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STABLE f-HARMONIC MAPS ON SPHERE

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Abstract. In this paper, we prove that any stable f-harmonic map ψ from $\mathbb{S}^2$ to N is a holomorphic or anti-holomorphic map, where N is a Kählerian manifold with non-positive holomorphic bisectional curvature and f is a smooth positive function on the sphere $\mathbb{S}^2$ with $\operatorname{Hess} f \leq 0$. We also prove that any stable f-harmonic map ψ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold N is constant.

1. Introduction

Let $\psi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds and let f be a smooth positive function on M. The map ψ is said to be f-harmonic if it is a critical point of the f-energy functional

$$E_f(\psi) = \frac{1}{2}\int_\Omega f(x)\,|d\psi|^2\,v_g \tag{1}$$

for every compact domain $\Omega \in M$. The Euler-Lagrange equation associated to the f-energy functional is

$$\tau_f(\psi) \equiv \operatorname{trace}_g \nabla f\, d\psi = f\,\tau(\psi) + d\psi(\operatorname{grad} f) = 0, \tag{2}$$

where $\tau(\psi) = \operatorname{trace}_g \nabla d\psi$ is the tension field of ψ vanishing of which means ψ is a harmonic map ([4], [5]). Then ψ is f-harmonic if and only if the f-tension field vanishes, i.e., $\tau_f(\psi) = 0$ (for more details on the concept of f-harmonic maps see [7], [15], [17]).

Note that in [1] Mitsunori Ara has introduced another generalization of harmonic maps, p-harmonic maps and exponential harmonic maps, as a critical point of the F-energy functional:

$$E_F(\psi) = \frac{1}{2}\int_\Omega F(\frac{|d\psi|^2}{2})\,v_g, \tag{3}$$

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for every compact domain $\Omega \in M$, and has studied the stability of F-harmonic maps (see [2] and [3]). For not to be confused with the definition of f-harmonic, we observe that any F-harmonic map $\psi : (M,g) \to (N,h)$ without critical points, i.e., $|d\psi| \neq 0$, is an f-harmonic map with $f = F'(\frac{|d\psi|^2}{2})$, in this case, the F-tension field is given by:

$$\tau_F(\psi) = F'(\frac{|d\psi|^2}{2})\,\tau(\psi) + d\psi\Big(\operatorname{grad}\big(F'(\frac{|d\psi|^2}{2})\big)\Big).$$

The index form for the f-harmonic maps is defined by ([7], [17]):

$$I(v,w) = \int_M h(J_f^\psi(v), w)\, v_g \tag{4}$$

for all $v, w \in \Gamma(\psi^{-1}TN)$, where

$$\begin{aligned} J_f^\psi(v) =& -f \operatorname{trace} R^N(v, d\psi)d\psi - \operatorname{trace}(\nabla_f^\psi)^2 v \\ =& -f R^N(v, d\psi(e_i))d\psi(e_i) - \nabla^\psi_{e_i} f \nabla^\psi_{e_i} v + f \nabla^\psi_{\nabla^M_{e_i} e_i} v, \end{aligned} \tag{5}$$

R^N is the curvature tensor of (N,h) defined by

$$R^N(U,V)W = \nabla^N_U \nabla^N_V W - \nabla^N_V \nabla^N_U W - \nabla^N_{[U,V]} W \tag{6}$$

for all $U, V, W \in \Gamma(TN)$, ∇^N is the Levi-Civita connection of (N,h), ∇^ψ denote the pull-back connection on $\psi^{-1}TN$, and v_g is the volume form of (M,g) (see [4], [13]). If ψ is a f-harmonic map and for any vector field v along ψ, the index form satisfies $I(v,v) \geq 0$, then ψ is called a stable f-harmonic map.

