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THERMAL STRESSES IN A SEMI-INFINITE SOLID CYLINDER SUBJECTED TO INTERNAL HEAT GENERATION

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Abstract. The present paper deals with the determination of displacement and thermal stresses in a semi-infinite circular cylinder defined as $0 \leq r \leq b$, $0 \leq z < \infty$, due to internal heat generation within it. A circular cylinder is considered having arbitrary initial temperature and subjected to time dependent heat flux at the fixed circular boundary $(r = b)$ whereas the zero temperature at the lower surface $(z = 0)$ of the semi-infinite circular cylinder. The governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessel functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

1. Introduction

Wankhede [7] determined the quasi-static thermal stresses in thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and fixed circular edge thermally insulated without heat generation. Gogulwar and Deshmukh [2] studied the thermal stresses in thin circular plate with constant heat source. Recently Deshmukh et al. [1] has discussed the thermal stresses in a hollow circular disk due to internal heat generation within it for a finite length. The steady-state thermal stresses in circular plate subjected to an axi-symmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge has been considered by Nowacki [5]. Kulkarni et al. [3] has determined the thermal stresses in a thick circular plate due to internal heat generation within it. Kulkarni et al. [4] has determined quasi-static thermal stresses in a steady-state thick circular plate. In this article, we consider a non homogenous boundary value problem of heat conduction in a semi infinite circular cylinder defined as

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$0 \le r \le b,\ 0 \le z < \infty$, and discuss the thermal behavior due to heat generation. A circular cylinder is considered having arbitrary initial temperature and subjected to time dependent heat flux at the fixed circular boundary $(r = b)$ whereas the zero temperature at the lower surface $(z = 0)$ of the semi-infinite circular cylinder. The governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessel functions. The results for displacement and stresses have been computed numerically and illustrated graphically. To our knowledge no one has studied thermal stresses due to heat generation in a semi-infinite circular cylinder. This is new and novel contribution to the field. The results presented here will be useful in engineering problems particularly in aerospace engineering for stations of a missile body not influenced by nose tapering. The missile skill material is assumed to have physical properties independent of temperature, so that the temperature $T(r, z, t)$ is a function of radius, thickness and time only.

2. Formulation of the problem

Consider a semi-infinite circular cylinder occupying space D defined by $0 \le r \le b,\ 0 \le z < \infty$. Initially the cylinder is at arbitrary temperature $F(r, z)$. The time dependent heat flux $f(z, t)$ is applied on the fixed circular boundary $(r = b)$ whereas the zero temperature at the lower surface $(z = 0)$ of the semi-infinite circular cylinder. Heat generate within the semi infinite circular cylinder at the rate of $\frac{g(r,z,t)}{K}$. Under these conditions, the displacement and thermal stresses, in a semi-infinite circular cylinder due to heat generation are required to be determined.

2.1. Displacement potential and thermal stresses

Following Deshmukh et al. [1], one assumes that a circular cylinder of small thickness h is in a planar state of stress. In fact, the smaller the thickness of the circular cylinder compared to its diameter, the nearer to a planar state of stress is the actual state. The displacement equations of thermoelasticity have the form,

$$U_{i,kk} + \left(\frac{1+\nu}{1-\nu}\right) e_{,i} = 2\left(\frac{1+\nu}{1-\nu}\right) \cdot a_t \cdot T_{,i} \tag{1}$$

$$e = U_{k,k}; k, i = 1, 2. \tag{2}$$

where U_i - displacement component, e - dilation, T - temperature and ν and a_t are respectively, the Poisson's ratio and linear coefficient of thermal expansion of the semi infinite circular cylinder material.

Introducing

$$U_i = \psi_{,i}\,,\ i = 1, 2,$$

we have

$$\nabla_1^2 \psi = (1+\nu)\, a_t T, \qquad \nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{3}$$

$$\sigma_{ij} = 2\mu\left(\psi_{,ij} - \delta_{ij}\psi_{,kk}\right), \ i,j,k = 1,2, \tag{4}$$

where μ is the Lamé constant and δ_{ij} is the Kronecker delta symbol.

