Commun. Korean Math. Soc. **30** (2015), No. 4, pp. 457–469
http://dx.doi.org/10.4134/CKMS.2015.30.4.457

PSEUDOHERMITIAN LEGENDRE SURFACES OF SASAKIAN SPACE FORMS

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Abstract. From the point of view of pseudohermitian geometry, we classify Legendre surfaces of Sasakian space forms with non-minimal $\hat{C}$-parallel mean curvature vector field for the Tanaka-Webster connection.

1. Introduction

A Legendre submanifold in contact manifolds is one of important subjects in contact geometry and the geometry of submanifolds. In Sasakian manifolds there exist no Legendre submanifolds with parallel mean curvature vector other than the minimal ones (cf. [15]). If the mean curvature vector field satisfies $DH||\xi$ for the characteristic vector field ξ, then it is said to be *C-parallel*, where D is the normal connection. In [1], C. Baikoussis and D. E. Blair treated C-parallel Legendre surfaces of Sasakian space forms (with respect to Levi-Civita connection). In [10] we studied C-parallel mean curvature vector fields along slant curves in Sasakian 3-manifolds.

On the other hand, for a given contact form we have two compatible structures: one is a Riemannian structure (or metric) and the other is a pseudohermitian structures (or (almost) CR-structure). In pseudohermitian geometry we use Tanaka-Webster connection as a canonical connection in stead of Levi-Civita connection. In [9], we defined pseudohermitian parallel mean curvature vector field in 3-dimensional contact Riemannian manifolds for the Tanaka-Webster connection in normal bundle. Thus, we found that a Legendre curve in a 3-dimensional Sasakian manifold satisfying pseudohermitian parallel mean curvature vector field is a pseudohermitian circle.

In this paper, we find that there exists no Legendre surface with pseudohermitian parallel mean curvature vector field other than the minimal ones in

Received January 30, 2015; Revised September 2, 2015.

2010 *Mathematics Subject Classification.* Primary 53C42, 53B25, 53C25.

Key words and phrases. Legendre surfaces, parallel mean curvature vector, Sasakian space forms, Tanaka-Webster connection.

This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A3006596).

5-dimensional Sasakian space forms M^5 for the Tanaka-Webster connection $\hat{\nabla}$. So, we study Legendre surfaces in 5-dimensional Sasakian space forms M^5 with $\hat{C}$-parallel mean curvature vector field H for the Tanaka-Webster connection $\hat{\nabla}$.

Let H be the mean curvature vector of a Legendre surface N isometrically immersed in M^5. The main purpose of the present paper is to prove the following result. A vector field V normal to N is said to be *$\hat{C}$-parallel* for the Tanaka-Webster connection $\hat{\nabla}$ if $\hat{D}_X V || \xi$ for any vector field X tangent to N. If N is a Legendre surface in the 5-dimensional Sasakian space form M^5 and mean curvature vector H is $\hat{C}$-parallel for the Tanaka-Webster connection $\hat{\nabla}$, then N is minimal, or N is locally the Riemnanian product of two curves. Moreover, if the Sasakian space form M^5 is the unit sphere $S^5 \subset E^6$, then we give 2-parameter family of the Legendre surface N in S^5 in Theorem 4.3.

2. Preliminaries

2.1. Contact Riemannian manifolds

A $(2n+1)$-dimensional smooth manifold M^{2n+1} is called a *contact manifold*, if it admits a global 1-form η such that $\eta \wedge (\mathrm{d}\eta)^n \neq 0$ everywhere on M^{2n+1}. This 1-form η is called the *contact form* on M^{2n+1}.

Given a contact form η, there exist a unique vector field ξ, the *characteristic vector field*, which satisfies $\eta(\xi) = 1$ and $\mathrm{d}\eta(\xi, X) = 0$ for any vector field X.

Moreover, there exist an associated Riemannian metric g and a $(1,1)$-type tensor field φ such that

$$(1) \qquad \eta(X) = g(X, \xi), \quad \mathrm{d}\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi,$$

where X and Y are vector fields on M^{2n+1}. From (1), it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M^{2n+1} equipped with the structure tensors (η, ξ, φ, g) satisfying (1) is said to be a *contact Riemannian manifold*. We denote it by $M = (M^{2n+1}, \eta; \xi, \varphi, g)$. Given a contact Riemannian manifold M, we define an endomorphism field h by $\mathrm{h} = \frac{1}{2} L_\xi \varphi$, where L_ξ denotes the Lie derivative in the characteristic direction ξ. The endomorphism field h is called the *structural operator* of $(M, \eta; \varphi, \xi, g)$.

