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ENTIRE SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATION AND FERMAT TYPE q-DIFFERENCE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the differential-difference equation

$$(f(z+c)-f(z))^2+P(z)^2(f^{(k)}(z))^2=Q(z),$$

where $P(z)$, $Q(z)$ are nonzero polynomials. In addition, we also investigate Fermat type q-difference differential equations

$$f(qz)^2+(f^{(k)}(z))^2=1 \quad \text{and} \quad (f(qz)-f(z))^2+(f^{(k)}(z))^2=1.$$

If the above equations admit a transcendental entire solution of finite order, then we can obtain the precise expression of the solution.

1. Introduction and results

In this paper, we assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [6, 7]. In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$. And we denote by $S(r,f)$ any quantify satisfying $S(r,f)=o(T(r,f))$, as $r\to\infty$, outside of a possible exceptional set of finite logarithmic measure.

Recently, a number of papers (including [2, 3, 8, 14]) have focused on meromorphic solutions of complex difference equations and differential-difference equations. Fermat type functional equations were investigated by Gross [4, 5], Montel [11] and Yang [13]. Yang [13] investigated the Fermat type functional equation

$$a(z)f(z)^n+b(z)g(z)^m=1, \tag{1.1}$$

where $a(z)$, $b(z)$ are small functions with respect to $f(z)$ and obtained the following result.

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Theorem A (See [13]). *Let m, n be positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no nonconstant entire solutions $f(z)$ and $g(z)$ that satisfy* (1.1).

Theorem A implies that there are nonconstant entire solutions with the assumption of $m > 2$, $n > 2$ in (1.1). However, when $m = n = 2$ and $g(z)$ has a specific relationship with $f(z)$ in (1.1), the problem is worth to be considered.

Tang and Liao [12] investigated the entire solutions of the following equation

$$f(z)^2 + P(z)^2(f^{(k)}(z))^2 = Q(z), \tag{1.2}$$

where $P(z)$, $Q(z)$ are nonzero polynomials.

Liu and Yang [10] considered the finite order entire solutions of the differential-difference equation

$$f(z+c)^2 + (f^{(k)}(z))^2 = 1. \tag{1.3}$$

In this paper, we consider the entire solutions of the following differential-difference equation

$$(f(z+c) - f(z))^2 + P(z)^2(f^{(k)}(z))^2 = Q(z), \tag{1.4}$$

where $P(z)$, $Q(z)$ are nonzero polynomials, and obtain the following results.

Theorem 1.1. *Let $P(z)$, $Q(z)$ be nonzero polynomials. If the differential-difference equation* (1.4) *admits a transcendental entire solution of finite order, then $P(z) \equiv A$ (constant), $Q(z) \equiv q_1 q_2$ (constant) and k must be an odd. Thus, $f(z) = -\frac{q_1 e^{az+b} + q_2 e^{-(az+b)}}{4} + c_1$, where $a, b, c, c_1 \in \mathbb{C}$ are constants such that $a^k = \frac{2i}{A}$, $c = \frac{(2m+1)\pi i}{a}$, $m \in \mathbb{Z}$.*

Corollary 1.1. *If $P(z)$, $Q(z)$ are nonconstant polynomials, then there does not exist transcendental entire solution of finite order of the differential-difference equation* (1.4).

Corollary 1.2. *If $P(z)$, $Q(z)$ are nonzero polynomials and k is an even, then there does not exist transcendental entire solution of finite order of the differential-difference equation* (1.4).

Corollary 1.3. *Let $P(z)$, $Q(z)$ be nonzero polynomials. Then the differential-difference equation*

$$(f(z+c) - f(z))^2 + zP(z)^2(f^{(k)}(z))^2 = Q(z) \tag{1.5}$$

has no transcendental entire solution of finite order.

Example 1.1. If $P(z) \equiv 2i$, $c = \pi i$, $Q(z) \equiv 1$ and k is an odd, then $(f(z + \pi i) - f(z))^2 + (2i)^2(f^{(k)}(z))^2 = 1$ has a transcendental entire solution $f(z) = -\frac{e^{z+b} + e^{-(z+b)}}{4} = -\frac{1}{2}\cos(iz + ib)$, where b is a constant.