In the axially-symmetric case

$$\psi = \psi\left(r,z,t\right), \ T = T\left(r,z,t\right)$$

and the differential equation governing the displacement potential function $\psi\left(r,z,t\right)$ is expressed as

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = \left(1+\nu\right) a_t T. \tag{5}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial\psi}{\partial r}, \tag{6}$$

$$\sigma_{\theta\theta} = -2\mu\frac{\partial^2\psi}{\partial r^2}. \tag{7}$$

Also, in the planar state of stress within the circular cylinder,

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0. \tag{8}$$

The temperature of the semi-infinite circular cylinder satisfying the differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} + \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g\left(r,z,t\right)}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{9}$$

in $0 \le r \le b, \ 0 \le z < \infty$ with boundary conditions

$$k\frac{\partial T}{\partial r} = f\left(z,t\right) \quad \text{at } r = b, \ t > 0, \tag{10}$$

$$T = 0 \quad \text{at } z = 0, \ t > 0 \tag{11}$$

and the initial condition

$$T = F\left(r,z\right) \quad \text{when } t = 0, \tag{12}$$

where K and α are thermal conductivity and thermal diffusivity of the material of the semi-infinite circular cylinder respectively.

3. Solution

To obtain the expression for temperature distribution function $T\left(r,z,t\right)$ we introduce the Fourier transform and its inverse transform over the variable z in the range $0 \le z < \infty$ defined in [6] as

$$\overline{T}\left(r,\eta,t\right) = \int_0^\infty K\left(\eta,z'\right) T\left(r,z',t\right) dz', \tag{13}$$

$$T\left(r,z,t\right) = \int_0^\infty K\left(\eta,z\right) \overline{T}\left(r,\eta,t\right) d\eta, \tag{14}$$

where Kernel $K(\eta, z) = \sqrt{\frac{2}{\pi}} \sin \eta z$.

Taking the integral transform of system (9)-(12) by applying the transform equation (13), one obtains

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} - \eta^2 \overline{T} + \frac{\overline{g}}{K}(r, \eta, t) = \frac{1}{\alpha}\frac{\partial \overline{T}}{\partial t}, \tag{15}$$

$$k\frac{\partial \overline{T}}{\partial r} = \overline{f}(\eta, t) \text{ at } r = b, \tag{16}$$

$$\overline{T} = \overline{F}(r, \eta) \quad \text{for } t = 0. \tag{17}$$

Secondly, one define the finite Hankel transform and its inverse transform over the variable r in the range $0 \le r \le b$ as

$$\overline{\overline{T}}(\beta_m, \eta, t) = \int_0^b r' K_0(\beta_m, r') \; \overline{T}(r', \eta, t)\, dr', \tag{18}$$

$$\overline{T}(r, \eta, t) = \sum_{m=1}^{\infty} K_0(\beta_m, r) \; \overline{\overline{T}}(\beta_m, \eta, t), \tag{19}$$

where kernel is

$$K_0(\beta_0, r) = \frac{\sqrt{2}}{b}\frac{J_0(\beta_m r)}{J_0(\beta_m b)}, \tag{20}$$

and $\beta_1, \beta_2, \ldots,$ are the positive roots of the transcendental equation

$$J_1(\beta_m b) = 0. \tag{21}$$

Now one take the integral transform of the system (15)-(17) by applying the transform (18), one obtains

$$\frac{d\overline{\overline{T}}}{dt} + \alpha\left(\eta^2 + \beta_m\right)\overline{\overline{T}} = A(\beta_m, \eta, t), \tag{22}$$

$$\overline{\overline{T}}(\beta_m, \eta, t) = \overline{\overline{F}}(\beta_m, \eta) \quad \text{for } t = 0, \tag{23}$$

where $A(\beta_m, \eta, t) = \frac{\alpha}{k}\overline{\overline{g}}(\beta_m, \eta, t)$.