Then we may observe that h is symmetric and satisfies

$$(2) \qquad \begin{aligned} &\mathrm{h}\xi = 0, \quad \mathrm{h}\varphi = -\varphi \mathrm{h}, \\ &\nabla_X \xi = -\varphi X - \varphi \mathrm{h} X, \end{aligned}$$

where ∇ is Levi-Civita connection of (M, g).

For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Riemannian manifold M is said to be *normal* or *Sasakian*. It is known that a contact metric manifold M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

A Sasakian manifold is also characterized by the condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for all vector fields X and Y on the manifold M.

We define the Riemannian curvature tensor R by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z on M.

Let $(M; \eta, \xi, \varphi, g)$ be a Sasakian manifold. Then M is called a space of constant holomorphic sectional curvature k if M satisfies

$$g(R(X, \varphi X)\varphi X, X) = k$$

for any unit vector field $X \perp \xi$. A complete and simply connected Sasakian space of constant holomorphic sectional curvature is called a *Sasakian space form*. Tanno ([12]) classified Sasakian space forms.

The curvature tensor of a Sasakian space form $M(k)$ is given by

$$\begin{aligned} R(X, Y)Z = {} & \frac{k+3}{4}(g(Y, Z)X - g(X, Z)Y) \\ & + \frac{k-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & - g(Y, Z)\eta(X)\xi + g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\}. \end{aligned} \tag{3}$$

Let N^m be a submanifold in a contact manifold M^{2n+1}. If η restricted to N^m vanishes, then N^m is called an integral submanifold. In particular, when $m = n$, it is called a *Legendre submanifold*.

2.2. Pseudohermitian structure and Tanaka-Webster connection

For a contact Riemannian manifold $M = (M^{2n+1}; \eta, \xi, \varphi, g)$, the tangent space T_pM of M at a point $p \in M$ can be decomposed into the direct sum $T_pM = D_p \oplus \{\xi\}_p$, with $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a $2n$-dimensional distribution orthogonal to ξ, which is called the *contact distribution*. We see that the restriction $J = \varphi|_D$ of φ to D defines an almost complex structure on D. Then the associated almost CR-structure of the contact Riemannian manifold M is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX \mid X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM. Then we see that each fiber $\mathcal{H}_p$ is of complex dimension n, $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$, and $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that the *associated* $CR-$ *structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY),$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) a *contact strongly pseudo-convex, pseudohermitian structure* on M. Now, we review the *Tanaka-Webster connection* ([11], [14]) on a contact strongly pseudo-convex CR-manifold $M = (M; \eta, L)$ with the associated contact Riemannian structure (η, ξ, φ, g). The Tanaka-Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (2), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + P(X, Y), \tag{4}$$

where

$$P(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{5}$$

We see that the Tanaka-Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY. \tag{6}$$

In particular, the K-contact manifold (5) and the above equation (6) are reduced to

$$\begin{aligned} &P(X, Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ &\hat{T}(X, Y) = 2g(X, \varphi Y)\xi. \end{aligned} \tag{7}$$

Furthermore, it was proved in ([13]) that:

Proposition 2.1. *The Tanaka-Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M^{2n+1}; \eta, \xi, \varphi, g)$ with the associated* (*integrable*) *CR-structure is the unique linear connection satisfying the following conditions*:

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$;
(iii-1) $\hat{T}(X, Y) = -\eta([X, Y])\xi$, X, $Y \in D$;
(iii-2) $\hat{T}(\xi, \varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in D$.

We define the pseudohermitian curvature tensor R of $\hat{\nabla}$ (or the Tanaka-Webster curvature tensor) on a contact strongly pseudo-convex CR manifold by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z \tag{8}$$

for all vector fields X, Y, Z in M. Actually, for Sasakian space forms $M^{2n+1}(k)$ the holomorphic sectional curvature for $\hat{k}$ is $\hat{k} = k + 3$ (see [4]).