Let M be a $2n$-dimensional differentiable manifold. An almost Hermitian structure on M is by definition a pair (J,g) of an almost complex structure J and a Riemannian metric g satisfying

$$J^2 X = -X, \quad g(JX, JY) = g(X,Y) \tag{7}$$

for all $X, Y \in \Gamma(TM)$. A manifold with such a structure (J,g) is called an almost Hermitian manifold. An almost Hermitian manifold (M,J,g) is Kählerian if and only if its almost complex structure J is parallel with respect to the Levi-Civita connection (see [13]), that is

$$\nabla_X JY = J\nabla_X Y, \quad X, Y \in \Gamma(TM). \tag{8}$$

Let X and Y be two unit vectors at a point in M. Then the holomorphic bisectional curvature is defined by

$$\operatorname{BHR}(X,Y) = g(R^M(X,JX)Y,JY). \tag{9}$$

2. Main results

Let $\psi : (M,J,g) \to (N,J',h)$ be a smooth map between two almost Hermitian manifolds. The map ψ is called $\pm$holomorphic (holomorphic or anti-holomorphic) if $d\psi \circ J = \pm J' \circ d\psi$ (see [13], [21]). Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$. Note that $\mathbb{S}^2$ admits a complex structure, and every Riemannian metric on an oriented 2-dimensional manifold is a Kähler metric with respect to the naturally induced complex structure (see [13]). If M is the sphere $\mathbb{S}^2$, then we prove the following:

Theorem 1. *Let (N, J', h) be a Kähler manifold with non-positive holomorphic bisectional curvature, and let f be a smooth positive function on the sphere $\mathbb{S}^2$ with $\operatorname{Hess} f \leq 0$. Then, any stable f-harmonic map $\psi : \mathbb{S}^2 \to N$ is a holomorphic or anti-holomorphic map.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^2$. For a vector field $v \in \Gamma(T\mathbb{S}^2)$ such that $(\nabla_{e_i} v)_{x_0} = 0$ $(i = 1, 2)$, we have

$$\nabla^\psi_{e_i} f \nabla^\psi_{e_i} J' d\psi(v) = J' \nabla^\psi_{\operatorname{grad} f} d\psi(v) + f J' \nabla^\psi_{e_i} \nabla^\psi_{e_i} d\psi(v).$$

Using the property $\nabla^\psi_X d\psi(Y) = \nabla^\psi_Y d\psi(X) + d\psi([X, Y])$ (see [4]), we have

$$\begin{aligned}\nabla^\psi_{e_i} f \nabla^\psi_{e_i} J' d\psi(v) = {} & J' \nabla^\psi_v d\psi(\operatorname{grad} f) + J' d\psi([\operatorname{grad} f, v]) \\ & + f J' \nabla^\psi_{e_i} \nabla^\psi_v d\psi(e_i) + f J' \nabla^\psi_{e_i} d\psi([e_i, v]).\end{aligned}$$

By the definition of the curvature tensor, we get the following result

$$\begin{aligned}\nabla^\psi_{e_i} f \nabla^\psi_{e_i} J' d\psi(v) = {} & J' \nabla^\psi_v d\psi(\operatorname{grad} f) - J' d\psi(\nabla_v \operatorname{grad} f) \\ & + f J' R^N(d\psi(e_i), d\psi(v)) d\psi(e_i) + f J' \nabla^\psi_v \nabla^\psi_{e_i} d\psi(e_i) \\ & + f J' d\psi([e_i, [e_i, v]]).\end{aligned}$$

By the property $R^N(d\psi(X), d\psi(Y))V = -R^N(d\psi(Y), d\psi(X))V$, we obtain

$$\begin{aligned}\nabla^\psi_{e_i} f \nabla^\psi_{e_i} J' d\psi(v) = {} & J' \nabla^\psi_v d\psi(\operatorname{grad} f) - J' d\psi(\nabla_v \operatorname{grad} f) \\ & - f J' R^N(d\psi(v), d\psi(e_i)) d\psi(e_i) + f J' \nabla^\psi_v \tau(\psi) \\ & + f J' \nabla^\psi_v d\psi(\nabla_{e_i} e_i) + f J' d\psi(\nabla_{e_i} \nabla_{e_i} v) \\ & - f J' d\psi(\nabla_{e_i} \nabla_v e_i) \\ = {} & J' \nabla^\psi_v d\psi(\operatorname{grad} f) - J' d\psi(\nabla_v \operatorname{grad} f) \\ & - f J' R^N(d\psi(v), d\psi(e_i)) d\psi(e_i) + J' \nabla^\psi_v f \tau(\psi) \\ & - v(f) J' \tau(\psi) + f J' d\psi(\nabla_v \nabla_{e_i} e_i) \\ & + f J' d\psi(\nabla_{e_i} \nabla_{e_i} v) - f J' d\psi(\nabla_{e_i} \nabla_v e_i).\end{aligned}$$