Solution of the equation (22) is obtained as

$$\overline{\overline{T}}(\beta_m, \eta, t) = e^{-\alpha\left(\beta_m^2+\eta^2\right)t}\left[\overline{\overline{F}}(\beta_m, \eta) + \int_{t'=0}^{t} e^{\alpha\left(\beta_m+\eta^2\right)t'} A(\beta_m, \eta, t)\, dt'\right]. \tag{24}$$

The resulting double transform of temperature is inverted successively by means of the inversion formulas (19) and (14). Then we obtain the expression of temperature $T(r, z, t)$ as

$$T(r, z, t) = \frac{2}{\sqrt{\pi} b}\int_0^{\infty}\sum_{m=1}^{\infty} \sin \eta z \frac{J_0(\beta_m r)}{J_0(\beta_m b)}\, e^{-\alpha\left(\beta_m^2+\eta^2\right)t} \tag{25}$$

$$\times \left[\frac{2}{\sqrt{\pi b}} \int_0^b \int_0^\infty r' \sin \eta z' \frac{J_0(\beta_m r)}{J_0(\beta_m b)} F(r', z')\, dr' dz' \right.$$
$$+ \int_{t'=0}^{t} e^{\alpha(\beta_m + \eta^2)t'} \frac{\alpha}{K} \frac{2}{\sqrt{\pi b}}$$
$$\left. \times \int_0^b \int_0^\infty r' \sin \eta z' \frac{J_0(\beta_m r)}{J_0(\beta_m b)} g(r' z' t')\, dz' dr' dt' \right] d\eta.$$

3.1. Displacement potential and thermal stresses

Using equation (25) in (5), one obtains

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \psi(r, z, t) \tag{26}$$
$$= (1+\nu)\, a_t \frac{2}{\sqrt{\pi b}} \int_0^\infty \sum_{m=1}^{\infty} \sin \eta z \frac{J_0(\beta_m r)}{J_0(\beta_m b)}\, e^{-\alpha(\beta_m^2 + \eta^2)t}$$
$$\times \left[\overline{\overline{F}}(\beta_m, \eta) + \int_0^t e^{\alpha(\beta_m + \eta^2)t'} A(\beta_m, \eta, t)\, dt' \right] d\eta.$$

Solving equation (26) by using the result

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) J_0(\beta_m r) = -\beta_m^2\, J_0(\beta_m r)$$

one obtains the expression of displacement potential function as

$$\psi(r, z, t) = -(1+\nu)\, a_t \frac{2}{\sqrt{\pi b}} \int_0^\infty \sum_{m=1}^{\infty} \sin \eta z \frac{1}{\beta_m^2} \frac{J_0(\beta_m r)}{J_0(\beta_m b)}\, e^{-\alpha(\beta_m^2 + \eta^2)t} \tag{27}$$
$$\times \left[\overline{\overline{F}}(\beta_m, \eta) + \int_0^t e^{\alpha(\beta_m + \eta^2)t'} A(\beta_m, \eta, t)\ dt' \right] d\eta.$$

Using equation (27) in equations (6) and (7), one obtains the expression of radial stress function and angular stress function as,

$$\sigma_{rr} = -4\mu(1+\nu)\, a_t \frac{1}{\sqrt{\pi b}} \left\{ \int_0^\infty \sum_{m=1}^{\infty} \sin \eta z \frac{1}{r\beta_m} \frac{J_1(\beta_m r)}{J_0(\beta_m b)}\, e^{-\alpha(\beta_m^2 + \eta^2)t} \right. \tag{28}$$
$$\left. \times \left[\overline{\overline{F}}(\beta_m, \eta) + \int_0^t e^{\alpha(\beta_m + \eta^2)t'} A(\beta_m, \eta, t)\, dt' \right] d\eta \right\},$$