Definition. If γ is a curve in a contact Riemannian manifold M which is parametrized by arc-length s, we say that γ is a Frenet curve of osculating order r when there exist orthonormal vector fields $E_1, E_2, \ldots, E_r$, along γ such that

$$\dot{\gamma} = E_1,\ \hat{\nabla}_{\dot{\gamma}} E_1 = \hat{\kappa}_1 E_2,\ \hat{\nabla}_{\dot{\gamma}} E_2 = -\hat{\kappa}_1 E_1 + \hat{\kappa}_2 E_3, \ldots,$$
$$\hat{\nabla}_{\dot{\gamma}} E_{r-1} = -\hat{\kappa}_{r-2} E_{r-2} + \hat{\kappa}_{r-1} E_r,\ \hat{\nabla}_{\dot{\gamma}} E_r = -\hat{\kappa}_{r-1} E_{r-1},$$

where $\hat{\kappa}_1, \hat{\kappa}_2, \ldots, \hat{\kappa}_{r-1}$ are positive c^∞ functions of s. $\hat{\kappa}_j$ is called the j-th pseudohermitian curvature of γ. A geodesic is a Frenet curve of osculating order 1 and a pseudohermitian circle is a Frenet curve of osculating order 2 with $\hat{\kappa}_1$ a constant. A pseudohermitian helix of order r is a Frenet curve of osculating order r, such that $\hat{\kappa}_1, \hat{\kappa}_2, \ldots, \hat{\kappa}_{r-1}$ are constants.

3. Legendre surfaces in Sasakian space forms

For a contact manifold M^{2n+1}, let $f : N^m \to M^{2n+1}$ be an isometric immersion. Then we have the basic formulas:

$$\hat{\nabla}^f_X Y = \hat{\nabla}^h_X Y + \hat{\sigma}(X,Y) \quad \text{and} \quad \hat{\nabla}^f_X V = -\hat{S}_V X + \hat{D}_X V, \tag{9}$$

where $X, Y \in TN^m$, $V \in T^{\perp}N^m$. $\hat{\sigma}$, $\hat{S}$ and $\hat{D}$ are the second fundamental form, the shape operator and the normal connection for $\hat{\nabla}$. The first formula is called the Gauss formula and the second formula is called the Weingarten formula for the Tanaka-Webster connection $\hat{\nabla}$. The pseudohermitian mean curvature vector field $\hat{H}$ is given by $\hat{H} = \frac{1}{m} tr\hat{\sigma}$. If $\hat{H} = 0$ at any point of N^m, then N^m is called pseudohermitian minimal.

From the equation (9) we can find the relation:

$$g(\hat{\sigma}(X,Y), V) = g(\hat{S}_V X, Y) = g(\hat{S}_V Y, X). \tag{10}$$

Let N^n be a Legendre submanifold of a Sasakian manifold M^{2n+1} and let $e_i (i = 1, \ldots, n)$ be an orthonormal frame field along N^n such that e_i's are tangent to N^n, $\varphi e_1 = e_{n+1}, \ldots, \varphi e_n = e_{2n}$, $\xi = e_{2n+1}$. By the equation (7), we can see that

$$P(X,Y) = 0 = \hat{T}(X,Y) \tag{11}$$

for $X, Y \in TN$, and $\hat{\sigma} = \sigma$. Using the equation (10) we get $\hat{S}_V X = S_V X$, for $V \in T^{\perp}N, X \in TN$. Moreover, we have

$$S_{\varphi Y} X = -\varphi\sigma(X,Y) = S_{\varphi X} Y, \quad S_\xi = 0. \tag{12}$$

Then by using a straightforward computation the equations of Gauss and Codazzi of Legrendre submanifolds for Tanaka-Webster connection are given:

$$g(\hat{R}^h(X,Y)Z,W) = g(\hat{R}(X,Y)Z,W) + g([S_{\varphi Z}, S_{\varphi W}]X,Y), \tag{13}$$

$$(\hat{\nabla}_X \sigma)(Y,Z) = (\hat{\nabla}_Y \sigma)(X,Z). \tag{14}$$

The sectional curvature $K(X,Y)$ of N^n determined by an orthonormal pair X, Y is given by

$$K(X,Y) = \frac{k+3}{4} + \Sigma_{\alpha=1}^{n}\{g(S_\alpha X, X)g(S_\alpha Y, Y) - g(S_\alpha X, Y)^2\}. \tag{15}$$

From the equation (11) we have:

Lemma 3.1. *Let N^n be a Legendre submanifold of a Sasakian manifold M^{2n+1}. The pseudohermitian mean curvature vector $\hat{H}$ is equal to the mean curvature vector H (i.e., $\hat{H} = H$). Moreover, N^n is pseudohermitian minimal if and only if it is minimal.*

