sum_{j=0}^{n}(c_j, d_j)x^j \in D[x; \bar{\sigma}]$. Then we have

(1)
$$\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i, b_i)\bar{\sigma}^i(c_j, d_j)\right)x^k = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i, b_i)(\sigma^i(c_j), d_j)\right)x^k = 0.$$

Put $f(x) = \sum_{i=0}^{m}b_i x^i, g(x) = \sum_{j=0}^{n}d_j x^j \in S[x]$, then we get $f(x)g(x) = 0$ by Eq. (1). Since $S[x]$ is a domain, $f(x) = 0$ or $g(x) = 0$. Note that $\sigma(s) = s$ for all $s \in S$, since $\sigma(1) = 1$. We will freely use this fact without reference in the following procedure.

If $f(x) = 0$, then we have $\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i(\sigma^i(c_j) + d_j), 0)\right)x^k = 0$ by Eq. (1), and so we get

(2) $$0 = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}a_i(\sigma^i(c_j) + d_j)\right)x^k = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}a_i\sigma^i(c_j + d_j)\right)x^k.$$

We let $p'(x) = \sum_{i=0}^{m}a_i x^i, q'(x) = \sum_{j=0}^{n}(c_j + d_j)x^j \in R[x; \sigma]$. Then we get $p'(x)q'(x) = 0$ by Eq. (2). Since R is strongly σ-IFP, $p'(x)((r + s)x^t)q'(x) = 0$ for any $r + s \in R$ and $t \geq 0$, and we obtain

$$0 = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}a_i\sigma^i((r + s)\sigma^t(c_j + d_j))\right)x^{k+t}$$

$$= \sum_{k=0}^{m+n}\left(\sum_{i+j=k}a_i(\sigma^i(r) + s)(\sigma^{i+t}(c_j) + d_j)\right)x^{k+t}.$$

Hence,

$$\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i,0)\bar{\sigma}^i\left((r,s)\bar{\sigma}^t((c_j,d_j))\right)\right)x^{k+t}$$

$$=\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i,0)(\sigma^i(r),s)(\sigma^{i+t}(c_j),d_j)\right)x^{k+t}$$

$$=\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i(\sigma^i(r)+s)(\sigma^{i+t}(c_j)+d_j),0)\right)x^{k+t}=0,$$

showing that $p(x)(r,s)x^t q(x)=0$ for any $(r,s)\in D$ and $t\geq 0$.

In case of $g(x)=0$, we can show that $p(x)(r,s)x^t q(x)=0$ for any $(r,s)\in D$ and $t\geq 0$ by the similar argument to above.

Consequently, the Dorroh extension D of R by S is strongly $\bar{\sigma}$-IFP if R is strongly σ-IFP. $\qquad\square$

By Proposition 2.7(2-ii) and Theorem 3.1, we have the following.

Corollary 3.2 ([19, Proposition 3.17]). (1) *For a central idempotent e of a ring R, R is strongly IFP if and only if eR and $(1-e)R$ are strongly IFP.*

(2) *Let R be an algebra over a commutative domain S, and D be the Dorroh extension of R by S. Then R is strongly IFP if and only if D is strongly IFP.*

Recall that if σ is an endomorphism of a ring R, then the map $\bar{\sigma}:R[x]\to R[x]$ defined by

$$\bar{\sigma}\left(\sum_{i=0}^{m}a_i x^i\right)=\sum_{i=0}^{m}\sigma(a_i)x^i$$

is an endomorphism of $R[x]$. This map extends σ clearly.