Barnett et al. [1] have stated a q-difference analogue of the logarithmic derivative lemma. However, they mainly investigated the zero-order meromorphic solutions of q-difference equations. In what follows, we will consider the entire solutions of finite order, not limited to zero-order in Fermat type q-difference differential equations.

Liu and Cao [9] have considered Fermat type q-difference differential equation

$$f'(z)^2 + (f(qz))^2 = 1, \tag{1.6}$$

and obtained the following result.

Theorem B (See [9])**.** *The transcendental entire solutions with finite order of equation* (1.6) *satisfy* $f(z) = \sin(z + B)$ *when* $q = 1$*, and* $f(z) = \sin(z + k\pi)$ *or* $f(z) = -\sin(z + k\pi + \frac{\pi}{2})$ *when* $q = -1$*. There are no transcendental entire solutions with finite order when* $q \neq \pm 1$*.*

In this paper, we consider an improvement of Theorem B and obtain the following results, which are also viewed as q-difference analogue of equation (1.2).

Theorem 1.2. *The transcendental entire solutions with finite order of q-difference differential equation*

$$f(qz)^2 + (f^{(k)}(z))^2 = 1, \tag{1.7}$$

must satisfy the following cases,

(i) $f(z) = \cos(iaz + ib)$, $a^k = -i$*, k is an odd and b is a constant when* $q = 1$*;*

(ii) $f(z) = \pm \sin(iaz)$, $a^k = i$*, k is an odd or* $f(z) = \pm \cos(iaz)$, $a^k = -i$*, k is an odd or* $f(z) = \pm \sin(iaz - \frac{\pi}{4})$, $a^k = 1$*, k is an even or* $f(z) = \pm \sin(iaz + \frac{\pi}{4})$, $a^k = -1$*, k is an even when* $q = -1$*. There are no transcendental entire solutions with finite order when* $q \neq \pm 1$*.*

Theorem 1.3. *The transcendental entire solutions with finite order of q-difference differential equation*

$$(f(qz) - f(z))^2 + (f^{(k)}(z))^2 = 1, \tag{1.8}$$

must satisfy $f(z) = \pm\frac{1}{2}\sin(iaz) + c$*, c is a constant,* $a^k = 2i$*, k is an odd and* $q = -1$*. There are no transcendental entire solutions with finite order when* $q \neq -1$ *or k is an even.*

2. Lemmas for the proof of theorems

Lemma 2.1 (See [15, Theorem 1.62])**.** *Let* $f_j(z)$ $(j = 1, 2, \ldots, n)$ *be meromorphic functions,* $f_k(z)$ $(k = 1, 2, \ldots, n-1)$ *be nonconstant functions, satisfying* $\sum_{j=1}^{n} f_j(z) \equiv 1$ *where* $n \geq 3$*. If* $f_n(z) \not\equiv 0$ *and*

$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j(z)}\right) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j(z)) < (\lambda + o(1))T(r, f_k(z)) \ (r \in I), \tag{2.1}$$

where $\lambda < 1$ *and* $k = 1, 2, \ldots, n-1$*, then* $f_n(z) \equiv 1$*.*

Lemma 2.2. *Let $Q(z)$ be nonzero polynomial and satisfy*

$$Q(z+c)-Q(z)\equiv c_1Q'(z)+c_2Q''(z)+\cdots+c_kQ^{(k)}(z), \tag{2.2}$$

where $c_1,c_2,\dots,c_k\in\mathbb{C}\setminus\{0\}$, $k\in\mathbb{N}$ and $c\neq c_1$. Then $Q(z)\equiv A$ (constant).

Proof. Suppose that $Q(z)\not\equiv A$, then $\deg Q(z)\geq 1$.

Denote

$$Q(z)=a_qz^q+a_{q-1}z^{q-1}+\cdots+a_0\ (a_q\neq 0).$$

Then

$$Q(z+c)=a_q(z+c)^q+a_{q-1}(z+c)^{q-1}+\cdots+a_0,$$
$$Q'(z)=qa_qz^{q-1}+(q-1)a_{q-1}z^{q-2}+\cdots+a_1,$$
$$Q(z+c)-Q(z)=qa_qcz^{q-1}+(a_qC_q^2c^2+a_{q-1}C_{q-1}^1c)z^{q-2}+\cdots.$$