First of all, for a Legendre surface N in a Sasakian space form M^5 we consider parallel mean curvature vector fields. Let X_1, X_2 be a local orthonomal basis of vector fields on N such that $H = \frac{trS_{\varphi X_1}}{2}\varphi X_1$. Using the equations (10) and (12), if

$$S_{\varphi X_1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad \text{then} \quad S_{\varphi X_2} = \begin{bmatrix} b & c \\ c & -b \end{bmatrix}, \tag{16}$$

where a, b, c are functions on M. Let $X_i (i = 1, \ldots, 5)$ be an orthonormal frame field along N^2 such that X_1, X_2 are tangent to N^2, $\varphi X_1 = X_3$, $\varphi X_2 = X_4$, $\xi = X_5$, then we denote by $\{\omega_i\}$, $i = 1, \ldots, 5$, the dual frame field of the frame $\{X_i\}$. Thus we get $\hat{\nabla} X_i = \Sigma_{i=1}^{5}\omega_i^j X_j$. Hence we have

$$\hat{\nabla}_{X_i} H = X_i\left(\frac{trS_{\varphi X_1}}{2}\right)\varphi X_1 + \frac{trS_{\varphi X_1}}{2}(\omega_1^2(X_i)\varphi X_2 + \varphi\sigma(X_i, X_1)). \tag{17}$$

From the above equation, we find:

Proposition 3.2. *Let N be a Legendre surface in a Sasakian space form M^5. There does not exist non-minimal pseudohermitian parallel mean curvature vector field for the Tanaka-Webster connection $\hat{\nabla}$.*

Now, we consider $\hat{C}$-parallel mean curvature vector field as following.

Definition. A vector field V normal to N is said to be *$\hat{C}$-parallel* for Tanaka-Webster connection $\hat{\nabla}$ if $\hat{D}_X V || \xi$ for the normal connection $\hat{D}$ of $\hat{\nabla}$ and any vector field X tangent to M.

From the above Definition if we compute $\hat{C}$-parallel mean curvature vector field (i.e., $\hat{D}_{X_i} H || \xi$, $i = 1, 2$) on Legendre surfaces in Sasakian space forms M^5 then we have $X_i(trS_{\varphi X_1}) = 0$ and $(trS_{\varphi X_1})\omega_1^2(X_i) = 0$. If $trS_{\varphi X_1} = 0$, then N is minimal. If $trS_{\varphi X_1} = constant \neq 0$, then $\omega_1^2 = 0$ and N is flat. Hence we have:

Lemma 3.3. *Let N be a Legendre surface in a Sasakian space form M^5. If the mean curvature vector field H of N is $\hat{C}$-parallel for Tanaka-Webster connection $\hat{\nabla}$, then N is minimal or N is flat.*

Corollary 3.4. *Let N be a Legendre surface in a Sasakian space form M^5. If N is flat, then the shape operators $S_{\varphi X_1}$ and $S_{\varphi X_2}$ are parallel.*

Proof. From the Codazzi equation (14),

$$a_2 = b_1,\ b_2 = c_1,\ c_2 = -b_1, \tag{18}$$

where $X_i f = f_i$ for any function f on N. So $a + c$ is a constant. Since N is flat, using the equation (15) we get $\frac{k+3}{4} + ac - 2b^2 - c^2 = 0$. Differentiating the above equation for X_i, $i = 1, 2$, then we have

$$ca_i - 4bb_i + (a - 2c)c_i = 0,\ i = 1, 2. \tag{19}$$

From (18) we get

$$(3c - a)c_1 - 4bc_2 = 0,\ (3c - a)c_2 + 4bc_1 = 0.$$

The determinant of this system is $D = (3c - a)^2 + 16b^2$. When $D \neq 0$, c is a constant. Since $a + c$ is a constant, a is a constant. Moreover, from (18), b is a constant. If $D = 0$, we have $b = 0$ and $a = 3c = \mathit{constant}$. Therefore we can see that the shape operators $S_{\varphi X_1}$ and $S_{\varphi X_2}$ are parallel. □

Remark 3.5. If $\hat{D}_X V = 0$ for the normal connection $\hat{D}$ and any vector field X tangent to M, then a vector field V normal to N is said to be *$\hat{D}$-parallel* for the Tanaka-Webster connection $\hat{\nabla}$. From the equation (17), we have $g(\hat{\nabla}_{X_i} H, \xi) = 0$. Therefore N has $\hat{C}$-parallel mean curvature vector field if and only if N has $\hat{D}$-parallel mean curvature vector field.