Theorem 3.3. *Let R be a ring with an endomorphism σ such that $\sigma^t=1_R$ for some positive integer t, where 1_R denotes the identity endomorphism of R. Then R is strongly σ-IFP if and only if $R[x]$ is strongly $\bar{\sigma}$-IFP.*

Proof. We apply the proof of [15, Theorem 3.6]. Assume that R is strongly σ-IFP, and let $p(y)=f_0(x)+f_1(x)y+\cdots+f_m(x)y^m$, $q(y)=g_0(x)+g_1(x)y+\cdots+g_n(x)y^n$ in $R[x][y;\bar{\sigma}]$ such that $p(y)q(y)=0$. Let

$$f_i(x)=a_{i_0}+a_{i_1}x+\cdots+a_{i_{w_i}}x^{w_i}\text{ and }g_j(x)=b_{j_0}+b_{j_1}x+\cdots+b_{j_{v_j}}x^{v_j}$$

for each $0\leq i\leq m$, and $0\leq j\leq n$, where $a_{i_0},\ldots,a_{w_i}$, $b_{j_0},\ldots,b_{v_j}\in R$. Set

$$C=\{a_{i_0},\ldots,a_{i_{w_i}},b_{j_0},\ldots,b_{j_{v_j}}\mid 0\leq i\leq m\text{ and }0\leq j\leq n\}.$$

Let $R[y;\sigma]$ be the skew polynomial ring with an indeterminate y over R, subject to $yr=\sigma(r)y$ for $r\in R$. We can rewrite $p(y)$ and $q(y)$ by

$$h_0(y)+h_1(y)x+\cdots+h_u(y)x^u\text{ and }l_0(y)+l_1(y)x+\cdots+l_s(y)x^s\in R[y;\sigma][x],$$

where $h_i(y), l_j(y) \in R[y; \sigma]$, and $R[y; \sigma][x]$ is the polynomial ring with an indeterminate x over $R[y; \sigma]$. Set $h(x) = p(y)$ and $l(x) = q(y)$, then $h(x)l(x) = 0$.

Take a positive integer k such that k is larger than the maximal integer in

$$\{\deg h_i(y), \deg l_j(y) \mid 0 \leq i \leq u \text{ and } 0 \leq j \leq s\},$$

where the degree of zero polynomial is taken to be 0. Set

$$h(y^{tk}) = h_0(y) + h_1(y)y^{tk} + \cdots + h_u(y)y^{mtk} \text{ and}$$
$$l(y^{tk}) = l_0(y) + l_1(y)y^{tk} + \cdots + l_s(y)y^{ntk}.$$

Then $h(y^{tk}), l(y^{tk}) \in R[y; \sigma]$. Note that the set of coefficients of the $h_i(y)$'s (resp., $l_j(y)$'s) equals the set of coefficients of $h(y^{tk})$ (resp., $l(y^{tk})$). Moreover this set is equal to the set C. Therefore, since $p(y)q(y) = 0 = h(x)l(x)$ and $\sigma^{tk} = 1_R$, we get $h(y^{tk})l(y^{tk}) = 0$ in $R[y; \sigma]$, noting that $\bar{\sigma}(r) = \sigma(r)$ for $r \in R$. Then we have $h(y^{tk})ry^\gamma l(y^{tk}) = 0$ for any $r \in R$ and $\gamma \geq 0$, since R is strongly σ-IFP. Then

$$\sum_{\iota=0}^{m+n} \left(\sum_{i+j=\iota} a_{\lambda_i} \sigma^{\lambda_i} (r\sigma^\gamma(b_{\epsilon_j})) \right) y^{\iota+\gamma} = 0 \text{ for all } \lambda_i \text{ and } \epsilon_j.$$

Thus $\sum_{i+j=\iota} h_i(y)(ry^\gamma)l_j(y) = 0$ for each $0 \leq \iota \leq m + n$ since $\sigma^{tk} = 1_R$, entailing $p(y)(ry^\gamma)q(y) = 0$ for any $r \in R$ and $\gamma \geq 0$. Therefore $R[x]$ is strongly $\bar{\sigma}$-IFP.

The converse comes from the fact that the class of strongly σ-IFP rings is closed under subrings, recalling that $\bar{\sigma}(r) = \sigma(r)$ for $r \in R$. □

The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^n a_i x^i$ with the usual addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x; x^{-1}]$. The map $\bar{\sigma} : R[x; x^{-1}] \to R[x; x^{-1}]$ defined by $\bar{\sigma}(\sum_{i=k}^n a_i x^i) = \sum_{i=k}^n \sigma(a_i)x^i$ extends σ and is also an endomorphism of $R[x; x^{-1}]$.