Comparing the coefficients of z^{q-1} on both sides of (2.4), we see that $qa_qc=qa_qc_1$, that is $qa_q(c-c_1)=0$. From $\deg Q(z)=q\geq 1$, $c\neq c_1$ and $a_q\neq 0$, we can get a contradiction. $\square$

3. Proof of theorems

Proof of Theorem 1.1. Assume that $f(z)$ is a transcendental entire solution of finite order of (1.4), then

$$(f(z+c)-f(z)+iP(z)f^{(k)}(z))(f(z+c)-f(z)-iP(z)f^{(k)}(z))=Q(z). \tag{3.1}$$

Thus, both $f(z+c)-f(z)+iP(z)f^{(k)}(z)$ and $f(z+c)-f(z)-iP(z)f^{(k)}(z)$ have finitely many zeros. Combining (3.1) with the Hadamard factorization theorem [15, Theorem 2.5], we assume that

$$f(z+c)-f(z)+iP(z)f^{(k)}(z)=Q_1(z)e^{h(z)}$$

and

$$f(z+c)-f(z)-iP(z)f^{(k)}(z)=Q_2(z)e^{-h(z)},$$

where $h(z)$ is a nonconstant polynomial, otherwise $f(z)$ is a polynomial, and $Q(z)=Q_1(z)Q_2(z)$, where $Q_1(z)$, $Q_2(z)$ are nonzero polynomials. Thus, we have

$$f(z+c)-f(z)=\frac{Q_1(z)e^{h(z)}+Q_2(z)e^{-h(z)}}{2} \tag{3.2}$$

and

$$f^{(k)}(z)=\frac{Q_1(z)e^{h(z)}-Q_2(z)e^{-h(z)}}{2iP(z)}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} f^{(k)}(z+c)&=\frac{Q_1(z+c)e^{h(z+c)}-Q_2(z+c)e^{-h(z+c)}}{2iP(z+c)}\\ &=\frac{(iP(z)p_1(z)+Q_1(z))e^{h(z)}+(iP(z)p_2(z)-Q_2(z))e^{-h(z)}}{2iP(z)}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} p_1(z) = {} & Q_1(z)(h'(z)^k + M_{k-1}(h'(z), h''(z), \ldots, h^{(k)}(z))) \\ & + Q_1'(z) M_{k-1}(h'(z), h''(z), \ldots, h^{(k-1)}(z)) + \cdots \\ & + Q_1^{(k-1)}(z) M_1(h'(z)) + Q_1^{(k)}(z), \\ p_2(z) = {} & Q_2(z)((-1)^k h'(z)^k + N_{k-1}(h'(z), h''(z), \ldots, h^{(k)}(z))) \\ & + (-1)^{k-1} Q_2'(z) N_{k-1}(h'(z), h''(z), \ldots, h^{(k-1)}(z)) + \cdots \\ & + (-1) Q_2^{(k-1)}(z) N_1(h'(z)) + Q_2^{(k)}(z), \end{aligned}$$

and M_j, N_j $(j = 1, 2, \ldots, k-1)$ are differential polynomials of $h'(z)$ with degree j respectively.

If $iP(z)p_1(z) + Q_1(z) \equiv 0$ and $iP(z)p_2(z) - Q_2(z) \not\equiv 0$, then (3.4) can be rewritten as

$$\begin{aligned} & \frac{P(z)Q_1(z+c)}{P(z+c)(iP(z)p_2(z) - Q_2(z))} e^{h(z)+h(z+c)} \\ \equiv {} & \frac{P(z)Q_2(z+c)}{P(z+c)(iP(z)p_2(z) - Q_2(z))} e^{h(z)-h(z+c)} + 1, \end{aligned} \tag{3.5}$$

compare the order of growth on both sides of (3.5), we see that (3.5) is a contradiction.

If $iP(z)p_2(z) - Q_2(z) \equiv 0$ and $iP(z)p_1(z) + Q_1(z) \not\equiv 0$, then (3.4) can be rewritten as

$$\begin{aligned} & \frac{P(z)Q_2(z+c)}{P(z+c)(iP(z)p_1(z) + Q_1(z))} e^{-h(z+c)-h(z)} \\ \equiv {} & \frac{P(z)Q_1(z+c)}{P(z+c)(iP(z)p_1(z) + Q_1(z))} e^{h(z+c)-h(z)} - 1, \end{aligned} \tag{3.6}$$

compare the order of growth on both sides of (3.6), we see that (3.6) is a contradiction.