In [1], C. Baikoussis and D. E. Blair treated C-parallel Legendre surfaces of Sasakian space forms (with respect to Levi-Civita connection). In fact, we can see the following:

Corollary 3.6. *Let N be a Legendre surface in a Sasakian space form M^5. N has C-parallel mean curvature vector field for the Levi-Civita connection ∇ if and only if N has $\hat{C}$-parallel mean curvature vector field for the Tanaka-Webster connection $\hat{\nabla}$.*

From Lemma 3.3, if N is not minimal, then N is flat. In this case, after rotating the basis $\{X_1, X_2\}$ through a constant angle, using the equations (10) and (12), we may assume that the weingarten maps are

$$S_1 = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \quad \text{and} \quad S_2 = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}, \tag{20}$$

where $S_i = S_{\varphi \overline{X_i}}$, $i = 1, 2$, and $\overline{X_1} = \cos\theta X_1 + \sin\theta X_2$, $\overline{X_2} = -\sin\theta X_1 + \cos\theta X_2$. Since $\hat{\nabla}^h_{X_i} X_j = \nabla^h_{X_i} X_j = 0$ we have $\hat{\nabla}^h_{\overline{X_i}} \overline{X_j} = \nabla^h_{\overline{X_i}} \overline{X_j} = 0$, and from $\hat{k} = k + 3$, we have

$$\frac{\hat{k}}{4} + ac - c^2 = 0. \tag{21}$$

We relabel $\overline{X_i}$ and $\varphi\overline{X_i}$ by X_i and $\varphi X_i, i=1,2$, respectively. From Proposition 2.1 and (9),

$$\begin{aligned}
&\hat\nabla_{X_1}X_1 = a\varphi X_1, \quad \hat\nabla_{X_1}X_2 = \hat\nabla_{X_2}X_1 = c\varphi X_2,\\
&\hat\nabla_{X_2}X_2 = c\varphi X_1 + d\varphi X_2, \quad \hat\nabla_{X_1}\varphi X_1 = -aX_1, \quad \hat\nabla_{X_1}\varphi X_2 = -cX_2,\\
&\hat\nabla_{X_2}\varphi X_1 = -cX_2,\ \hat\nabla_{X_2}\varphi X_2 = -cX_1 - dX_2, \quad \hat\nabla_{X_1}\xi = \hat\nabla_{X_2}\xi = 0.
\end{aligned} \tag{22}$$

Let $X_1 = E_1$. From (22) we have

$$\hat\nabla_{E_1}E_1 = \nabla_{E_1}E_1 = a\varphi E_1 = \kappa_1 E_2,$$

where $E_2 = \varepsilon\varphi E_1, \hat\kappa_1 = \kappa_1 = \varepsilon a$, and $\varepsilon = \pm 1$ according to $a > 0$ or $a < 0$.

$$\hat\nabla_{E_1}E_2 = -aX_1 = -\kappa_1 E_1.$$

Thus $\hat\kappa_2 = 0$ and X_1-curve is a pseudohermitian circle of M^5. If $a = 0$, the X_1-curve is a geodesic of M^5.

Now we put $X_2 = E_1$. From (22) we have

$$\hat\nabla_{E_1}E_1 = \nabla_{E_1}E_1 = c\varphi X_1 + d\varphi X_2 = \hat\kappa_1 E_2,$$

where $E_2 = \frac{c\varphi X_1 + d\varphi X_2}{\sqrt{c^2+d^2}}$, $\hat\kappa_1 = \sqrt{c^2+d^2}$. If $c^2+d^2 = 0$ (i.e., $c = d = 0$), then the X_2-curve is a geodesic. If $c^2+d^2 \neq 0$, then

$$\hat\nabla_{E_1}E_2 = \frac{1}{\sqrt{c^2+d^2}}\{-cdX_1 - (c^2+d^2)X_2\} = -\hat\kappa_1 E_1 + \hat\kappa_2 E_3,$$

where $E_3 = \varepsilon X_1, \hat\kappa_2 = \varepsilon\frac{cd}{\sqrt{c^2+d^2}}$ and $\varepsilon = \mp 1$ according to $cd > 0$ or $cd < 0$. If $cd = 0$ and $c^2+d^2 \neq 0$, then the X_2-curve is a pseudohermitian circle in M^5. If $cd \neq 0$, then $\hat\nabla_{E_1}E_3 = -\hat\kappa_2 E_2 + \hat\kappa_3 E_4$, where $E_4 = \frac{c\varphi X_2 - d\varphi X_1}{\sqrt{c^2+d^2}}$ and $\hat\kappa_3 = \frac{c^2}{\sqrt{c^2+d^2}}$.

