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# VOLUME RATIOS OF A HYPERSURFACE RELATIVE TO THE FLRW SPACE-TIME

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Abstract. We calculate volume ratio of a hypersurface orthogonal to a timelike geodesic relative to that of a hypersurface in the FLRW space-time.

## 1. Introduction

The accelerated expanding universe by the cosmological observations has been recently one of the most remarkable achievements. It is well known that the Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time as the homogeneous and isotropic universe plays a good role for the expansion of the universe by adding the cosmological constant referred to as the standard model of cosmology. Let $\bar{M} = (0, \infty) \times_f S$ be the FLRW space-time, where $S$ is a 3-dimensional Riemannian manifold of constant curvature and $f$ is a scale factor (also known as a warping function). For a perfect fluid with energy density $\rho$ and pressure $p$, the stress-energy tensor is given by $T = (\rho + p)U^* \otimes U^* + p\ g$, where $g$ is a Lorentzian metric and $U^*$ is metric dual to an observer field $U$ (a future-pointing timelike unit vector field on $\bar{M}$). The Friedmann equation for the FLRW space-time $-3\frac{f''(t)}{f(t)} = 4\pi(\rho + 3p)$ along an observer field given by a geodesic $\bar{\gamma} = (t, \bar{q})$ for $q \in S$ (cf. [6]) and the equation of state $w = \frac{p}{\rho} < -\frac{1}{3}$ gives a geometrical interpretation for the expanding universe in terms of the Ricci curvature $\mathrm{Ric}(\bar{\gamma}'(t), \bar{\gamma}'(t)) = -3\frac{f''(t)}{f(t)}$. So the negative Ricci curvature indicates the expanding universe. The upperbounds of the volume expansion rates in a Lorentzian manifold in [2], [3] and [7] are based on the inverse of the timelike convergent condition $\mathrm{Ric}(\gamma', \gamma') \geq 0$ along a timelike geodesic $\gamma$ which indicates “the gravity attracts on average”. So the negative part of Ricci curvature explains the upperbound of the volume expansion rate relative to the space-time of zero curvature.

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From cosmological viewpoints, the FLRW space-time as the model space for volume expansion and contraction rates is considered. By a possible influx of dark matter and gravitational mass into the FLRW space-time nearby an observer field, the FLRW space-time could be deformed to a space-time $M$ which can not be locally no longer homogeneous and isotropic with the nonzero shear tensor along an observer field. In other words, a space-time $M$ is a geometry deviated from the FLRW space-time by such influx of dark matter and gravitational mass. We calculate volume expansion and contraction rates of a spacelike hypersurface of $M$ relative to that of the FLRW space-time in a geometrical way as in [2], [7] whose methods are mainly due to the Riemannian relative volume comparison theories obtained by P. Petersen, G. Wei and C. Sprouse [8], [9] (cf. [4], [10]).

Let $M$ be an $n$-dimensional Lorentzian manifold and $\gamma_v$ be a unit speed timelike radial geodesic $\gamma_v(t) = \exp_p tv$ with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ for all $v \in T_pM$. Let $\mathrm{Fut}(T_pM)$ be the set of all future directed timelike vectors $v \in T_pM$ such that $\exp_p(v)$ is defined for a fixed point $p \in M$. Put

$$H(r_0) = \{v \in \mathrm{Fut}(T_pM) \mid \langle v, v\rangle = -r_0^2\}$$

for $0 < r_0 < r < \mathrm{cut}_v(p)$ and denote by $H^*(r_0)$ a compact subset of $H(r_0)$. Consider a geodesic variation along $\gamma_v$ starting from $p$ which produces level hypersurfaces of geodesic sphere $\exp_p H^*(1)$. Then we get the following differential equation ([3], [4])

$$\theta' + \theta^2 + s^M - s(H_t) + 3\mathrm{Ric}(\gamma_v', \gamma_v') = 0, \tag{1}$$

where we denote by $s^M$, $s(H_t)$ the scalar curvature of $M$ at the point $\gamma_v(t)$, the scalar curvature of level hypersurface $H_t = \exp_p H(t)\,(r_0 < t)$, respectively and $\theta(t)$ is the mean curvature of $H_t$ along $\gamma_v(t)$.