Thus, we have $iP(z)p_1(z) + Q_1(z) \not\equiv 0$ and $iP(z)p_2(z) - Q_2(z) \not\equiv 0$. Then (3.4) can be rewritten as

$$\begin{aligned} & \frac{P(z)Q_1(z+c)}{P(z+c)(iP(z)p_2(z) - Q_2(z))} e^{h(z)+h(z+c)} \\ & - \frac{P(z)Q_2(z+c)}{P(z+c)(iP(z)p_2(z) - Q_2(z))} e^{h(z)-h(z+c)} \\ & - \frac{iP(z)p_1(z) + Q_1(z)}{iP(z)p_2(z) - Q_2(z)} e^{2h(z)} \equiv 1. \end{aligned} \tag{3.7}$$

Since $h(z)$ is a nonconstant polynomial, we know that both $e^{h(z)+h(z+c)}$ and $e^{2h(z)}$ are not constants. From Lemma 2.1, we see that

$$P(z+c)(iP(z)p_2(z) - Q_2(z)) e^{h(z+c)-h(z)} \equiv -P(z)Q_2(z+c),$$

thus $h(z) = az + b$, where a is a nonzero constant, b is a constant.

Then, we have

$$p_1(z) = Q_1(z)a^k + kQ_1'(z)a^{k-1} + \cdots + Q_1^{(k)}(z), \tag{3.8}$$

$$p_2(z) = (-1)^k Q_2(z)a^k + (-1)^{k-1}kQ_2'(z)a^{k-1} + \cdots + Q_2^{(k)}(z). \tag{3.9}$$

By Lemma 2.1 and (3.7), we obtain

$$\begin{aligned} e^{ac} = e^{h(z+c)-h(z)} &\equiv \frac{P(z+c)(iP(z)p_1(z) + Q_1(z))}{P(z)Q_1(z+c)} \\ &\equiv -\frac{P(z)Q_2(z+c)}{P(z+c)(iP(z)p_2(z) - Q_2(z))}, \end{aligned}$$

that is,

$$e^{ac}P(z)Q_1(z+c) \equiv P(z+c)(iP(z)p_1(z) + Q_1(z)), \tag{3.10}$$

$$-e^{-ac}P(z)Q_2(z+c) \equiv P(z+c)(iP(z)p_2(z) - Q_2(z)). \tag{3.11}$$

From (3.8)-(3.11), we see that $P(z) \equiv A(\neq 0)$ and

$$e^{ac}Q_1(z+c) \equiv (iAa^k + 1)Q_1(z) + iA(kQ_1'(z)a^{k-1} + \cdots + Q_1^{(k)}(z)), \tag{3.12}$$

$$\begin{aligned} -e^{-ac}Q_2(z+c) \equiv\ & ((-1)^k iAa^k - 1)Q_2(z) \\ & + iA((-1)^{k-1}kQ_2'(z)a^{k-1} + \cdots + Q_2^{(k)}(z)). \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we get

$$e^{ac} = iAa^k + 1, \ -e^{-ac} = (-1)^k iAa^k - 1,$$

then k must be an odd, $a^k = \frac{2i}{A}$ and $c = \frac{(2m+1)\pi i}{a}$, $m \in \mathbb{Z}$.

Therefore, (3.12) and (3.13) can be rewritten as

$$Q_1(z+c) - Q_1(z) \equiv -iA(ka^{k-1}Q_1'(z) + \cdots + Q_1^{(k)}(z)), \tag{3.14}$$

$$Q_2(z+c) - Q_2(z) \equiv iA(-ka^{k-1}Q_2'(z) + \cdots + Q_2^{(k)}(z)). \tag{3.15}$$

Since $c \neq -iAka^{k-1}$, by Lemma 2.2, we see that $Q_1(z) \equiv q_1$(constant), $Q_2(z) \equiv q_2$(constant).

