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CERTAIN SUMMATION FORMULAS FOR HUMBERT'S DOUBLE HYPERGEOMETRIC SERIES Ψ_2 AND Φ_2

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Abstract. The main objective of this paper is to establish certain explicit expressions for the Humbert functions

$$\Phi_2(a,\, a+i\,;\, c\,;\, x,\, -x) \quad \text{and} \quad \Psi_2(a\,;\, c,\, c+i\,;\, x,\, -x)$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$. Several new and known summation formulas for Φ_2 and Ψ_2 are considered as special cases of our main identities.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers and nonpositive integers, respectively. We start with recalling two Humbert's functions defined as follows (see, *e.g.*, [1, 2, 3, 6, 10]):

$$\Psi_2(a\,;\, b,\, c\,;\, x,\, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{(b)_n\,(c)_k}\, \frac{x^n\, y^k}{n!\, k!} \tag{1}$$

and

$$\Phi_2(a,\, b;\; c\,;\, x,\, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_n\,(b)_k}{(c)_{n+k}}\, \frac{x^n\, y^k}{n!\, k!}. \tag{2}$$

It is noted that the double series (1) and (2) converge absolutely at any x, $y \in \mathbb{C}$.

Only a few relations between these Humbert's functions and hypergeometric or generalized hypergeometric series are available in the literature. We recall here the following ones (see, *e.g.*, [2, 3, 10]):

$$\Psi_2(\gamma\,;\, \gamma,\, \gamma\,;\, x,\, y) = e^{x+y}\, {}_0F_1\left[\begin{matrix} -\,; \\ \gamma\,; \end{matrix}\; x\,y\right]; \tag{3}$$

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$$\Psi_2(\alpha\,;\,\gamma,\,\gamma\,;\,x,\,-x) = {}_2F_3\left[\begin{matrix} \tfrac{1}{2}\alpha,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\,; \\ \gamma,\ \tfrac{1}{2}\gamma,\ \tfrac{1}{2}\gamma+\tfrac{1}{2}\,; \end{matrix}\ -x^2\right]; \tag{4}$$

$$\Psi_2(a\,;\,b,\,c\,;\,x,\,x) = {}_3F_3\left[\begin{matrix} a,\ \tfrac{1}{2}(c+b-1),\ \tfrac{1}{2}(c+b)\,; \\ b,\ c,\ b+c-1\,; \end{matrix}\ 4x\right]; \tag{5}$$

$$\Phi_2(\alpha,\,\gamma-\alpha;\ \gamma\,;\,x,\,y) = e^y\,{}_1F_1\left[\begin{matrix} \alpha\,; \\ \gamma\,; \end{matrix}\ x-y\right]; \tag{6}$$

$$\Phi_2(\beta,\,\beta';\ \gamma\,;\,x,\,x) = {}_1F_1\left[\begin{matrix} \beta+\beta'\,; \\ \gamma\,; \end{matrix}\ x\right]; \tag{7}$$

$$\Phi_2(a\,;\,a,\,c\,;\,x,\,-x) = {}_1F_2\left[\begin{matrix} a\,; \\ \tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2}\,; \end{matrix}\ \tfrac{x^2}{4}\right]. \tag{8}$$

Recently, Manako [5] obtained the following identity: For $b,\,c \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $|x| \neq 0$,

$$\Psi_2(a\,;\,b,\,c\,;\,x,\,y) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\,{}_2F_1\left[\begin{matrix} -k,\ -k-b+1\,; \\ c\,; \end{matrix}\ \frac{y}{x}\right]\frac{x^k}{k!}, \tag{9}$$

whose several special cases are also considered.

Very recently, Rathie [8] presented the following identity:

$$\Phi_2(a,\,b;\ c\,;\,x,\,y) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m}\,{}_2F_1\left[\begin{matrix} -m,\,b\,; \\ 1-a-m\,; \end{matrix}\ \frac{y}{x}\right]\frac{x^m}{m!}. \tag{10}$$

Here we consider some special cases of (9) and (10). If we take $y = x$ in (9) and use Gauss's summation theorem (see, *e.g.*, [7, p. 49] and [9, p. 64]):

$$\begin{gathered} {}_2F_1\left[\begin{matrix} a,\,b\,; \\ c\,; \end{matrix}\ 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \\ \left(\Re(c-a-b) > 0;\ c \in \mathbb{C}\setminus\mathbb{Z}_0^-\right), \end{gathered} \tag{11}$$

after some simplification, we get the result (5).