As a generalization of the FLRW space-time, consider a Lorentzian warped product $\bar{M} = (0, \infty) \times_f H(r_0)$. Then a unit speed timelike radial geodesic $\bar{\gamma}_{\bar{v}}(t)$ with $\bar{\gamma}_{\bar{v}}(0) = \bar{p}$ and $\bar{\gamma}_{\bar{v}}'(0) = \bar{v}$ for each $\bar{v} \in T_{\bar{p}}\bar{M}$ is orthogonal to the hypersurface $H(r_0)$ at time $t = r_0$. Since a Jacobi tensor $\bar{A}(t)$ along $\bar{\gamma}_{\bar{v}}(t)$ is given by $\bar{A}(t) = f(t)\mathrm{Id}$ with the zero shear tensor (note that the fiber is totally umbilic and the curvature tensor is isotropic along $\bar{\gamma}_{\bar{v}}$), we get the following differential equation along $\bar{\gamma}_{\bar{v}}$ ([4])

$$\bar{\theta}' + \bar{\theta}^2 = \frac{(n-1)(n-2)(f')^2 + (n-1)ff''}{f^2} \tag{2}$$

and the Ricci curvature along $\bar{\gamma}_{\bar{v}}$ is given by $\mathrm{Ric}(\bar{\gamma}_{\bar{v}}', \bar{\gamma}_{\bar{v}}') = -\frac{(n-1)f''}{f}$.

To compare the volume ratio, we need the following linear isometry (3). Let $M$ be a globally hyperbolic Lorentzian manifold of dimension $n$ and $\gamma$ be a unit speed timelike radial geodesic orthogonal to the achronal spacelike hypersurface $H^*_{r_0} = \exp_p H^*(r_0)$ for a fixed point $p \in M$. Let $A$, $\bar{A}$ be an $H^*_{r_0}$, $H^*(r_0)$-Jacobi tensor along $\gamma_v$, $\bar{\gamma}_{\bar{v}}$, respectively. Assume a linear isometry

$$\imath : T_{\gamma_v(r_0)}H^*_{r_0} \to T_{\bar{\gamma}_{\bar{v}}(r_0)}H^*(r_0) \tag{3}$$

such that $H^*(r_0) = \exp_{\bar{\gamma}_{\bar{v}}(r_0)} \circ \imath \circ \exp^{-1}_{\gamma_v(r_0)} H^*_{r_0}$ and $\imath(\gamma'_v(r_0)) = \bar{\gamma}'_{\bar{v}}(r_0)$, $\imath(e_i) = \bar{e}_i$ for an orthonormal basis $\{e_1, e_2, \ldots, e_{n-1}\}$ of $T_{\gamma(r_0)}H^*_{r_0}$ and its parallel basis $\{E_1, E_2, \ldots, E_{n-1}\}$ along $\gamma_v$ with $E_i(r_0) = e_i$ for each $i$. And we can apply the above linear isometry for all directions $v \in T_pM$ with $\imath(\gamma'_v(r_0)) = \bar{\gamma}'_{\bar{v}}(r_0)$. So from now on, we omit the direction $v$.

Now we can get the following upperbound (4) of the volume expansion rate using the similar calculations as in [2], [7]. Let $dH^*(1)$ be the volume element of $H^*(1)$. Then the volume element of a level hypersurface $H^*_t = \exp_p H^*(t)$ along $\gamma(t)$ is given by $\det A(t)\, dH^*(1)$. Let $\bar{M} = (0, \infty) \times_f H(r_0)$ be a Lorentzian warped product with $\dim \bar{M} = n$. Assume that $\theta(r_0) \leq \bar{\theta}(r_0)$ and $\bar{\theta}(t) = \frac{(n-1)f'}{f} \geq 0$. Then we get the upperbound of the volume expansion rate

$$\frac{\det(A(R))}{\det(A(r_0))} \leq \Big(\frac{f(R)}{f(r_0)}\Big)^{n-1} \exp((R - r_0)\sqrt{n-1}\,k_\gamma(1, R)^{\frac{1}{2}}), \tag{4}$$

where $k_\gamma(1, R) = \frac{1}{R - r_0}\int_{r_0}^R \tilde{\rho}\, dt$ with $\tilde{\rho} = \max\{0, -\mathrm{Ric}(\gamma', \gamma') - \frac{(n-1)f''}{f}\}$ and $0 < r_0 < R < \min\{\mathrm{cut}_{\bar{v}}(\bar{p}), \mathrm{cut}_v(p)\}$.

As an example of (4), consider a complete and simply connected $n$-dimensional Lorentzian manifold $\bar{M}(k)$ of constant curvature $k > 0$, whose Jacobi tensor along a unit speed timelike geodesic $\bar{\gamma}(t)$ with $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma}'(0) = \bar{v}$ is given by

$$\bar{A}(t) = \frac{1}{\sqrt{k}} \sinh\sqrt{k}t\, \mathrm{Id}$$

with the initial conditions $\bar{A}(0) = 0$ and $\bar{A}'(0) = \mathrm{Id}$. Note that the Jacobi equation along $\bar{\gamma}$ is $\bar{x}'' - k\bar{x} = 0$ with $\bar{x}(0) = 0$ and $\bar{x}'(0) = 1$, where $\bar{x} = (\det \bar{A})^{\frac{1}{n-1}}$. Thus for $k_\gamma(1, R) = \frac{1}{R - r_0}\int_{r_0}^R \tilde{\rho}\, dt$ with $\tilde{\rho} = \max\{0, -\mathrm{Ric}(\gamma', \gamma') + (n-1)k\}$, we have for $0 < r_0 < R$