Using the f-harmonicity of ψ, we get

$$\begin{aligned}\nabla^\psi_{e_i} f \nabla^\psi_{e_i} J' d\psi(v) = {} & - J' d\psi(\nabla_v \operatorname{grad} f) - f J' R^N(d\psi(v), d\psi(e_i)) d\psi(e_i) \\ & - v(f) J' \tau(\psi) + f J' d\psi(\operatorname{Ricci} v) + f J' d\psi(\nabla^2 v),\end{aligned}$$

where $\operatorname{Ricci} v = R(v, e_i) e_i$ is the Ricci tensor of $\mathbb{S}^2$. We conclude that

$$\begin{aligned}-h(\operatorname{trace}(\nabla^\psi_f)^2 J' d\psi(v), J' d\psi(v)) = {} & h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ & + f h(R^N(d\psi(v), d\psi(e_i)) d\psi(e_i), d\psi(v)) \\ & + v(f) h(\tau(\psi), d\psi(v)) \\ & - f h(d\psi(\operatorname{Ricci} v), d\psi(v))\end{aligned}$$

$$- fh(d\psi(\operatorname{trace}\nabla^2 v), d\psi(v)). \tag{10}$$

By the definition of Jacobi operator (5), we have

$$\begin{aligned} h(J_f^\psi(J'd\psi(v)), J'd\psi(v)) = {} & h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ & + fh(R^N(d\psi(v), d\psi(e_i))d\psi(e_i), d\psi(v)) \\ & - fh(R^N(J'd\psi(v), d\psi(e_i))d\psi(e_i), J'd\psi(v)) \\ & + v(f)h(\tau(\psi), d\psi(v)) \\ & - fh(d\psi(\operatorname{Ricci} v), d\psi(v)) \\ & - fh(d\psi(\operatorname{trace}\nabla^2 v), d\psi(v)). \end{aligned} \tag{11}$$

Let $\lambda(x) = \langle \alpha, x\rangle_{\mathbb{R}^3}$ for all $x \in \mathbb{S}^2$, where $\alpha \in \mathbb{R}^3$ and let $v = \operatorname{grad}\lambda$. It is easily seen that v is the projection of a parallel vector field in $\mathbb{R}^3$ into the sphere. Note that

$$v = \langle \alpha, e_i\rangle e_i, \quad (\nabla_{e_i} v)_{x_0} = 0 \tag{12}$$

$$\nabla_X v = -\lambda X, \quad \operatorname{trace}\nabla^2 v = -v \tag{13}$$

for all $X \in \Gamma(T\mathbb{S}^2)$, where ∇ is the Levi-Civita connection on $\mathbb{S}^2$ with respect to the standard metric of the sphere (see [21]). From equations (11), (12), (13) with the formula $\operatorname{Ricci} v = v$ (see [5]), at point x_0

$$\begin{aligned} h(J_f^\psi(J'd\psi(v)), J'd\psi(v)) = {} & h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ & + fh(R^N(d\psi(v), d\psi(e_i))d\psi(e_i), d\psi(v)) \\ & - fh(R^N(J'd\psi(v), d\psi(e_i))d\psi(e_i), J'd\psi(v)) \\ & + v(f)h(\tau(\psi), d\psi(v)). \end{aligned}$$

Set $e_1 = e$ and $e_2 = Je$. By the f-harmonicity of ψ, consequently

$$\begin{aligned} & \operatorname{trace} h(J_f^\psi(J'd\psi(v)), J'd\psi(v)) \\ = {} & \operatorname{trace} h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) + fh(R^N(d\psi(Je), d\psi(e))d\psi(e), d\psi(Je)) \\ & + fh(R^N(d\psi(e), d\psi(Je))d\psi(Je), d\psi(e)) \\ & - fh(R^N(J'd\psi(e), d\psi(e))d\psi(e), J'd\psi(e)) \\ & - fh(R^N(J'd\psi(Je), d\psi(e))d\psi(e), J'd\psi(Je)) \\ & - fh(R^N(J'd\psi(e), d\psi(Je))d\psi(Je), J'd\psi(e)) \\ & - fh(R^N(J'd\psi(Je), d\psi(Je))d\psi(Je), J'd\psi(Je)) - fh(\tau(\psi), \tau(\psi)). \end{aligned}$$