Setting $c = b$, $y = -x$ in (9) and making use of the following Kummer's summation theorem (see, *e.g.*, [7, p. 68]):

$$\begin{gathered} {}_2F_1\left[\begin{matrix} a,\,b\,; \\ 1+a-b\,; \end{matrix}\ -1\right] = \frac{\Gamma\left(1+\tfrac{1}{2}a\right)\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma\left(1+\tfrac{1}{2}a-b\right)} \\ \left(\Re(b) < 1;\ 1+a-b \in \mathbb{C}\setminus\mathbb{Z}_0^-\right), \end{gathered} \tag{12}$$

we obtain the result (4).

If we take $y = x$ in (10) and use Gauss's summation theorem (11), we get the result (7).

Setting $b = a$, $y = -x$ in (10) and using Kummer's summation theorem (12), we obtain the result (8).

Here, in this paper, we aim at finding certain explicit expressions for the Humbert functions

$$\Phi_2(a,\, a+i\,;\, c\,;\, x,\, -x) \quad \text{and} \quad \Psi_2(a\,;\, c,\, c+i\,;\, x,\, -x)$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$. The results are derived, with the help of (9) and (10), by using the following generalizations of the Kummer's summation theorem given earlier by Lavoie *et al.* [4]:

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 1+a-b+i\,; \end{matrix}\; -1\right] &= \frac{\Gamma\big(\frac12\big)\,\Gamma(1-b)\,\Gamma(1+a-b+i)}{2^a\,\Gamma\big(1-b+\frac12(i+|i|)\big)} \\ &\times \left\{ \frac{\mathcal{A}_i(a,b)}{\Gamma\big(\frac12 a-b+\frac12 i+1\big)\,\Gamma\big(\frac12 a+\frac12 i+\frac12-\big[\frac{i+1}{2}\big]\big)} + \frac{\mathcal{B}_i(a,b)}{\Gamma\big(\frac12 a-b+\frac12 i+\frac12\big)\,\Gamma\big(\frac12 a+\frac12 i-\big[\frac{i}{2}\big]\big)} \right\} \end{aligned} \tag{13}$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$. Here $[x]$ is the greatest integer less than or equal to x and its absolute value is denoted by $|x|$. The coefficients $\mathcal{A}_i(a, b)$ and $\mathcal{B}_i(a, b)$ are obtained from the following table.

Several interesting special cases of our main identities including known results (4) and (8) are also considered.

2. Main summation formulas for Ψ_2 and Φ_2

Here we establish two generalized formulas for Ψ_2 and Φ_2 asserted by the following two theorems.

Theorem 1. *The following formula for Ψ_2 holds true: For $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$,*

$$\begin{aligned} \Psi_2(a\,;\, c,\, c+i\,;\, x,\, -x) &= \frac{\sqrt{\pi}\,\Gamma(c)\,\Gamma(c+i)}{\Gamma\big(c+\frac12(i+|i|)\big)} \sum_{k=0}^{\infty} \frac{(a)_k\,(2x)^k}{\big(c+\frac12(i+|i|)\big)_k\; k!} \\ &\times \left\{ \frac{\mathcal{A}_i}{\Gamma\big(\frac12 k+c+\frac12 i\big)\,\Gamma\big(-\frac12 k+\frac12 i+\frac12-\big[\frac{i+1}{2}\big]\big)} + \frac{\mathcal{B}_i}{\Gamma\big(\frac12 k+c-\frac12+\frac12 i\big)\,\Gamma\big(-\frac12 k+\frac12 i-\big[\frac{i}{2}\big]\big)} \right\} \end{aligned} \tag{14}$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$. Here the coefficients $\mathcal{A}_i := \mathcal{A}_i(-k, -k-c+1)$ and $\mathcal{B}_i := \mathcal{B}_i(-k, -k-c+1)$ in the table.

Theorem 2. *The following formula for Φ_2 holds true: For $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$,*

$$\begin{aligned} \Phi_2(a,\, a+i;\; c\,;\, x,\, -x) &= \frac{\sqrt{\pi}\,\Gamma(1-a)\,\Gamma(1-a-i)}{\Gamma\big(1-a+\frac12(|i|-i)\big)} \sum_{m=0}^{\infty} \frac{(-2x)^m}{(c)_m\; m!} \\ &\times \left\{ \frac{\mathcal{C}_i}{\Gamma\big(1-\frac12 m-a-\frac12 i\big)\,\Gamma\big(-\frac12 m+\frac12 i+\frac12-\big[\frac{i+1}{2}\big]\big)} + \frac{\mathcal{D}_i}{\Gamma\big(\frac12-\frac12 m-a+\frac12 i\big)\,\Gamma\big(-\frac12 m+\frac12 i-\big[\frac{i}{2}\big]\big)} \right\} \end{aligned} \tag{15}$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$. Here the coefficients $\mathcal{C}_i := \mathcal{A}_i(-m, a+i)$ and $\mathcal{D}_i := \mathcal{B}_i(-m, a+i)$ in the table.