$$\frac{\det(A(R))}{\det(A(r_0))} \leq \Big(\frac{\sinh\sqrt{k}R}{\sinh\sqrt{k}r_0}\Big)^{n-1} \exp((R - r_0)\sqrt{n-1}\,k_\gamma(1, R)^{\frac{1}{2}}),$$

which indicates that $\tilde{\rho} = \max\{0, -\mathrm{Ric}(\gamma', \gamma') + (n-1)k\}$ does mainly control the upperbound of the volume expansion rate of the level hypersurfaces as follows.

**Theorem 1.** *Let $\bar{M} = (0, \infty) \times_f H(r_0)$ be a Lorentzian warped product with $\dim \bar{M} = n$ and $\bar{\gamma}_{\bar{v}}(t)$ be a unit speed timelike radial geodesic with $\bar{\gamma}_{\bar{v}}(0) = \bar{p}$ and $\bar{\gamma}'_{\bar{v}}(0) = \bar{v}$ for each $\bar{v} \in T_{\bar{p}}\bar{M}$ orthogonal to the hypersurface $H(r_0)$ at time $t = r_0$. Assume that $\theta(r_0) \leq \bar{\theta}(r_0)$ and $\bar{\theta}(t) = \frac{(n-1)f'}{f} \geq 0$. Then we get the upperbound of the volume expansion rate*

$$\frac{\det(A(R))}{\det(A(r_0))} \leq \Big(\frac{f(R)}{f(r_0)}\Big)^{n-1} \exp((R - r_0)\mu_\gamma(1, R)^{\frac{1}{2}}),$$

*where* $\mu_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^{R} \mu\, dt$ *with* $\mu = \max\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)(n-2)(f')^2+(n-1)ff''}{f^2}\}$ *and* $0 < r_0 < R < \min\{cut_{\bar{v}}(\bar{p}), cut_v(p)\}$.

*If* $-(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) \leq -\mathrm{Ric}(\gamma',\gamma')$, *then we get the sharper upperbound than* (4) *from*

$$\begin{aligned} &\max\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)(n-2)(f')^2 + (n-1)ff''}{f^2}\} \\ \leq\ &\max\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)f''}{f}\} \\ \leq\ &\max\{0, -\mathrm{Ric}(\gamma',\gamma') - \frac{(n-1)f''}{f}\}. \end{aligned}$$

The upperbound of the volume expansion rate calculated in [2] using $\bar{\theta}'(t) + \bar{\theta}^2(t) = 0$ with $\bar{\theta}(0) = V > 0$ is now given by

$$\frac{\det(A(R))}{\det(A(r_0))} \leq \Big(\frac{VR+1}{Vr_0+1}\Big)\exp((R-r_0)\mu_\gamma(1,R)^{\frac{1}{2}}),$$

where $\mu_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^{R} \mu\, dt$ with $\mu = \max\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma'))\}$.

Finally we obtain the lowerbound of the volume contraction rate as follows.

**Theorem 2.** *Let* $\bar{M} = (0,\infty) \times_f H(r_0)$ *be a Lorentzian warped product with* $\dim \bar{M} = n$ *and* $\bar{\gamma}_{\bar{v}}(t)$ *be a unit speed timelike radial geodesic with* $\bar{\gamma}_{\bar{v}}(0) = \bar{p}$ *and* $\bar{\gamma}'_{\bar{v}}(0) = \bar{v}$ *for each* $\bar{v} \in T_{\bar{p}}\bar{M}$ *orthogonal to the hypersurface* $H(r_0)$ *at time* $t = r_0$. *Assume that* $\theta(r_0) \geq \bar{\theta}(r_0)$ *and* $\theta(t) \geq 0$. *Then we get the lowerbound of the volume contraction rate*

$$\Big(\frac{f(R)}{f(r_0)}\Big)^{n-1}\exp((R-r_0)(-\mu_\gamma(1,R)^{\frac{1}{2}})) \leq \frac{\det(A(R))}{\det(A(r_0))},$$

*where* $\mu_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^{R} -\mu\, dt$ *with* $\mu = \min\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)(n-2)(f')^2+(n-1)ff''}{f^2}\}$ *and* $0 < r_0 < R < \min\{cut_{\bar{v}}(\bar{p}), cut_v(p)\}$.

