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OPERATIONAL CALCULUS ASSOCIATED WITH CERTAIN FAMILIES OF GENERATING FUNCTIONS

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Abstract. In this paper, we discuss how the operational calculus can be exploited to the theory of mixed generating functions. We use operational methods associated with multi-variable Hermite polynomials, Laguerre polynomials and Bessels functions to drive identities useful in electromagnetism, fluid mechanics etc. Certain special cases giving bilateral generating relations related to these special functions are also discussed.

1. Introduction

The appropriate combination of methods, relevant to generalized operational calculus and to special functions can be a very useful tool to treat a large body of problems both in physics and mathematics. The exponential operator techniques with the principle of quasi monomiality can be used for a more general insight into the theory of ordinary polynomials and for their extension. The idea of monomiality came from the concept of poweroid suggested by Steffensen [15]. The monomiality principle is reformulated and developed by Dettoli [5].

According to the principle of monomiality the polynomials $p_n(x)$ $(n \in \mathbb{N},\ x \in \mathbb{C})$ are called quasi-monomials, if two operators $\hat{M}$ and $\hat{P}$, can be defined in such a way that

$$\begin{aligned} \hat{M}\{p_n(x)\} &= p_{n+1}(x), \\ \hat{P}\{p_n(x)\} &= np_{n-1}(x). \end{aligned} \tag{1.1}$$

The operators $\hat{M}$ and $\hat{P}$ are called multiplicative and derivative operators and can be recognized as raising and lowering operators acting on the polynomials $p_n(x)$. Obviously $\hat{M}$ and $\hat{P}$ satisfy the commutative relation

$$[\hat{P},\ \hat{M}] = 1 \tag{1.2}$$

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and thus display a Weyl group structure. Further consequence of (1.1) is the eigen property of $\hat{M}\hat{P}$

$$\hat{M}\hat{P}\{p_n(x)\} = np_n(x). \tag{1.3}$$

The polynomials $p_n(x)$ are obtained by taking the action of $\hat{M}$ on $p_0(x)$

$$p_n(x) = \hat{M}^n\, p_0(x), \tag{1.4}$$

(in the following we shall always set $p_0(x) = 1$) and consequently the exponential generating function of $p_n(x)$ is

$$G(x,t) = \sum_{n=0}^{\infty} p_n(x)\, \frac{t^n}{n!} = \exp(t\hat{M})\{1\}. \tag{1.5}$$

The principle of monomility for Hermite and Lageurre polynomials can be exploited in many useful and flexible ways.

In the next section, we exploit the operational techniques to find the generating function for Hermite polynomials $H_n(x, y, z; \tau_1, \tau_2)$. The reason of interest for this family of Hermite polynomials is due to their mathematical importance and the fact that these polynomials give rise to the eigenstates of the quantum harmonic oscillator.

2. Generating function for $\mathbf{H_n(x, y, z; \tau_1, \tau_2)}$

First, we recall the definition of 2-variable 1-parameter Hermite polynomials (2V1PHP) [8]:

$$\exp\left(xt + y\tau\ t^2\right) = \sum_{n=0}^{\infty} H_n(x, y; \tau)\ t^n. \tag{2.1}$$

In this section, we define the 3-variable 2-parameter analogue of 2V1PHP $H_n(x, y; \tau)$ as follows [11]:

$$H_n(x, y, z; \tau_1, \tau_2) = n! \sum_{r=0}^{\left[\frac{n}{3}\right]} \frac{(z\tau_2\ )^r H_{n-3r}(x, y; \tau_1)}{r!(n-3r)!}, \tag{2.2}$$

which is equivalent to the following generating function for $H_n(x, y, z; \tau_1, \tau_2)$

$$\exp\left(xt + y\tau_1\ t^2 + z\tau_2\ t^3\right) = \sum_{n=0}^{\infty} H_n(x, y, z; \tau_1, \tau_2)\ t^n. \tag{2.3}$$

These polynomials are quasi-monomials under the action of the operators

$$\begin{aligned} \hat{M} &= x + 2y\tau_1 \frac{\partial}{\partial x} + 3z\tau_2 \frac{\partial^2}{\partial x^2}, \\ \hat{P} &= \frac{\partial}{\partial x}, \end{aligned} \tag{2.4}$$

which play the role of multiplicative and derivative operators respectively in the sense that

$$\hat{M}\{H_n(x,y,z;\tau_1,\tau_2)\} = H_{n+1}(x,y,z;\tau_1,\tau_2),$$
$$\hat{P}\{H_n(x,y,z;\tau_1,\tau_2)\} = nH_{n-1}(x,y,z;\tau_1,\tau_2).$$

We can explicitly write the polynomials $H_n(x,y,z;\tau_1,\tau_2)$ in terms of the operators (2.4) as follows:

$$H_n(x,y,z;\tau_1,\tau_2) = \Big(x + 2y\tau_1\frac{\partial}{\partial x} + 3z\tau_2\frac{\partial^2}{\partial x^2}\Big)^n. \tag{2.5}$$