Proposition 3.4. *Let R be a ring with an endomorphism σ such that $\sigma^t = 1_R$ for some positive integer t. Then R is strongly σ-IFP if and only if $R[x; x^{-1}]$ is strongly $\bar{\sigma}$-IFP.*

Proof. Assume that R is strongly σ-IFP, and let $p(y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m$, $q(y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n$ in $R[x; x^{-1}][y; \bar{\sigma}]$ such that $p(y)q(y) = 0$. Let

$$f_i(x) = \sum_{u=s_i}^{w_i} a_u x^u \text{ and } g_j(x) = \sum_{v=k_j}^{z_j} b_v x^v$$

for each $0 \leq i \leq m$, and $0 \leq j \leq n$, where $a_{s_i}, \ldots, a_{w_i}, b_{k_j}, \ldots, b_{z_j} \in R$ and $s_i, w_i, k_j, z_j \in \mathbb{Z}$. Take positive integers s and k such that $s = \max\{|s_i| \mid i = 0, 1, \ldots, m\}$ and $k = \max\{|k_j| \mid j = 0, 1, \ldots, n\}$. Let $p(y) = x^s f(y) = f_0'(x) + f_1'(x)y + \cdots + f_m'(x)y^m$, $q(y) = x^k g(y) = g_0'(x) + g_1'(x)y + \cdots + g_n'(x)y^n$,

where $f_i'(x) = f_i(x)x^s$ and $g_j'(x) = g_j(x)x^k$. By the similar computation to the proof of Theorem 3.3, we can show that $R[x; x^{-1}]$ is strongly $\bar{\sigma}$-IFP. $\square$

By Theorem 3.3 and Proposition 3.4, we obtain the following result which includes [19, Theorem 3.14].

Corollary 3.5. *For a ring R, the following are equivalent*:
(1) R *is strongly IFP.*
(2) $R[x]$ *is strongly IFP.*
(3) $R[x; x^{-1}]$ *is strongly IFP.*

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* is defined, and *regular* means if it is both left and right regular (and hence not a zero divisor). Assume that M is a multiplicatively closed subset of R consisting of central regular elements. Let σ be an automorphism of R and assume $\sigma(m) = m$ for every $m \in M$. Then $\sigma(m^{-1}) = m^{-1}$ in $M^{-1}R$ and the induced map $\bar{\sigma} : M^{-1}R \to M^{-1}R$ defined by $\bar{\sigma}(u^{-1}a) = u^{-1}\sigma(a)$ is also an automorphism.

Proposition 3.6. *Let R be a ring with an automorphism σ and assume that there exists a multiplicatively closed subset M of R consisting of central regular elements and $\sigma(m) = m$ for every $m \in M$. Then R is a strongly σ-IFP ring if and only if $M^{-1}R$ is a strongly $\bar{\sigma}$-IFP ring.*

Proof. Let R be a strongly σ-IFP, and suppose that $P(x)Q(x) = 0$ for $P(x) = u^{-1}(a_0 + a_1x + \cdots + a_kx^k), Q(x) = v^{-1}(b_0 + b_1x + \cdots + b_nx^n) \in M^{-1}R[x; \bar{\sigma}]$. Then

$$0 = P(x)Q(x)$$
$$= u^{-1}(a_0v^{-1} + a_1\sigma(v)^{-1}x + \cdots + a_k\sigma^k(v)^{-1}x^k)(b_0 + b_1x + \cdots + b_nx^n).$$

Since $\sigma^i(v)^{-1} = v^{-1}$ for $0 \leq i \leq k$ by hypothesis, we have

$$0 = P(x)Q(x)$$
$$= (uv)^{-1}(a_0 + a_1x + \cdots + a_kx^k)(b_0 + b_1x + \cdots + b_nx^n).$$