Let $K(U, V) = h(R^N(U, V)U, V)$ where $U, V \in \Gamma(TN)$. Then we get

$$\begin{aligned} & \operatorname{trace} h(J_f^\psi(J'd\psi(v)), J'd\psi(v)) \\ = {} & \operatorname{trace} h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ & - 2fK(d\psi(e), d\psi(Je)) + 2fK(d\psi(e), J'd\psi(Je)) \end{aligned}$$

$$+fK(d\psi(e), J'd\psi(e)) + fK(d\psi(Je), J'd\psi(Je)) - fh(\tau(\psi), \tau(\psi)). \tag{14}$$

Let $\omega = d\psi(Je) - J'd\psi(e)$ and $\eta = d\psi(Je) + J'd\psi(e)$. Then

$$\begin{aligned} h(R^N(\omega, J\omega)\eta, J\eta) = & -2K(d\psi(e), d\psi(Je)) + 2K(d\psi(e), J'd\psi(Je)) \\ & + K(d\psi(e), J'd\psi(e)) + K(d\psi(Je), J'd\psi(Je)) \end{aligned} \tag{15}$$

(see [21]). Substituting the formula (14) into (13) gives

$$\begin{aligned} \operatorname{trace} h(J_f^{\psi}(J'd\psi(v)), J'd\psi(v)) = & \operatorname{trace} h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ & + fh(R^N(\omega, J\omega)\eta, J\eta) - fh(\tau(\psi), \tau(\psi)). \end{aligned}$$

From the last equation, $\operatorname{Hess} f \leq 0$ (see [4], [18]) and the stability condition, we obtain

$$\begin{aligned} \int_{\mathbb{S}^2} fh(R^N(\omega, J\omega)\eta, J\eta)\, dx = & \operatorname{trace} I(J'd\psi(v), J'd\psi(v)) \\ & - \operatorname{trace} \int_{\mathbb{S}^2} h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v))\, dx \\ & + \int_{\mathbb{S}^2} fh(\tau(\psi), \tau(\psi))\, dx \geq 0. \end{aligned} \tag{16}$$

Hence, by the condition of non-positive holomorphic bisectional curvature

$$\int_{\mathbb{S}^2} fh(R^N(\omega, J\omega)\eta, J\eta)\, dx < 0.$$

This contradicts (16). □

From Theorem 1, we have:

Corollary 1. *Let N be a Kähler manifold with non-positive holomorphic bisectional curvature. Then any stable harmonic map $\psi : \mathbb{S}^2 \to N$ is holomorphic or anti-holomorphic map.*

Note that Corollary 1 yields the following proposition.

Proposition 1 ([10])**.** *Let M be a closed Riemann surface and N a simply-connected Kähler manifold with $H_2(N) = \mathbb{Z}$ generated by a holomorphic map $P^1 \to N$. Then any map $\phi : M \to N$ with minimum energy is holomorphic or anti-holomorphic.*

Remark 1. Under the condition of Theorem 1, we conclude that, any stable f-harmonic map ψ from $\mathbb{S}^2$ to Kähler manifold (N, J', h) is harmonic ($\operatorname{grad} f \in \operatorname{Ker} d\psi$).

Theorem 2. *Any stable f-harmonic map ψ from a sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant, where f is a smooth positive function on $\mathbb{S}^n$ such that* $\operatorname{Hess} f \leq 0$.