(29)
$$\sigma_{\phi\phi} = -4\mu(1+\nu)\, a_t \frac{1}{\sqrt{\pi b}} \left\{ \int_0^\infty \sum_{m=1}^{\infty} \sin \eta z \frac{1}{J_0(\beta_m b)} \left(J_0(\beta_m r) - \frac{J_1(\beta_m r)}{\beta_m r} \right) \right.$$
$$\left. \times\, e^{-\alpha(\beta_m^2 + \eta^2)t} \left[\overline{\overline{F}}(\beta_m, \eta) + \int_0^t e^{\alpha(\beta_m + \eta^2)t'} A(\beta_m, \eta, t)\, dt' \right] d\eta \right\}.$$

4. Special case

4.1. Setting

$$F(r,z) = z^2 \times (r^2 - b^2)^2$$
$$f(z,t) = z^2 \times e^{-\omega t}, \omega > 0$$
$$g(r,z,t) = g_{pi} \cdot \delta\,(r - r_1) \cdot \delta\,(z - z_1) \cdot \delta\,(t - \tau)\,,$$

with $\omega = 10$, $t \to \tau = 5$, $g_{pi} = 50$ where r is the radius measured in meter, δ is the Dirac-delta function, $\omega > 0$. The heat source $g(r,z,t)$ is an instantaneous point heat source of strength $g_{pi} = 50J/m$ situated at the center of the semi-infinite circular cylinder along radial direction and releases its heat instantaneously at the time $t \to \tau = 5$.

4.2. Dimensions

Radius of a semi-infinite circular plate $b = 1\,m$.
Central circular paths of semi-infinite circular cylinder $r_1 = 0.5\,m$.

4.3. Material properties

The numerical calculation has been carried out for a Copper (Pure) thin circular cylinder with the material properties as,
Thermal diffusivity $\alpha = 112.34 \times 10^{-6}(m^2 s^{-1})$.
Thermal conductivity $K = 386(W/mk)$.
Density $\rho = 8954kg/m^3$.
Specific heat $c_p = 383J/kgK$,
Poisson ratio$\nu = 0.35$,
Coefficient of linear thermal expansion $a_t = 16.5 \times 10^{-6}\,\frac{1}{K}$,
Lam constant$\mu = 26.67$.

4.4. Roots of transcendental equation

The $\beta_1 = 3.8317,\ \ \beta_2 = 7.0156,\ \ \beta_3 = 10.1735,\ \ \beta_4 = 13.3237,\ \ \beta_5 = 16.470,$

$\beta_6 = 19.6159,\ \ \beta_7 = 22.7601,\ \ \beta_8 = 25.9037,\ \ \beta_9 = 29.0468,\ \ \beta_{10} = 32.18$

are the roots of transcendental equation $J_1(\beta\, b) = 0$.

We set for convenience, $X = \frac{2}{10^8\sqrt{\pi b}}$, $Y = \frac{2\,(1+\nu)a_t}{10^7\sqrt{\pi b}}$ and $Z = \frac{4(1+\nu)a_t\mu}{10^7\sqrt{\pi b}}$.

From Fig. 1, it is observed that, the temperature is maximum at the centre of the cylinder due to arbitrary initial heat supply and goes on decreasing towards the mid point of the radius and it fluctuate due to point heat source towards the outer circular edge of the cylinder

From Fig. 2, it is observed that the initial displacement function is negligible and goes on increasing towards the mid point of radius and afterward it is monotonically increases due to point heat source.