Note that $\exp((R-r_0)(-\mu_\gamma(1,R)^{\frac{1}{2}}))$ is less than or equal to 1. So it could be considered as the contracting term. If $s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma') \leq -\frac{(n-1)(n-2)(f')^2+(n-1)ff''}{f^2}$, then we have

$$\Big(\frac{f(R)}{f(r_0)}\Big)^{n-1} \leq \frac{\det(A(R))}{\det(A(r_0))}.$$

As in [2], let's denote $\frac{1}{t+\frac{1}{V}}$ by $\bar{\theta}(t)$ which satisfies $\bar{\theta}'(t) + \bar{\theta}^2(t) = 0$ and $\bar{\theta}(0) = V > 0$. Then we get by Remark 1 at the end of this paper

$$\Big(\frac{VR+1}{Vr_0+1}\Big)\exp((R-r_0)(-\mu_\gamma(1,R)^{\frac{1}{2}})) \leq \frac{\det(A(R))}{\det(A(r_0))},$$

where $\mu_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^{R} -\mu\, dt$ with $\mu = \min\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma'))\}$.

## 2. Preliminaries

**Definition 1** (cf. [5])**.** Let $\gamma$ be a unit-speed geodesic orthogonal to a hypersurface $H$ at $\gamma(r_0)$ with $N_{\gamma(r_0)} = \gamma'(r_0)$. A smooth $(1,1)$ tensor field $A : (\gamma')^{\perp} \to (\gamma')^{\perp}$ is called an *H-Jacobi tensor* along $\gamma$ if it satisfies

$$A'' + R(A, \gamma')\gamma' = 0, \quad \mathrm{ker}A \cap \mathrm{ker}A' = \{0\}, \quad A(r_0) = \mathrm{Id}, \quad A'(r_0) = S_{-N},$$

where Id is the identity endomorphism of $(\gamma')^{\perp}$.

Put $B = A'A^{-1}$ for an $H$-Jacobi tensor $A$ along $\gamma$, then we have

$$B' = A''A^{-1} - A'A^{-1}A'A^{-1} = -R_{\gamma'} - B \circ B, \tag{5}$$

where we put $R(A, \gamma')\gamma' = R_{\gamma'}A$. The *expansion* $\theta$ can be written as

$$\theta = \mathrm{tr}(B) = \frac{(\det(A))'}{\det(A)}. \tag{6}$$

The shape operator $S_{-\gamma'(t)}$ of each level hypersurface $H_t$ is given by

$$A'A^{-1}(t) = S_{-\gamma'(t)} = S_t \tag{7}$$

as in [1] and we denote by $\theta(t) = \mathrm{tr}S_{-\gamma'(t)}$ the mean curvature of $H_t$ along $\gamma(t)$. The *shear tensor* $\sigma$ of $A$ along $\gamma$ is defined by

$$\sigma = B - \frac{\theta}{n-1}\mathrm{Id}.$$

Note that a variation tensor field $A$ is a Lagrange tensor (Proposition 1 in [1]). So the vorticity $\frac{1}{2}(B - B^*)$ is zero, where $*$ denotes the adjoint. Taking the trace of (5), we get the *Raychaudhuri equation*

$$\theta' + \frac{\theta^2}{n-1} + \mathrm{Ric}(\gamma', \gamma') + \mathrm{tr}\sigma^2 = 0, \tag{8}$$

where $\mathrm{Ric}(\gamma', \gamma') = \sum_{i=1}^{n-1} g(R(e_i, \gamma')\gamma', e_i)$ for an orthonormal basis $\{e_i\}_{i=1}^{n-1}$ of $\gamma'^{\perp}$.

Putting $x = \det A^{\frac{1}{n-1}}$, we see

$$x' = \frac{1}{n-1}x\theta, \quad x'' = \frac{1}{n-1}(\theta' + \frac{\theta^2}{n-1})x. \tag{9}$$

So we obtain the *Jacobi equation* by (8) and (9)

$$x'' + \frac{1}{n-1}(\mathrm{Ric}(\gamma', \gamma') + \mathrm{tr}\sigma^2)x = 0. \tag{10}$$

## 3. Proofs

Mathematically we adopt the methods of relative volume comparison theories studied in [8], [9] and [10]. The upperbound (4) can be obtained by using Raychaudhuri equation with the calculations ([2], [7]) for some interval satisfying $\mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') = -\frac{(n-1)f''}{f} \geq \mathrm{Ric}(\gamma', \gamma')$. We introduce here basic methods for the calculations. The Raychaudhuri equation along a geodesic $\bar{\gamma}(t) = (t, \bar{q})$ in $\bar{M} = (0, \infty) \times_f H(r_0)$ is given by

$$\bar{\theta}' + \frac{\bar{\theta}^2}{n-1} = -\mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') = \frac{(n-1)f''}{f}, \tag{11}$$

since we have the $H(r_0)$-Jacobi tensor $\bar{A}(t) = f(t)\mathrm{Id}$ and the zero shear tensor $\sigma = 0$ along $\bar{\gamma}$. The Raychaudhuri equation along a geodesic $\gamma(t)$ in $M$ is

$$\theta' + \frac{\theta^2}{n-1} + \mathrm{Ric}(\gamma', \gamma') + \mathrm{tr}\sigma^2 = 0. \tag{12}$$