From (1.4) and (3.3), we have

$$f(z) = \frac{q_1 e^{az+b} + q_2 e^{-(az+b)}}{2iAa^k} + c_1 = -\frac{q_1 e^{az+b} + q_2 e^{-(az+b)}}{4} + c_1,$$

where $a(\neq 0), b, c_1 \in \mathbb{C}$ are constants. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. As in the beginning of the proof of Theorem 1.1, we have

$$f(qz) = \frac{e^{h(z)} + e^{-h(z)}}{2} \tag{3.16}$$

and

$$f^{(k)}(z)=\frac{e^{h(z)}-e^{-h(z)}}{2i}, \tag{3.17}$$

where $h(z)$ is a nonconstant polynomial. Combining (3.16) with (3.17), we obtain

$$f^{(k)}(qz)=\frac{e^{h(qz)}-e^{-h(qz)}}{2i}=\frac{h_1(z)e^{h(z)}+h_2(z)e^{-h(z)}}{2q^k}, \tag{3.18}$$

where

$$\begin{aligned}h_1(z)&=h'^k(z)+M_{k-1}(h^{(k)}(z),\dots,h'(z)),\\ h_2(z)&=(-1)^k h'^k(z)+N_{k-1}(h^{(k)}(z),\dots,h'(z))\end{aligned}$$

and M_{k-1},N_{k-1} are polynomials of $h'(z),\dots,h^{(k)}(z)$ with degree $k-1$. By (3.18), we obtain

$$-\frac{ih_1(z)}{q^k}e^{h(qz)+h(z)}-\frac{ih_2(z)}{q^k}e^{h(qz)-h(z)}+e^{2h(qz)}\equiv 1. \tag{3.19}$$

From Lemma 2.1, if $h(qz)+h(z)=A$, then we have $-\frac{ih_1(z)}{q^k}e^A\equiv 1$ and $\frac{ih_2(z)}{q^k}e^{-A}\equiv 1$, which implies $h(z)=az+b$, where a is a nonzero constant, b is a constant. Thus, from $h(z)=az+b$, $-\frac{ia^k}{q^k}e^A=1$ and $\frac{i(-1)^k a^k}{q^k}e^{-A}=1$, we have $q=-1,\ (-1)^k a^{2k}=1$. If k is an odd and $a^k=i$, then $e^A=e^{-A}=-1$, $b=\frac{1}{2}A=\frac{(2m+1)\pi i}{2},\ m\in\mathbb{Z}$, from (1.7) and (3.17), we have

$$\begin{aligned}f(z)&=\frac{e^{az+b}+e^{-(az+b)}}{2ia^k}=-\frac{e^{az+b}+e^{-(az+b)}}{2}\\ &=-\cos(iaz+ib)=\pm\sin(iaz).\end{aligned}$$

If k is an odd and $a^k=-i$, then $e^A=e^{-A}=1$, $b=\frac{1}{2}A=m\pi i,\ m\in\mathbb{Z}$, from (1.7) and (3.17), we have

$$\begin{aligned}f(z)&=\frac{e^{az+b}+e^{-(az+b)}}{2ia^k}=\frac{e^{az+b}+e^{-(az+b)}}{2}\\ &=\cos(iaz+ib)=\pm\cos(iaz).\end{aligned}$$

If k is an even and $a^k=1$, then $e^A=i,\ e^{-A}=-i$, $b=\frac{1}{2}A=(m+\frac{1}{4})\pi i,\ m\in\mathbb{Z}$, from (1.7) and (3.17), we have

$$\begin{aligned}f(z)&=\frac{e^{az+b}-e^{-(az+b)}}{2ia^k}=\frac{e^{az+b}-e^{-(az+b)}}{2i}\\ &=-\sin(iaz+ib)=\pm\sin\left(iaz-\frac{\pi}{4}\right).\end{aligned}$$

If k is an even and $a^k=-1$, then $e^A=-i,\ e^{-A}=i$, $b=\frac{1}{2}A=(m-\frac{1}{4})\pi i,\ m\in\mathbb{Z}$, from (1.7) and (3.17), we have

$$f(z)=\frac{e^{az+b}-e^{-(az+b)}}{2ia^k}=-\frac{e^{az+b}-e^{-(az+b)}}{2i}$$

$$= \sin(iaz + ib) = \pm \sin\left(iaz + \frac{\pi}{4}\right).$$

From Lemma 2.1, if $h(qz) - h(z) = B$, then we have $\frac{ih_1(z)}{q^k}e^{-B} \equiv 1$ and $-\frac{ih_2(z)}{q^k}e^{B} \equiv 1$, which implies $h(z) = az + b$, where a is a nonzero constant, b is a constant. Thus, from $h(z) = az + b$, $\frac{ia^k}{q^k}e^{-B} = 1$ and $-\frac{i(-1)^k a^k}{q^k}e^{B} = 1$, we have $q = 1$, $a^k = -i$ and k must be an odd. From (1.7) and (3.17), we have