TABLE 1. Table for $\mathcal{A}_i(a,b)$ and $\mathcal{B}_i(a,b)$

i	$\mathcal{A}_i(a,b)$	$\mathcal{B}_i(a,b)$
5	$-4(6+a-b)^2+2b(6+a-b)$ $+b^2+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)$ $-b^2-34(6+a-b)-b+62$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)$ $-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)$ $-b^2+16(a-b-4)-b+12$

Proof. We will prove only our first main formula (14). If we set $y=-x$ and replace b and c by c and $c+i$ ($i=0,\,\pm1,\,\pm2,\,\ldots,\,\pm5$) in (9), respectively, we get

$$\Psi_2(a\,;\,c,\,c+i\,;\,x,\,-x)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{x^k}{k!}\,{}_2F_1\left[\begin{matrix}-k,\,-k-c+1\,;\\ c+i\,;\end{matrix}\;-1\right]. \tag{16}$$

Now it is easy to see that the ${}_2F_1$ appearing on the right-hand side of (16) can be evaluated with the help of the generalizations of Kummer's summation theorem (13). Then, after some simplification, we arrive at the right-hand side of (14). This completes the proof of (14).

Making use of (10) and using the same argument as above, one can establish our second main formula (15). □

3. Special cases

We begin by observing the following special cases of (14) and (15): The expression in the braces in (14):

$$\frac{\mathcal{A}_i}{\Gamma\left(\frac{1}{2}k+c+\frac{1}{2}i\right)\,\Gamma\left(-\frac{1}{2}k+\frac{1}{2}i+\frac{1}{2}-\left[\frac{i+1}{2}\right]\right)}$$

is seen to be zero when k is a nonnegative even integer and $i=\pm 1$. Similarly the expression in the braces in (14):

$$\frac{\mathcal{B}_i}{\Gamma\left(\frac{1}{2}k+c-\frac{1}{2}+\frac{1}{2}i\right)\,\Gamma\left(-\frac{1}{2}k+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}$$

is zero when k is a nonnegative integer and $i=\pm 1$. Taking into account these facts, we can easily get some special cases of (14) and (15).

Here we consider to record the following special cases of our main formulas (14) and (15).

Special cases of (14): Setting $i=0$ in (14) yields immediately the known result (4). Taking $i=\pm 1,\ \pm 2$ in (14), we get the following interesting identities:

$$\begin{aligned}\Psi_2(a\,;\,c,\,c+1\,;\,x,\,-x) &= {}_2F_3\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,;\\ c,\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{2}c+1\,;\end{matrix}\ -x^2\right]\\ &\quad+\frac{ax}{c(c+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}a+1\,;\\ c+1,\ \frac{1}{2}c+1,\ \frac{1}{2}c+\frac{3}{2}\,;\end{matrix}\ -x^2\right];\end{aligned}\tag{17}$$

$$\begin{aligned}\Psi_2(a\,;\,c,\,c-1\,;\,x,\,-x) &= {}_2F_3\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,;\\ c-1,\ \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2}\,;\end{matrix}\ -x^2\right]\\ &\quad-\frac{ax}{c(c-1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}a+1\,;\\ c,\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{2}c+1\,;\end{matrix}\ -x^2\right];\end{aligned}\tag{18}$$

$$\begin{aligned}\Psi_2(a\,;\,c,\,c+2\,;\,x,\,-x) &= {}_2F_3\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,;\\ c+1,\ \frac{1}{2}c+1,\ \frac{1}{2}c+\frac{3}{2}\,;\end{matrix}\ -x^2\right]\\ &\quad+\frac{2ax}{c(c+2)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}a+1\,;\\ c+1,\ \frac{1}{2}c+\frac{3}{2},\ \frac{1}{2}c+2\,;\end{matrix}\ -x^2\right];\end{aligned}\tag{19}$$

$$\begin{aligned}\Psi_2(a\,;\,c,\,c-2\,;\,x,\,-x) &= {}_2F_3\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,;\\ c-1,\ \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2}\,;\end{matrix}\ -x^2\right]\\ &\quad-\frac{2ax}{c(c-2)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}a+1\,;\\ c-1,\ \frac{1}{2}c+\frac{1}{2},\ \frac{1}{2}c+1\,;\end{matrix}\ -x^2\right].\end{aligned}\tag{20}$$

It is remarked in passing that the formulas (17) to (20) are expressed in terms of ${}_2F_3$ as in the (4). Similar expressions of $\Psi_2(a\,;\,c,\,c+i\,;\,x,\,-x)$ can be obtained for $i = \pm 3,\,\pm 4,\,\pm 5$.