So we have the following inequality

$$\theta' + \frac{\theta^2}{n-1} \leq -\mathrm{Ric}(\gamma', \gamma'), \tag{13}$$

since $\mathrm{tr}\sigma^2$ is not negative.

Put $\psi(t) = \max\{0, \theta(t) - \bar{\theta}(t)\}$. The subtraction (11) from (13) gives

$$\psi' + \frac{\psi^2}{n-1} + \frac{2\psi\bar{\theta}}{n-1} \leq \tilde{\rho}, \tag{14}$$

where $\tilde{\rho} = \max\{0, -\mathrm{Ric}(\gamma', \gamma') - \frac{(n-1)f''}{f}\}$. Multiply the inequality (14) by $\psi^{2p-2}$ and integrate to get

$$\int_{r_0}^{R} \psi^{2p}\, dt \leq (n-1)^p \int_{r_0}^{R} \tilde{\rho}^p\, dt \tag{15}$$

for $p \geq 1$ under the assumption $\theta(r_0) \leq \bar{\theta}(r_0)$ as in [7].

Since $\theta = \frac{(\det(A))'}{\det(A)}$, we see

$$\log\Big(\frac{\det(A(R))}{\det(A(r_0))}\Big) = \int_{r_0}^{R} \frac{(\det(A))'}{\det(A)}\, dt = \int_{r_0}^{R} \theta\, dt \leq \int_{r_0}^{R} \bar{\theta}\, dt + \int_{r_0}^{R} \psi\, dt$$

and

$$\int_{r_0}^{R} \bar{\theta}\, dt = (n-1)\int_{r_0}^{R} \frac{f'}{f}\, dt = \log\Big(\frac{f(R)}{f(r_0)}\Big)^{n-1}.$$

So we have

$$\frac{\det(A(R))}{\det(A(r_0))} \leq e^{\int_{r_0}^{R} \bar{\theta}\, dt} e^{\int_{r_0}^{R} \psi\, dt} \leq \Big(\frac{f(R)}{f(r_0)}\Big)^{n-1} e^{\int_{r_0}^{R} \psi\, dt}.$$

Using Hölder inequality together with (15) as in [7], we get the upperbound of $\int_{r_0}^{R} \psi\, dt$ of (4).

So the upperbound (4) indicates that if $\mathrm{Ric}(\gamma',\gamma') \geq -\frac{(n-1)f''}{f}$, then the volume expansion rate $\frac{\det(A(R))}{\det(A(r_0))}$ is less than equal to $\left(\frac{f(R)}{f(r_0)}\right)^{n-1}$. Also the upperbound (4) can be viewed as a generalization of the case with $f(t) = Vt+n-1$ so that $\bar{\theta}(t) = (n-1)\frac{f'(t)}{f(t)} = (n-1)\frac{V}{Vt+n-1} = \frac{n-1}{t+\frac{n-1}{V}}$

$$\frac{\det(A(R))}{\det(A(r_0))} \leq \Big(\frac{VR+n-1}{Vr_0+n-1}\Big)^{n-1}\exp((R-r_0)\sqrt{n-1}k_\gamma(1,R)^{\frac{1}{2}}), \tag{16}$$

where $k_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^R \tilde{\rho}(t)\,dt$, $\tilde{\rho}(t) = \max\{0, -\mathrm{Ric}(\gamma',\gamma')\}$ and $V = \bar{\theta}(r_0) = \frac{(\det A(r_0))'}{\det A(r_0)} > 0$ obtained in [2] and [7].