$$f(z) = \frac{e^{az+b} + e^{-(az+b)}}{2ia^k} = \frac{e^{az+b} + e^{-(az+b)}}{2} = \cos(iaz + ib).$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. As in the beginning of the proof of Theorem 1.1, we have

$$f(qz) - f(z) = \frac{e^{h(z)} + e^{-h(z)}}{2} \tag{3.20}$$

and

$$f^{(k)}(z) = \frac{e^{h(z)} - e^{-h(z)}}{2i}, \tag{3.21}$$

where $h(z)$ is a nonconstant polynomial. Combining (3.20) with (3.21), we obtain

$$-\frac{ih_1(z) + 1}{q^k}e^{h(qz)+h(z)} - \frac{ih_2(z) - 1}{q^k}e^{h(qz)-h(z)} + e^{2h(qz)} \equiv 1, \tag{3.22}$$

where

$$h_1(z) = h'^k(z) + M_{k-1}(h^{(k)}(z), \ldots, h'(z)),$$
$$h_2(z) = (-1)^k h'^k(z) + N_{k-1}(h^{(k)}(z), \ldots, h'(z))$$

and M_{k-1}, N_{k-1} are polynomials of $h'(z), \ldots, h^{(k)}(z)$ with degree $k - 1$.

If $ih_1(z) + 1 \equiv 0$ and $ih_2(z) - 1 \not\equiv 0$, from (3.22), we have

$$-\frac{ih_2(z) - 1}{q^k}e^{h(qz)-h(z)} + e^{2h(qz)} \equiv 1. \tag{3.23}$$

Clearly, we find that $2h(qz)$ and $h(qz) - h(z)$ are not constants synchronously. Thus (3.23) is impossible.

If $ih_2(z) - 1 \equiv 0$ and $ih_1(z) + 1 \not\equiv 0$, from (3.22), we have

$$-\frac{ih_1(z) + 1}{q^k}e^{h(qz)+h(z)} + e^{2h(qz)} \equiv 1, \tag{3.24}$$

and $2h(qz)$ and $h(qz) + h(z)$ are not constants simultaneously, then (3.24) is also impossible.

Thus, we have $ih_1(z) + 1 \not\equiv 0$ and $ih_2(z) - 1 \not\equiv 0$.

From (3.22) and Lemma 2.1, if $h(qz)+h(z) = A$, then we have $-\frac{ih_1(z)+1}{q^k}e^{A} \equiv 1$ and $\frac{ih_2(z)-1}{q^k}e^{-A} \equiv 1$, which implies $h(z) = az + b$, where a is a nonzero constant, b is a constant. Thus, from $h(z) = az + b$, $-\frac{ia^k+1}{q^k}e^A = 1$ and $\frac{i(-1)^k a^k-1}{q^k}e^{-A} = 1$, we have $q = -1$, $a^k = 2i$, $e^A = e^{-A} = -1$, $b = \frac{1}{2}A = \frac{(2m+1)\pi i}{2}$, $m \in \mathbb{Z}$ and k must be an odd. By (1.8) and (3.21), we have

$$\begin{aligned} f(z) &= \frac{e^{az+b} + e^{-(az+b)}}{2ia^k} + c = -\frac{e^{az+b} + e^{-(az+b)}}{4} + c \\ &= -\frac{1}{2}\cos(iaz + ib) + c = \pm\frac{1}{2}\sin(iaz) + c, \end{aligned}$$

where $a(\neq 0), b, c \in \mathbb{C}$ are constants.

From (3.22) and Lemma 2.1, if $h(qz)-h(z) = B$, then we have $-\frac{ih_2(z)-1}{q^k}e^B \equiv 1$ and $\frac{ih_1(z)+1}{q^k}e^{-B} \equiv 1$, which implies $h(z) = az + b$, where a is a nonzero constant, b is a constant. Thus, from $h(z) = az + b$, $-\frac{i(-1)^k a^k-1}{q^k}e^B = 1$ and $\frac{ia^k+1}{q^k}e^{-B} = 1$, we have $a = 0$, which is impossible. This completes the proof of Theorem 1.3. □

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