Special cases of (15): Setting $i = 0$ in (15) yields at once the known identity (8). Taking $i = \pm 1,\,\pm 2$ in (15), we obtain the following interesting identities:

(21)
$$\Phi_2(a,\,a+1\,;\,c\,;\,x,\,-x) = {}_1F_2\left[\begin{matrix} a+1\,; \\ \frac{1}{2}c,\,\frac{1}{2}c+\frac{1}{2}\,; \end{matrix}\ \frac{x^2}{4}\right] - \frac{x}{c}\,{}_1F_2\left[\begin{matrix} a+1\,; \\ \frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}c+1\,; \end{matrix}\ \frac{x^2}{4}\right];$$

(22)
$$\Phi_2(a,\,a-1\,;\,c\,;\,x,\,-x) = {}_1F_2\left[\begin{matrix} a\,; \\ \frac{1}{2}c,\,\frac{1}{2}c+\frac{1}{2}\,; \end{matrix}\ \frac{x^2}{4}\right] + \frac{x}{c}\,{}_1F_2\left[\begin{matrix} a\,; \\ \frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}c+1\,; \end{matrix}\ \frac{x^2}{4}\right];$$

(23)
$$\Phi_2(a,\,a+2\,;\,c\,;\,x,\,-x) = {}_2F_3\left[\begin{matrix} a+1,\,\frac{1}{2}a+\frac{3}{2}\,; \\ \frac{1}{2}c,\,\frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}a+\frac{1}{2}\,; \end{matrix}\ \frac{x^2}{4}\right] - \frac{2x}{c}\,{}_1F_2\left[\begin{matrix} a+2\,; \\ \frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}c+1\,; \end{matrix}\ \frac{x^2}{4}\right]$$

and

(24)
$$\Phi_2(a,\,a+2\,;\,c\,;\,x,\,-x) = {}_1F_2\left[\begin{matrix} a+1\,; \\ \frac{1}{2}c,\,\frac{1}{2}c+\frac{1}{2}\,; \end{matrix}\ \frac{x^2}{4}\right] - \frac{2x}{c}\,{}_1F_2\left[\begin{matrix} a+2\,; \\ \frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}c+1\,; \end{matrix}\ \frac{x^2}{4}\right] + \frac{2x^2}{c(c+1)}\,{}_1F_2\left[\begin{matrix} a+2\,; \\ \frac{1}{2}c+1,\,\frac{1}{2}c+\frac{3}{2}\,; \end{matrix}\ \frac{x^2}{4}\right],$$

which are found to have two forms;

(25)
$$\Phi_2(a,\,a-2\,;\,c\,;\,x,\,-x) = {}_2F_3\left[\begin{matrix} a-1,\,\frac{1}{2}a+\frac{1}{2}\,; \\ \frac{1}{2}c,\,\frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}a-\frac{1}{2}\,; \end{matrix}\ \frac{x^2}{4}\right] + \frac{2x}{c}\,{}_1F_2\left[\begin{matrix} a\,; \\ \frac{1}{2}c+\frac{1}{2},\,\frac{1}{2}c+1\,; \end{matrix}\ \frac{x^2}{4}\right]$$

and

$$
\begin{aligned}
\Phi_2(a,\, a-2\,;\, c\,;\, x,\, -x) &= {}_1F_2\left[\begin{matrix} a-1\,; \\ \tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2}\,; \end{matrix}\ \tfrac{x^2}{4}\right] \\
&\quad + \frac{2x}{c}\,{}_1F_2\left[\begin{matrix} a\,; \\ \tfrac{1}{2}c+\tfrac{1}{2},\ \tfrac{1}{2}c+1\,; \end{matrix}\ \tfrac{x^2}{4}\right] \\
&\quad + \frac{2x^2}{c(c+1)}\,{}_1F_2\left[\begin{matrix} a\,; \\ \tfrac{1}{2}c+1,\ \tfrac{1}{2}c+\tfrac{3}{2}\,; \end{matrix}\ \tfrac{x^2}{4}\right],
\end{aligned} \tag{26}
$$

which are found to have two forms. Similar expressions of $\Phi_2(a\,;\,c,\,c+i\,;\,x,\,-x)$ can be obtained for $i = \pm 3,\ \pm 4,\ \pm 5$.

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