$$\hat\nabla_{E_1}E_4 = -\frac{c^2}{\sqrt{c^2+d^2}}X_1 = -\hat\kappa_3 E_3,$$

since $c \neq 0$ and $\hat\kappa_4 = 0$, the X_2-curve is a pseudohermitian helix of order 4. Hence we have:

Theorem 3.7. *Let N be a Legendre surface of the Sasakian space form M^5. If the mean curvature vector H of N is $\hat C$-parallel for Tanaka-Webster connection $\hat\nabla$ then N is minimal, or N is a locally product of two curves as follows*:

(i) *two geodesics, or*

(ii) *two pseudohermitian circles or a pseudohermitian circle and a geodesic, or*

(iii) *a pseudohermitian helix of order* 4 *and a pseudohermitian circle or a geodesic.*

In [1], they studied Legendre surfaces of Sasakian space form M^5 satisfying C-parallel mean curvature vector field (with respect to the Levi-Civita connection). In fact, for a Legendre surface N in a Sasakian space form M^5, N has C-parallel mean curvature vector field for the Levi-Civita connection ∇ if and only if N has $\hat{C}$-parallel mean curvature vector field for the Tanaka-Webster connection $\hat{\nabla}$. In Theorem 3.7, we construct Legendre surfaces N in Sasakian space forms satisfying $\hat{C}$-parallel mean curvature vector field from the point of view of pseudohermitian geometry. We can see that it is very different from [1].

4. Legendre surfaces in S^5

In this section, we particularly consider the case that the Sasakian space form M^5 is the unit sphere S^5. Let

$$x : N \to S^5 \subset E^6 \tag{23}$$

be an isometric immersion of a Legendre surface N in S^5.

From Lemma 3.3, if N is not minimal, then N is flat. In this case the second fundamental form σ of N in E^6 is given as follows:

$$\begin{aligned} &\sigma(X_1, X_1) = a\varphi X_1 - x, \\ &\sigma(X_1, X_2) = c\varphi X_2, \\ &\sigma(X_2, X_2) = c\varphi X_1 + d\varphi X_2 - x, \end{aligned} \tag{24}$$

where x is the position vector of N in E^6. From (22) and (24) we get

$$\begin{aligned} &\hat{\nabla}'_{X_1} X_1 = a\varphi X_1 - x, \hat{\nabla}'_{X_1} \varphi X_1 = -aX_1, \\ &\hat{\nabla}'_{X_2} X_1 = c\varphi X_2, \hat{\nabla}'_{X_i} x = X_i, \\ &\hat{\nabla}'_{X_2} X_2 = c\varphi X_1 + d\varphi X_2 - x, \hat{\nabla}'_{X_2} \varphi X_1 = -cX_2, \\ &\hat{\nabla}'_{X_2} \varphi X_2 = -cX_1 - dX_2, \end{aligned} \tag{25}$$

where $\hat{\nabla}'$ is the connection of E^6.

Let $X_1 = E_1$. From (25) we have

$$\hat{\nabla}'_{E_1} E_1 = a\varphi X_1 - x = \hat{\kappa}_1 E_2,$$

where $E_2 = \frac{a\varphi X_1 - x}{\sqrt{a^2+1}}$ and $\hat{\kappa}_1 = \sqrt{a^2+1}$.

$$\hat{\nabla}'_{E_1} E_2 = -\sqrt{a^2+1} X_1 = -\hat{\kappa}_1 E_1.$$

Thus $\hat{\kappa}_2 = 0$ and X_1-curve is a pseudohermitian circle in E^6.

Now we put $X_2 = E_1$. From (25) we have

$$\hat{\nabla}'_{E_1} E_1 = c\varphi X_1 + d\varphi X_2 - x = \hat{\kappa}_1 E_2,$$

where $E_2 = \frac{c\varphi X_1 + d\varphi X_2 - x}{\sqrt{c^2+d^2+1}}$, $\hat{\kappa}_1 = \sqrt{c^2+d^2+1}$.

$$\hat{\nabla}'_{E_1} E_2 = -\sqrt{c^2+d^2+1} X_2 - \frac{cd}{\sqrt{c^2+d^2+1}} X_1 = -\hat{\kappa}_1 E_1 + \hat{\kappa}_2 E_3,$$

where $E_3 = \varepsilon X_1$, and $\hat{\kappa}_2 = -\varepsilon \frac{cd}{\sqrt{c^2+d^2+1}}$, $\varepsilon = \mp 1$ according to $cd > 0$ or $cd < 0$.