*Proof of Theorem 1.* Put $\psi(t) = \max\{0,\, \theta(t) - \bar{\theta}(t)\}$. The subtraction (2) from (1) gives

$$\psi' + \psi^2 + 2\psi\bar{\theta} \leq \mu, \tag{17}$$

where $\mu = \max\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)(n-2)(f')^2+(n-1)ff''}{f^2}\}$. Thus we get

$$\psi' + \psi^2 + 2\psi\bar{\theta} \leq \tilde{\rho}.$$

Multiply the inequality (17) by $\psi^{2p-2}$ and integrate to get

$$\int_{r_0}^R \psi^{2p}\,dt \leq \int_{r_0}^R \mu^p\,dt \tag{18}$$

for $p \geq 1$ under the assumptions $\bar{\theta}(t) \geq 0$ and $\theta(r_0) \leq \bar{\theta}(r_0)$. By Hölder inequality, we get

$$\begin{aligned}\frac{1}{R-r_0}\int_{r_0}^R \psi\,dt &\leq \frac{1}{R-r_0}\Big(\int_{r_0}^R \psi^{2p}\,dt\Big)^{\frac{1}{2p}}(R-r_0)^{\frac{1}{q}}\\ &\leq \frac{1}{R-r_0}\Big(\int_{r_0}^R \mu^p\,dt\Big)^{\frac{1}{2p}}(R-r_0)^{\frac{1}{q}}\\ &\leq \Big(\frac{1}{R-r_0}\int_{r_0}^R \mu^p\,dt\Big)^{\frac{1}{2p}}\end{aligned}$$

for $\frac{1}{2p} + \frac{1}{q} = 1$ $(2p > 1)$. Put $\mu_\gamma(p,R) = \frac{1}{R-r_0}\int_{r_0}^R \mu^p\,dt$, then

$$\frac{1}{R-r_0}\int_{r_0}^R \psi\,dt \leq \Big(\frac{1}{R-r_0}\int_{r_0}^R \mu^p\,dt\Big)^{\frac{1}{2p}} = (\mu_\gamma(p,R))^{\frac{1}{2p}}. \tag{19}$$

Then we obtain

$$\frac{\det(A(R))}{\det(A(r_0))} \leq e^{\int_{r_0}^R \bar{\theta}\,dt} e^{\int_{r_0}^R \psi\,dt} \leq \Big(\frac{f(R)}{f(r_0)}\Big)^{n-1}\exp((R-r_0)(\mu_\gamma(p,R)^{\frac{1}{2p}}))$$

for $\mu_\gamma(p,R) = \frac{1}{R-r_0}\int_{r_0}^R \mu^p\,dt$. Using Hölder inequality, we get

$$\int_{r_0}^R \frac{1}{R-r_0}\mu\,dt \leq \Big(\int_{r_0}^R \mu^p\,dt\Big)^{\frac{1}{p}}\Big(\int_{r_0}^R (\frac{1}{R-r_0})^q\,dt\Big)^{\frac{1}{q}} = \Big(\frac{1}{R-r_0}\int_{r_0}^R \mu^p\,dt\Big)^{\frac{1}{p}}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Hence for any positive $p>1$,

$$\mu_\gamma(1,R)=\frac{1}{R-r_0}\int_{r_0}^{R}\mu\,dt\le\Big(\frac{1}{R-r_0}\int_{r_0}^{R}\mu^p\,dt\Big)^{\frac{1}{p}}=(\mu_\gamma(p,R))^{\frac{1}{p}},\tag{20}$$

which means that $\mu_\gamma(1,R)=\inf\{(\mu_\gamma(p,R))^{\frac{1}{p}}\,|\,p>1\}$. So we get the upper-bound of $\int_{r_0}^{R}\psi\,dt$ of Theorem 1. □

*Proof of Theorem 2.* Put $\psi(t)=\min\{0,\,\theta(t)-\bar\theta(t)\}$. Then

$$\theta\ge\bar\theta+\psi.\tag{21}$$

The subtraction (2) from (1) gives

$$\psi'-\psi^2+2\psi\theta\ge\mu,$$

where $\mu=\min\{0,-(s^M-s(H_t)+3\mathrm{Ric}(\gamma',\gamma'))-\frac{(n-1)(n-2)(f')^2+(n-1)ff''}{f^2}\}$ (cf. (17)). Thus we have

$$-\psi'+\psi^2-2\psi\theta\le-\mu.\tag{22}$$

Multiply (22) by $\psi^{2p-2}$ and integrate to get

$$-\int_{r_0}^{R}\psi'\psi^{2p-2}\,dt+\int_{r_0}^{R}\psi^{2p}\,dt-2\int_{r_0}^{R}\psi^{2p-1}\theta\,dt\le\int_{r_0}^{R}-\mu\psi^{2p-2}\,dt.\tag{23}$$

Since $\psi^{2p-1}=(\psi^2)^{p-1}\psi\le0$, $\theta(t)\ge0$ and $\psi(r_0)=0$ from the assumptions of Theorem 2, we get

$$-\int_{r_0}^{R}\psi'\psi^{2p-2}\,dt=-\frac{1}{2p-1}\psi^{2p-1}\Big|_{r_0}^{R}\ge0,\quad -2\int_{r_0}^{R}\psi^{2p-1}\theta\ dt\ge0.$$