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^n$. Set $\lambda(x) = \langle \alpha, x\rangle_{\mathbb{R}^{n+1}}$ for all $x \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ and let $v = \operatorname{grad}\lambda$. Note that $v = \langle \alpha, e_i\rangle e_i$, $(\nabla_{e_i} v)_{x_0} = 0$, $\nabla_X v = -\lambda X$ and $\operatorname{trace}\nabla^2 v = -v$ for all $X \in \Gamma(T\mathbb{S}^n)$, where ∇ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric of the sphere (see [21]). At point x_0, we have

$$\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i} d\psi(v) = \nabla^{\psi}_{\operatorname{grad} f} d\psi(v) + f\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i} d\psi(v). \tag{17}$$

The first term of (17) is given by

$$\begin{aligned}\nabla^{\psi}_{\operatorname{grad} f} d\psi(v) &= \nabla^{\psi}_{v} d\psi(\operatorname{grad} f) + d\psi([\operatorname{grad} f, v])\\ &= \nabla^{\psi}_{v} d\psi(\operatorname{grad} f) - d\psi(\nabla_v \operatorname{grad} f),\end{aligned} \tag{18}$$

and the second term of (17) is given by

$$\begin{aligned}f\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i} d\psi(v) &= f\nabla^{\psi}_{e_i}\nabla^{\psi}_{v} d\psi(e_i) + f\nabla^{\psi}_{e_i} d\psi([e_i, v])\\ &= fR^N(d\psi(e_i), d\psi(v))d\psi(e_i) + f\nabla^{\psi}_{v}\nabla^{\psi}_{e_i} d\psi(e_i)\\ &\quad + fd\psi([e_i,[e_i,v]]).\end{aligned} \tag{19}$$

From the definition of tension field (see [4]), we get

$$\begin{aligned}f\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i} d\psi(v) = &-fR^N(d\psi(v), d\psi(e_i))d\psi(e_i) + f\nabla^{\psi}_{v}\tau(\psi)\\ &+ f\nabla^{\psi}_{v} d\psi(\nabla_{e_i} e_i) + fd\psi(\nabla_{e_i}\nabla_{e_i} v) - fd\psi(\nabla_{e_i}\nabla_v e_i)\\ = &-fR^N(d\psi(v), d\psi(e_i))d\psi(e_i) + \nabla^{\psi}_{v} f\tau(\psi) - v(f)\tau(\psi)\\ &+ fd\psi(\nabla_v\nabla_{e_i} e_i) + fd\psi(\nabla_{e_i}\nabla_{e_i} v) - fd\psi(\nabla_{e_i}\nabla_v e_i).\end{aligned} \tag{20}$$

By equations (17), (18), (20) and the f-harmonicity of ψ, we have

$$\begin{aligned}\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i} d\psi(v) = &-d\psi(\nabla_v \operatorname{grad} f) - fR^N(d\psi(v), d\psi(e_i))d\psi(e_i) - v(f)\tau(\psi)\\ &+ fd\psi(\nabla_v\nabla_{e_i} e_i) + fd\psi(\nabla_{e_i}\nabla_{e_i} v) - fd\psi(\nabla_{e_i}\nabla_v e_i).\end{aligned} \tag{21}$$

From the definition of Jacobi operator (5) and equation (21), we have

$$\begin{aligned}J^{\psi}_f(d\psi(v)) = d\psi(\nabla_v \operatorname{grad} f) &+ v(f)\tau(\psi) - fd\psi(\nabla_v\nabla_{e_i} e_i)\\ &- fd\psi(\nabla_{e_i}\nabla_{e_i} v) + fd\psi(\nabla_{e_i}\nabla_v e_i),\end{aligned} \tag{22}$$

and by the definition of Ricci tensor (see [4]), we get the following result

$$\begin{aligned}J^{\psi}_f(d\psi(v)) = d\psi(\nabla_v \operatorname{grad} f) &+ v(f)\tau(\psi) - fd\psi(\operatorname{Ricci} v)\\ &- fd\psi(\operatorname{trace}\nabla^2 v).\end{aligned} \tag{23}$$

Since $\operatorname{trace}\nabla^2 v = -v$ and $\operatorname{Ricci} v = (n-1)v$ (see [4], [18], [21]), we conclude

$$\begin{aligned}h(J^{\psi}_f(d\psi(v)), d\psi(v)) = h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) &+ v(f)h(\tau(\psi), d\psi(v))\\ &- (n-2)fh(d\psi(v), d\psi(v)).\end{aligned} \tag{24}$$