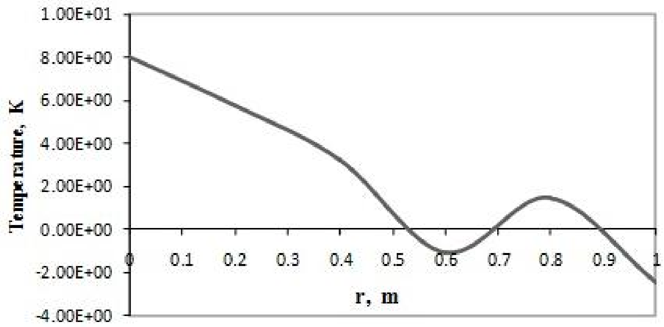

FIGURE 1. Temperature distribution

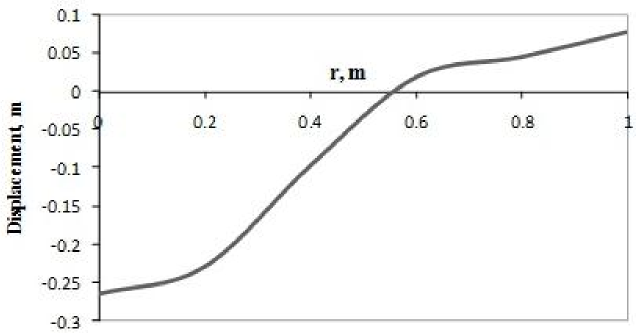

FIGURE 2. Displacement potential function

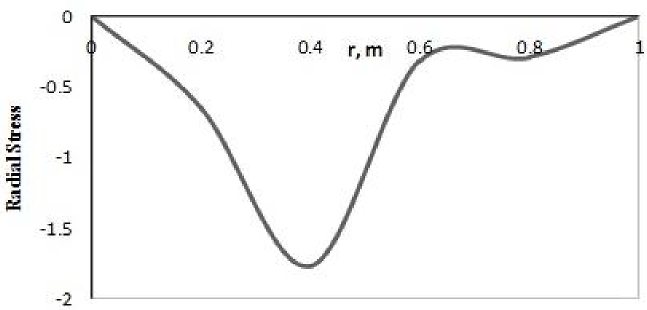

FIGURE 3. Radial stress function

From Fig. 3, it can be observed that the radial stress function is negligible and no such stresses are observed along the radial direction due to internal heat source as well as initial arbitrary heat supply.

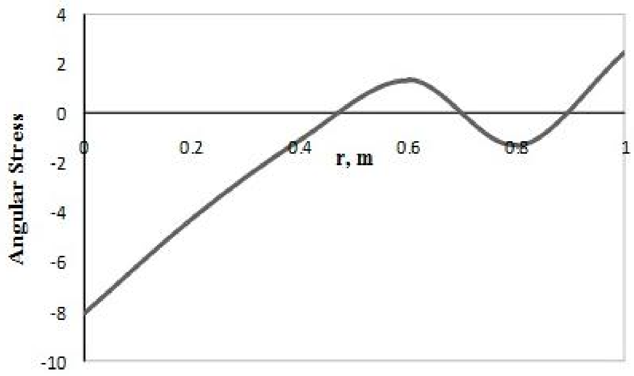

FIGURE 4. Angular stress function

From Fig. 4, it can be observed that angular stress function increases due to arbitrary initial heat supply and show fluctuation after central part $r_1 = 0.5$, due to point heat source.

5. Discussion of the results

In this paper, a semi-infinite circular cylinder is considered having arbitrary initial temperature and subjected to time dependent heat flux at the fixed outer circular boundary. The governing heat conduction equation has been solved by using Integral transform method. As a special case the mathematical model is constructed for a semi-infinite thin circular cylinder made up of Copper (Pure) metal.

From the figures of temperature and displacement it can be observed that the displacements and temperature are in proportionate but in opposite direction. Also from the figures of stress functions it can observe that the compressive stress occurs at the center where as the tensile stress occurs at outer surfaces of the plate. It may conclude that due arbitrary time dependent heat flux on the outer circular boundary of the circular cylinder, the plate expands in radial direction.

The results obtained here are more useful in engineering problems particularly in the determination of state of strain in circular cylinder. Also any particular case of special interest can be derived by assigning suitable values to the parameter and function in the expressions (25), (27) and (29).

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