If $cd = 0$, then the X_2-curve is a pseudohermitian circle in E^6. If $cd \neq 0$, then the X_2-curve is a pseudohermitian helix of order 4 in E^6. Hence we have:

Proposition 4.1. *Let N be a Legendre surface of S^5 in E^6 with $\hat{C}$-parallel mean curvature vector H for $\hat{\nabla}$. Then N is minimal in S^5, or locally the product of two curves in E^6;*

(i) *two pseudohermitian circles or*

(ii) *a pseudohermitian circle and a pseudohermitian helix of order* 4.

On Legendre surface N we may choose local coordinates such that the immersion (23) is $x = x(u, v)$ with $x_u = X_1$ and $x_v = X_2$. Then from (21) we have $1 + ac - c^2 = 0$ and from (25), we find

$$\begin{aligned} &\text{(i) } x_{uuu} + (a^2+1)x_u = 0,\\ &\text{(ii) } x_{uuv} + c^2 x_v = 0,\\ &\text{(iii) } x_{uvv} + c^2 x_u + cd x_v = 0. \end{aligned} \tag{26}$$

We will find the general solution of the system (26). First of all, we solve the following ordinary differential equation.

Lemma 4.2. *The general solution of the ordinary differential equation*

$$f''' + (a^2+1)f' = 0$$

is

$$f(t) = c_1 \cos\sqrt{a^2+1}t + c_2 \sin\sqrt{a^2+1}t + c_3,$$

where c_1, c_2, c_3 and a are constants.

Theorem 4.3. *Let $x : N \to S^5 \subset E^6$ be an immersion of a Legendre surface N into S^5, with $\hat{C}$-parallel mean curvature vector field H for $\hat{\nabla}$. If N is not minimal in S^5, then N lies in $E^4 \subset E^6$ and the position vector $x = x(u, v)$ of N in E^6 is given by*

$$\begin{aligned} x = &\frac{1}{\sqrt{\lambda_1(\lambda_1+\lambda_2)}}\cos(\lambda_1 v + u)e_1 + \frac{1}{\sqrt{\lambda_2(\lambda_1+\lambda_2)}}\cos(\lambda_2 v - u)e_2\\ &+ \frac{1}{\sqrt{\lambda_1(\lambda_1+\lambda_2)}}\sin(\lambda_1 v + u)e_3 + \frac{1}{\sqrt{\lambda_2(\lambda_1+\lambda_2)}}\sin(\lambda_2 v - u)e_4, \end{aligned} \tag{27}$$

where $\lambda_1 = \frac{1}{2}(\sqrt{4+d^2} + d)$, $\lambda_2 = \frac{1}{2}(\sqrt{4+d^2} - d)$, $d =$ constant and $\{e_i\}$, $i = 1, \ldots, 4$, is an orthonormal basis of E^4 in E^6.

Proof. From the assumption the position vector $x = x(u, v)$ of N in E^6 satisfying the system (26) is given by

$$x = A_1(v)\cos\sqrt{a^2+1}u + A_2(v)\sin\sqrt{a^2+1}u + A_3, \tag{28}$$

where A_i, $i = 1, 2, 3$, are E^6-valued smooth functions of the variable v. From the Second equation of (26) and (28), we get $a = 0$, $c^2 = 1$ and

$$x = A_1(v)\cos u + A_2(v)\sin u + A_3, \tag{29}$$

where A_3 is a constant.

We may assume that $c = 1$ without loss of generality. From the third equation of (26) and (29), we have

$$A_1''(v) + A_1(v) - dA_2'(v) = 0,\quad A_2''(v) + A_2(v) + dA_1'(v) = 0. \tag{30}$$

If $d = 0$, the general solution of these equations are

$$A_1(v) = B_1 \cos v + B_2 \sin v,\quad A_2(v) = C_1 \cos v + C_2 \sin v,$$

where B_i, C_i, $i = 1, 2$, are constant vectors in E^6.