Hence (23) becomes

$$\int_{r_0}^{R}\psi^{2p}\,dt\le\int_{r_0}^{R}-\mu\psi^{2p-2}\,dt.$$

By Hölder inequality, we get

$$\int_{r_0}^{R}\psi^{2p}\,dt\le\int_{r_0}^{R}-\mu\psi^{2p-2}\,dt\le\Big(\int_{r_0}^{R}(-\mu)^p dt\Big)^{\frac{1}{p}}\Big(\int_{r_0}^{R}\psi^{2p}\,dt\Big)^{1-\frac{1}{p}}.\tag{24}$$

Dividing by $\Big(\int_{r_0}^{R}\psi^{2p}\,dt\Big)^{1-\frac{1}{p}}$, we get for $p>1$

$$\Big(\int_{r_0}^{R}\psi^{2p}\,dt\Big)^{\frac{1}{p}}\le\Big(\int_{r_0}^{R}(-\mu)^p\,dt\Big)^{\frac{1}{p}},\tag{25}$$

which holds trivially for $p=1$ from (23).

Using Hölder inequality, we see

$$\frac{1}{R-r_0}\int_{r_0}^{R}-\psi\,dt\le\frac{1}{R-r_0}\Big(\int_{r_0}^{R}(-\psi)^{2p}\,dt\Big)^{\frac{1}{2p}}(R-r_0)^{\frac{1}{q}}$$

for $\frac{1}{2p} + \frac{1}{q} = 1 (2p > 1)$. Thus we get

$$\frac{1}{R-r_0}\int_{r_0}^{R}\psi\,dt \geq \frac{-1}{R-r_0}\Big(\int_{r_0}^{R}(-\psi)^{2p}\,dt\Big)^{\frac{1}{2p}}(R-r_0)^{\frac{1}{q}}. \tag{26}$$

And we get by (25) and (26)

$$\begin{aligned}\frac{1}{R-r_0}\int_{r_0}^{R}\psi\,dt &\geq \frac{-1}{R-r_0}\Big(\int_{r_0}^{R}(-\mu)^{p}\,dt\Big)^{\frac{1}{2p}}(R-r_0)^{\frac{1}{q}}\\ &= -\Big(\frac{1}{R-r_0}\int_{r_0}^{R}(-\mu)^{p}\,dt\Big)^{\frac{1}{2p}}.\end{aligned}$$

Put

$$\mu_\gamma(p,R) = \frac{1}{R-r_0}\int_{r_0}^{R}(-\mu)^{p}\,dt,$$

then we have

$$\frac{1}{R-r_0}\int_{r_0}^{R}\psi\,dt \geq -\Big(\frac{1}{R-r_0}\int_{r_0}^{R}(-\mu)^{p}\,dt\Big)^{\frac{1}{2p}} = -(\mu_\gamma(p,R))^{\frac{1}{2p}}. \tag{27}$$

Note that $\theta = \mathrm{tr}(B) = \frac{(\det(A))'}{\det(A)}$ (6) and $\bar\theta = \frac{(n-1)f'}{f}$ from $\bar A = f\mathrm{Id}$. So we have

$$\int_{r_0}^{R}\bar\theta\,dt = (n-1)\int_{r_0}^{R}\frac{f'}{f}\,dt = \log\Big(\frac{f(R)}{f(r_0)}\Big)^{n-1}. \tag{28}$$

Since $\theta \geq \bar\theta + \psi$ (21), we see

$$\log\Big(\frac{\det(A(R))}{\det(A(r_0))}\Big) = \int_{r_0}^{R}\frac{(\det(A))'}{\det(A)}\,dt = \int_{r_0}^{R}\theta\,dt \geq \int_{r_0}^{R}\bar\theta\,dt + \int_{r_0}^{R}\psi\,dt.$$

Thus it follows from (27) and (28) that

$$\frac{\det(A(R))}{\det(A(r_0))} \geq e^{\int_{r_0}^{R}\bar\theta\,dt}e^{\int_{r_0}^{R}\psi\,dt} \geq \Big(\frac{f(R)}{f(r_0)}\Big)^{n-1}\exp((R-r_0)(-\mu_\gamma(p,R)^{\frac{1}{2p}})). \tag{29}$$