By (24) and the f-harmonicity of ψ, it follows that

$$\operatorname{trace} h(J^{\psi}_f(d\psi(v)), d\psi(v)) = \operatorname{trace} h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v))$$

$$- fh(\tau(\psi),\tau(\psi)) - (n-2)f|d\psi|^2. \tag{25}$$

From the stable f-harmonic condition, Hess $f \le 0$, and equation (25), we get

$$-(n-2)\int_{\mathbb{S}^n} f|d\psi|^2 dx \ge 0. \tag{26}$$

Consequently, $|d\psi| = 0$, that is ψ is constant, because $n > 2$. □

Using Theorem 2, we obtain:

Corollary 2 ([20])**.** *Any stable harmonic map ψ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant.*

From Y. L. Xin [20, Theorem 3.1], we have:

Theorem 3. *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f-harmonic map $\psi : M \to \mathbb{S}^n$ must be constant, where f is a smooth positive function on M.*

Proof. With the same data of previous proof, we have

$$\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i}(v\circ\psi) = \nabla^{\psi}_{\operatorname{grad} f}(v\circ\psi) + f\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}(v\circ\psi). \tag{27}$$

The first term of (27) is given by

$$\nabla^{\psi}_{\operatorname{grad} f}(v\circ\psi) = -(\lambda\circ\psi)d\psi(\operatorname{grad} f), \tag{28}$$

and the second term of (27) is given by

$$\begin{aligned} f\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}(v\circ\psi) &= -f\nabla^{\psi}_{e_i}(\lambda\circ\psi)d\psi(e_i) \\ &= -fd\psi(\operatorname{grad}(\lambda\circ\psi)) - (\lambda\circ\psi)f\tau(\psi). \end{aligned} \tag{29}$$

By the definition of gradient operator, we get

$$-fd\psi(\operatorname{grad}(\lambda\circ\psi)) = -f\langle d\psi(e_i), v\circ\psi\rangle d\psi(e_i). \tag{30}$$

Substituting the formulas (28), (29), (30) into (27) gives

$$\begin{aligned} \nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i}(v\circ\psi) = &-(\lambda\circ\psi)d\psi(\operatorname{grad} f) - f\langle d\psi(e_i), v\circ\psi\rangle d\psi(e_i) \\ &-(\lambda\circ\psi)f\tau(\psi). \end{aligned} \tag{31}$$

From the f-harmonicity of ψ and equation (31), we have

$$\langle\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i}(v\circ\psi), v\circ\psi\rangle = -f\langle d\psi(e_i), v\circ\psi\rangle\langle d\psi(e_i), v\circ\psi\rangle. \tag{32}$$

Since the sphere $\mathbb{S}^n$ has constant curvature, we obtain

$$\begin{aligned} &\langle fR^{\mathbb{S}^n}(v\circ\psi, d\psi(e_i))d\psi(e_i), v\circ\psi\rangle \\ &= f|d\psi|^2\langle v\circ\psi, v\circ\psi\rangle - f\langle d\psi(e_i), v\circ\psi\rangle\langle d\psi(e_i), v\circ\psi\rangle. \end{aligned} \tag{33}$$

By the definition of Jacobi operator, (32) and (33), we get

$$\begin{aligned} \langle J^{\psi}_f(v\circ\psi), v\circ\psi\rangle = 2f\langle d\psi(e_i), v\circ\psi\rangle\langle d\psi(e_i), v\circ\psi\rangle \\ - f|d\psi|^2\langle v\circ\psi, v\circ\psi\rangle. \end{aligned} \tag{34}$$

Consequently

$$\operatorname{trace} \angle J_f^{\psi}(v \circ \psi), v \circ \psi\rangle = -(n-2) f |d\psi|^2. \tag{35}$$

Hence Theorem 3 follows from (35) and the stable f-harmonicity condition of ψ with $n > 2$. □

From Theorem 3, we deduce:

Corollary 3 ([11], [14], [16]). *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable harmonic map $\psi : M \to \mathbb{S}^n$ must be constant.*

The result of Corollary 3 is obtained previously by P. F. Leung [14], Y. Ohnita [16], R. Howard and S. W. Wei [11].

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