If $d \neq 0$, from (30) we have

$$A_i^{(iv)}(v) + (d^2 + 2)A_i''(v) + A_i(v) = 0,\ i = 1, 2. \tag{31}$$

We can find the following general solution of these differential equations:

$$\begin{aligned} A_1(v) &= B_1 \cos \lambda_1 v + B_2 \sin \lambda_1 v + B_3 \cos \lambda_2 v + B_4 \sin \lambda_2 v,\\ A_2(v) &= C_1 \cos \lambda_1 v + C_2 \sin \lambda_1 v + C_3 \cos \lambda_2 v + C_4 \sin \lambda_2 v, \end{aligned} \tag{32}$$

where $\lambda_1 = \frac{1}{2}(\sqrt{4 + d^2} + d)$, $\lambda_2 = \frac{1}{2}(\sqrt{4 + d^2} - d)$ and B_i, C_i, $i = 1, \dots, 4$, are constant vectors in E^6.

Substituting (32) into (30) and using that $\cos \lambda_1 v$, $\sin \lambda_1 v$, $\cos \lambda_2 v$, $\sin \lambda_2 v$ are linearly independent, we get

$$B_1 = -C_2,\ B_2 = C_1,\ B_3 = C_4,\ B_4 = -C_3.$$

Substituting (32) into (28), the position vector x of N is given by (27), where $e_1, \dots, e_4$ are the constant vectors in $E^4 \subset E^6$ and

$$\begin{aligned} e_1 &= \sqrt{\lambda_1(\lambda_1 + \lambda_2)}B_1,\ e_2 = \sqrt{\lambda_2(\lambda_1 + \lambda_2)}B_3,\\ e_3 &= \sqrt{\lambda_1(\lambda_1 + \lambda_2)}B_2,\ e_4 = \sqrt{\lambda_2(\lambda_1 + \lambda_2)}B_4. \end{aligned}$$

Therefore we get at the point $x(0.0)$

$$\begin{aligned} x &= \frac{1}{\sqrt{\lambda_1(\lambda_1 + \lambda_2)}}e_1 + \frac{1}{\sqrt{\lambda_2(\lambda_1 + \lambda_2)}}e_2,\\ x_u &= \frac{1}{\sqrt{\lambda_1(\lambda_1 + \lambda_2)}}e_3 - \frac{1}{\sqrt{\lambda_2(\lambda_1 + \lambda_2)}}e_4,\\ x_v &= \sqrt{\frac{\lambda_1}{\lambda_1 + \lambda_2}}e_3 + \sqrt{\frac{\lambda_2}{\lambda_1 + \lambda_2}}e_4,\\ x_{uu} &= -\frac{1}{\sqrt{\lambda_1(\lambda_1 + \lambda_2)}}e_1 - \frac{1}{\sqrt{\lambda_2(\lambda_1 + \lambda_2)}}e_2,\\ x_{vv} &= -\lambda_1\sqrt{\frac{\lambda_1}{\lambda_1 + \lambda_2}}e_1 - \lambda_2\sqrt{\frac{\lambda_2}{\lambda_1 + \lambda_2}}e_2,\\ x_{uv} &= -\frac{\lambda_1}{\sqrt{\lambda_1(\lambda_1 + \lambda_2)}}e_1 + \frac{\lambda_2}{\sqrt{\lambda_2(\lambda_1 + \lambda_2)}}e_2, \end{aligned} \tag{33}$$

$$x_{uvv} = -\lambda_1\sqrt{\frac{\lambda_1}{\lambda_1+\lambda_2}}e_3 + \lambda_2\sqrt{\frac{\lambda_2}{\lambda_1+\lambda_2}}e_4.$$

On the other hand, form (25) we get

$$g(x,x) = g(x_u,x_u) = g(x_v,x_v) = 1,\ g(x,x_u) = g(x,x_v) = g(x_u,x_v) = 0. \tag{34}$$

Moreover,

$$\begin{aligned} &g(x,x_{uu}) = g(x,x_{vv}) = g(x_u,x_{uvv}) = -g(x_{uv},x_{uv}) = -1,\\ &g(x_{uvv},x_v) = -g(x_{vv},x_{uv}) = -d,\ g(x_{vv},x_{vv}) = 2+d^2,\\ &g(x_{uvv},x_{uvv}) = 1+d^2, \end{aligned} \tag{35}$$

and all others are zero.

From (33) and (34), we have $A_3 = 0$ and finally we obtain the position vector (27). Also, combining (33) with the above equations (34) and (35), we can see that $g(e_i,e_j) = \delta_{ij}$. □

Remark 4.4. In [1], they found that if Legendre surfaces in Sasakian space forms with C-parallel mean curvature vector field (for Levi-Civita connection ∇) are not minimal, then N lies fully in E^6.

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