Again by Hölder inequality, we get

$$\begin{aligned}\int_{r_0}^{R}\frac{1}{R-r_0}|\mu|\,dt &\leq \Big(\int_{r_0}^{R}|\mu|^{p}\,dt\Big)^{\frac{1}{p}}\Big(\int_{r_0}^{R}(\frac{1}{R-r_0})^{q}\,dt\Big)^{\frac{1}{q}}\\ &= \Big(\frac{1}{R-r_0}\int_{r_0}^{R}|\mu|^{p}\,dt\Big)^{\frac{1}{p}}\end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = 1$. Hence for $p > 1$, we see

$$-\Big(\frac{1}{R-r_0}\int_{r_0}^{R}(-\mu)^{p}\,dt\Big)^{\frac{1}{2p}} \leq -\Big(\frac{1}{R-r_0}\int_{r_0}^{R}|\mu|\,dt\Big)^{\frac{1}{2}} = -\mu_\gamma(1,R)^{\frac{1}{2}}, \tag{30}$$

which means that $-\mu_\gamma(1,R) = \sup\{(-\mu_\gamma(p,R))^{\frac{1}{p}}\,|\,p > 1\}$. The lowerbound of Theorem 2 follows from (29) and (30), that is,

$$\Big(\frac{f(R)}{f(r_0)}\Big)^{n-1}\exp((R-r_0)(-\mu_\gamma(1,R)^{\frac{1}{2}})) \leq \frac{\det(A(R))}{\det(A(r_0))}. \qquad \square$$

*Remark* 1. Instead of the equation (2), consider

$$\bar{\theta}'(t) + \bar{\theta}^2(t) = 0, \tag{31}$$

whose solution is denoted by $\bar{\theta}(t) = \frac{1}{t+\frac{1}{V}}$ with $\bar{\theta}(0) = V > 0$ as in [2]. Put $\psi(t) = \min\{0, \theta(t) - \bar{\theta}(t)\}$. By the subtraction (1) from (31), we have

$$\psi' - \psi^2 + 2\psi\theta \geq \mu, \tag{32}$$

where $\mu = \min\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma', \gamma'))\}$. Since

$$\int_{r_0}^{R} \bar{\theta}\, dt = \int_{r_0}^{R} \frac{d}{dt}(\log(Vt+1))\, dt = \log \frac{VR+1}{Vr_0+1},$$

we get from the same arguments of the proof of Theorem 2

$$\Big(\frac{VR+1}{Vr_0+1}\Big)\exp((R-r_0)(-\mu_\gamma(1,R)^{\frac{1}{2}})) \leq \frac{\det(A(R))}{\det(A(r_0))},$$

where $\mu_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^{R} -\mu\, dt$.

If $-(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) \geq -\mathrm{Ric}(\gamma',\gamma')$, then we get

$$\psi' - \psi^2 + 2\psi\theta \geq \mu \geq \tilde{\mu}, \tag{33}$$

where $\tilde{\mu} = \min\{0, -\mathrm{Ric}(\gamma',\gamma')\}$. Thus we get

$$\Big(\int_{r_0}^{R} -\psi^{2p}\, dt\Big)^{\frac{1}{p}} \leq \Big(\int_{r_0}^{R} (-\mu)^p\, dt\Big)^{\frac{1}{p}} \leq \Big(\int_{r_0}^{R} (-\tilde{\mu})^p\, dt\Big)^{\frac{1}{p}},$$

which leads to

$$\begin{aligned} &\Big(\frac{VR+1}{Vr_0+1}\Big)\exp((R-r_0)(-\tilde{\mu}_\gamma(1,R)^{\frac{1}{2}})) \\ \leq &\Big(\frac{VR+1}{Vr_0+1}\Big)\exp((R-r_0)(-\mu_\gamma(1,R)^{\frac{1}{2}})) \leq \frac{\det(A(R))}{\det(A(r_0))}, \end{aligned}$$

where $\tilde{\mu}_\gamma(1,R) = \frac{1}{R-r_0}\int_{r_0}^{R} -\tilde{\mu}\, dt$ with $\tilde{\mu} = \min\{0, -\mathrm{Ric}(\gamma',\gamma')\}$.

Consider

$$\bar{\theta}' + \bar{\theta}^2 = \frac{(n-1)(n-2)(f')^2 + (n-1)ff''}{f^2} \geq \frac{(n-1)f''}{f}$$

and subtract it from (1), we have

$$\psi' - \psi^2 + 2\psi\theta \geq \mu, \tag{34}$$

where $\mu = \min\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)f''}{f}\}$. Note that

$$\begin{aligned} &\tilde{\mu} = \min\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)(n-2)(f')^2 + (n-1)ff''}{f^2}\} \\ &\leq \mu = \min\{0, -(s^M - s(H_t) + 3\mathrm{Ric}(\gamma',\gamma')) - \frac{(n-1)f''}{f}\}. \end{aligned}$$

Hence $-\Big(\frac{1}{R-r_0}\int_{r_0}^{R} |\tilde{\mu}|\, dt\Big)^{\frac{1}{2}} \leq -\Big(\frac{1}{R-r_0}\int_{r_0}^{R} |\mu|\, dt\Big)^{\frac{1}{2}}$ (see (30)).

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