Let $p(x) = a_0 + a_1x + \cdots + a_kx^k$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$. Then we get $p(x)q(x) = 0$ and hence $p(x)R[x; \sigma]q(x) = 0$, since R is a strongly σ-IFP. Then for any $r^{-1}s \in M^{-1}R$ and $t \geq 0$,

$$P(x)(r^{-1}sx^t)Q(x)$$
$$= u^{-1}(a_0 + a_1x + \cdots + a_kx^k)(r^{-1}sx^t)v^{-1}(b_0 + b_1x + \cdots + b_nx^n)u^{-1}$$
$$= (urv)^{-1}p(x)(sx^t)q(x) = 0,$$

by using $\sigma^i(m)^{-1} = m^{-1}$ for $m \in M$. Thus $M^{-1}R$ is strongly $\bar{\sigma}$-IFP.

The converse can be obtained from that the class of strongly σ-IFP ring is closed under subrings. $\square$

For $n \geq 2$, denote the n by n full matrix ring over a ring R by $\mathrm{Mat}_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$.

For a ring R and $n \geq 2$, consider the rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \{ m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)}$$
$$\text{for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1 \},$$

which are subrings of $\mathrm{Mat}_n(R)$. Note that $V_n(R) \cong R[x]/(x^n)$, where (x^n) is an ideal of the polynomial ring $R[x]$ over R generated by x^n.

For a ring R with an endomorphism σ, the corresponding $(a_{ij}) \mapsto (\sigma(a_{ij}))$ induces endomorphisms of $\mathrm{Mat}_n(R)$, $D_n(R)$ and $V_n(R)$, respectively. We denote each of them by $\bar{\sigma}$.

Let A be any ring with an endomorphism σ. Then $\mathrm{Mat}_n(A)$ and $U_n(A)$ ($n \geq 2$) do not have IFP since the subring $U_2(A)$ is not Abelian. Moreover, $D_n(A)$ ($n \geq 4$) does not have IFP by [17, Example 1.3]. Therefore $\mathrm{Mat}_n(A)$, $U_n(A)$ for $n \geq 2$ and $D_n(A)$ for $n \geq 4$ are not strongly $\bar{\sigma}$-IFP by Proposition 2.4(1). We will freely use this fact without reference. On the other hand, $V_n(R)$ ($n \geq 2$) over a σ-rigid ring R is strongly $\bar{\sigma}$-reversible by [15, Theorem 3.1] and so strongly $\bar{\sigma}$-IFP.

But we have the following result.

Proposition 3.7. *If R is a σ-rigid ring, then $D_n(R)$ for $n = 2, 3$ is strongly $\bar{\sigma}$-IFP.*

Proof. It is enough to show that $D_3(R)$ over a σ-rigid ring R is strongly $\bar{\sigma}$-IFP since $D_2(R) = V_2(R)$. Note that $D_3(R)[x; \bar{\sigma}] \cong D_3(R[x; \sigma])$. Suppose that R is σ-rigid. Let $p(x) = \sum_{i=0}^m A_i x^i, q(x) = \sum_{j=0}^n B_j x^j \in D_3(R)[x; \bar{\sigma}]$ with $A_i = (a_{st}^{(i)})$ and $B_j = (b_{uv}^{(j)}) \in D_3(R)$. We can write $p(x) = (p_{st}(x)), q(x) = (q_{uv}(x)) \in D_3(R[x; \sigma])$ with $p_{st} = \sum_{i=0}^m a_{st}^{(i)} x^i, q_{uv} = \sum_{j=0}^n b_{uv}^{(j)} x^j \in R[x; \sigma]$. From $p(x)q(x) = 0$, we have the following equalities:

(3) $$p_{11}(x)q_{11}(x) = 0;$$

(4) $$p_{11}(x)q_{12}(x) + p_{12}(x)q_{11}(x) = 0;$$

(5) $$p_{11}(x)q_{13(x)} + p_{12}(x)q_{23}(x) + p_{13}(x)q_{11}(x) = 0;$$

(6) $$p_{11}(x)q_{23}(x) + p_{23}(x)q_{11}(x) = 0.$$

Note that $R[x;\sigma]$ is reduced. Recall that $ab = 0$ implies $aAb = 0$ and $ba = 0$, and that $ab^2 = 0$ implies that $ab = 0$ for any reduced ring A and $a, b \in A$. We will freely use these facts in the following procedure. Since R is strongly σ-IFP,

$$p_{11}(x)R[x;\sigma]q_{11}(x) = 0 \tag{7}$$

from Eq. (3). If we multiply Eq. (4) and Eq. (6) on the right-hand side by $q_{11}(x)$ respectively, then we have $p_{12}(x)q_{11}^2(x) = 0$ and $p_{23}(x)q_{11}^2(x) = 0$, and so $p_{12}(x)q_{11}(x) = 0$ and $p_{23}(x)q_{11}(x) = 0$ and moreover, $p_{11}(x)q_{12}(x) = 0$ and $p_{11}(x)q_{23}(x) = 0$. Thus

$$
\begin{aligned}
&p_{11}(x)R[x;\sigma]q_{12}(x) = 0, \ p_{12}(x)R[x;\sigma]q_{11}(x) = 0, \\
&p_{11}(x)R[x;\sigma]q_{23}(x) = 0 \ \text{ and } \ p_{23}(x)R[x;\sigma]q_{11}(x) = 0.
\end{aligned} \tag{8}
$$

If we multiply Eq. (5) on the right-hand side by $q_{11}(x)$ and $q_{23}(x)$ in turn, then we get $p_{13}(x)q_{11}(x) = 0$, $p_{12}(x)q_{23}(x) = 0$ and moreover $p_{11}(x)q_{13(x)} = 0$ by the same method as above, respectively. Hence,

$$
\begin{aligned}
p_{11}(x)R[x;\sigma]q_{13}(x) = 0, \ p_{12}(x)R[x;\sigma]q_{23}(x) = 0 \text{ and } p_{13}(x)R[x;\sigma]q_{11(x)} = 0.
\end{aligned} \tag{9}
$$

Consequently, Eqs. (8,9,10) imply that $D_3(R)$ is strongly $\bar{\sigma}$-IFP. $\square$

The hypothesis "R is a σ-rigid ring" in Proposition 3.7 cannot be weakened the condition "R is a strongly σ-IFP ring". For example, consider the strongly σ-IFP ring R which is not σ-rigid, as in Example 2.11(1). Then $D_2(R)$ is not σ-IFP by [1, Example 2.9], and hence $D_2(R)$ is not strongly σ-IFP by Proposition 2.4(1). Observe that Proposition 3.7 extends the results of [17, Propositions 1.2 and 1.6].

For an ideal I and an endomorphism σ of a ring R, if I is a σ-ideal (i.e., $\sigma(I) \subseteq I$) of R then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(a + I) = \sigma(a) + I$ for $a \in R$ is an endomorphism of a factor ring R/I. The class of strongly σ-IFP rings is not closed under homomorphic images. Indeed, if R is the ring of quaternions with integer coefficients with a monomorphism σ, then R is a domain, and so strongly σ-IFP; while for any odd prime integer q, we have $R/qR \cong \mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [6, Exercise 2A]. But $\mathrm{Mat}_2(\mathbb{Z}_q)$ is not strongly $\bar{\sigma}$-IFP as mentioned earlier, and thus the factor ring R/qR is not strongly $\bar{\sigma}$-IFP.

For a nonempty subset S of a ring R, we write $r_R(S) = \{c \in R \mid Sc = 0\}$ (resp., $\ell_R(S) = \{c \in R \mid cS = 0\}$) which is called the *right* (resp., *left*) *annihilator* of S in R. A ring R has IFP if and only if any one-sided annihilator is an ideal of R by [26, Lemma 1.2].

Proposition 3.8. (1) *Let R be a strongly σ-IFP ring. If I is the one-sided annihilator of any nonempty subset in R and $\sigma(I) \subseteq I$, then R/I is strongly $\bar{\sigma}$-IFP.*

(2) *Let σ be an endomorphism and I an ideal of a ring R. For a σ-ideal I of R such that R/I is strongly $\bar{\sigma}$-IFP, if I is a σ-rigid ring without identity, then R is strongly σ-IFP.*

Proof. (1) Assume that R is a strongly σ-IFP ring and let $I = r_R(S)$ for $\phi \neq S \subseteq R$. Then I is an ideal since R has IFP by Proposition 2.4(1). Let $\bar{p}(x)\bar{q}(x) = \bar{0}$ with $\bar{p}(x) = \sum_{i=0}^{m} \bar{a}_i x^i, \bar{q}(x) = \sum_{j=0}^{n} \bar{b}_j x^j \in \bar{R}[x; \bar{\sigma}]$ where $\bar{R} = R/I$ and $\bar{c} = c + I$ for $c \in R$. Then $Sp(x)q(x) = 0$, and so $Sp(x)(rx^t)q(x) = 0$ for any $r \in R$ and $t \geq 0$ by hypothesis, where $p(x) = \sum_{i=0}^{m} a_i x^i, q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma]$. Thus

$$S \left(\sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i \sigma^i (r\sigma^t(b_j)) \right) x^{k+t} \right) = 0$$

and so

$$\sum_{k=0}^{m+n} \left(\sum_{i+j=k} \bar{a}_i \bar{\sigma}^i (\bar{r}\bar{\sigma}^t(\bar{b}_j)) \right) x^{k+t} = \bar{0},$$

entailing that $\bar{p}(x)(\bar{r}x^t)\bar{q}(x) = \bar{0}$ for any $r \in R$ and $t \geq 0$. Therefore R/I is strongly $\bar{\sigma}$-IFP. The left annihilator case is similar.

(2) Assume that R/I is strongly $\bar{\sigma}$-IFP and I is a σ-rigid ring. Let $p(x)q(x) = 0$ for $p(x), q(x) \in R[x; \sigma]$. Then $p(x)R[x; \sigma]q(x) \subseteq I[x; \sigma]$ since R/I is strongly $\bar{\sigma}$-IFP, and $q(x)I[x; \sigma]p(x) = 0$ since $q(x)I[x; \sigma]p(x) \subseteq I[x; \sigma]$, $(q(x)I[x; \sigma]p(x))^2 = 0$ and $I[x; \sigma]$ is reduced. Thus, $(p(x)R[x; \sigma]q(x)I[x; \sigma])^2 = p(x)R[x; \sigma]q(x) I[x; \sigma]p(x)R[x; \sigma]q(x)I = 0$ and so $p(x)R[x; \sigma]q(x)I[x; \sigma] = 0$; hence

$$(p(x)R[x; \sigma]q(x))^2 \subseteq p(x)R[x; \sigma]q(x)I[x; \sigma] = 0$$

since $p(x)R[x; \sigma]q(x) \subseteq I[x; \sigma]$. Then $p(x)R[x; \sigma]q(x) = 0$ since $I[x; \sigma]$ is reduced. Therefore R is strongly σ-IFP. $\square$

The condition "I is a σ-rigid ring without identity" in the Proposition 3.8(2) cannot be dropped by the next example.

Example 3.9. Consider a ring $R = U_2(F)$ over a division ring F and an automorphism σ of R defined by $\sigma \left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right) \right) = \left(\begin{smallmatrix} a & -b \\ 0 & c \end{smallmatrix} \right)$. Then R is not clearly Abelian, and so R is not strongly σ-IFP by Corollary 2.5(1). The σ-ideal $I = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix} \right)$ of R is not σ-rigid as a ring without identity: Indeed, $A\sigma(A) = 0$ but $A \neq 0$ for $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \in I$. Note that the factor ring $R/I \cong F$ is reduced and $\bar{\sigma}$ is an identity map on R/I. That is, R/I is $\bar{\sigma}$-rigid and hence strongly $\bar{\sigma}$-IFP.

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