Furthermore, the polynomials $H_n(x,y,z;\tau_1,\tau_2)$ are the solution of

$$\frac{\partial}{\partial y}H_n(x,y,z;\tau_1,\tau_2) = \frac{\partial^2}{\partial x^2}H_n(x,y,z;\tau_1,\tau_2)$$

and

$$\frac{\partial}{\partial z}H_n(x,y,z;\tau_1,\tau_2) = \frac{\partial^3}{\partial x^3}H_n(x,y,z;\tau_1,\tau_2),$$

which gives the operational rule

$$H_n(x,y,z;\tau_1,\tau_2) = e^{y\tau_1\frac{\partial^2}{\partial x^2}+z\tau_2\frac{\partial^3}{\partial x^3}}(x^n), \tag{2.6}$$

which for $\tau_1 = \tau_2 = 1$, reduces to operational rule for polynomials $H_n(x,y,z)$ [4] as:

$$H_n(x,y,z) = e^{y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}}(x^n). \tag{2.7}$$

The previous considerations confirm that the most of the properties of families of polynomials, recognized as quasi monomials, can be deduced, quite straight forwardly, by using operational rules associated with the relevant multiplicative and derivative operators. Furthermore, they suggest that we can define families of isospectral problems by exploiting the correspondence:

$$\hat{M} \longleftrightarrow x$$
$$\hat{P} \longleftrightarrow \partial_x$$
$$p_n(x) \longleftrightarrow x^n$$

We can therefore use the polynomials $p_n(x)$ as a basis to introduce new functions with eigenvalues corresponding to the ordinary case.

To obtain the generating function, we consider the following relation [13]

$$\sum_{l=0}^{\infty} c^{k-l}{L_l}^{k-l}(-bc)x^l = e^{bx}\ (x+c)^k, \tag{2.8}$$

with ${L_l}^{\alpha}(x)$ being associated Laguerre polynomials [12],

$${L_n}^{\alpha}(x) = \Gamma(\alpha+n+1)\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!\Gamma(a+r+1)(n-r)!}.$$

We replace x by $\left(x+2y\tau_1\frac{\partial}{\partial x}+3z\tau_2\frac{\partial^2}{\partial x^2}\right)$ in (2.8) and use the identity (2.6) to get

$$\begin{aligned}(2.9)\qquad &\sum_{l=0}^{\infty} c^{k-l}{L_l}^{k-l}(-bc)H_n(x,y,z;\tau_1,\tau_2)\\ &= e^{b\left(x+2y\tau_1\frac{\partial}{\partial x}+3z\tau_2\frac{\partial^2}{\partial x^2}\right)}H_n(x+c,y,z;\tau_1,\tau_2).\end{aligned}$$

Now by decoupling the exponential on the right hand side of (2.9) by means of the rule [7]

$$e^{\hat{A}+\hat{B}}=e^{m^2/12}e^{-m/2\hat{A}^{1/2+\hat{A}}}e^{\hat{B}};\ [\hat{A},\hat{B}]=m\hat{A}^{1/2},$$

with

$$m=2\sqrt{3}\ b^{3/2}(z\tau_2)^{1/2},$$

where $\hat{A}$ and $\hat{B}$ are the operators and m a complex number, we find

$$\begin{aligned}(2.10)\qquad &\sum_{l=0}^{\infty} c^{k-l}{L_l}^{k-l}(-bc)H_n(x,y,z;\tau_1,\tau_2)\\ &= \exp\left(b^3z\tau_2\right)\exp\left(-3b^2z\tau_2\frac{\partial}{\partial x}+3bz\tau_2\frac{\partial^2}{\partial x^2}\right)\\ &\quad \exp\left(bx+2by\tau_1\frac{\partial}{\partial x}\right)H_k(x+c,y,z;\tau_1,\tau_2).\end{aligned}$$

Again decoupling the exponential on the right hand side of (2.10) by means of the rule [7]

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}+\hat{B}}e^{-m/2};\ [\hat{A},\hat{B}]=m,$$

and by using the identity

$$e^{\alpha}\frac{d^m}{dx^m}f(x)=f\left(x+m\alpha\frac{d^{m-1}}{dx^{m-1}}\right)e^{\alpha}\frac{d^m}{dx^m}$$

we obtain

$$\begin{aligned}(2.11)\qquad &\sum_{l=0}^{\infty} c^{k-l}{L_l}^{k-l}(-bc)H_l(x,y,z;\tau_1,\tau_2)\\ &= \exp\left(bx+b^2y\tau_1+b^3z\tau_2\right)\\ &\quad \exp\left(-3b^2z\tau_2\frac{\partial}{\partial x}+3bz\tau_2\frac{\partial^2}{\partial x^2}\right)H_k(x+2by\tau_1+c,y,z;\tau_1,\tau_2).\end{aligned}$$

The action of the exponential operators, on the polynomials $H_k(x,y,z;\tau_1,\tau_2)$ can be specified by (2.11) and

$$e^{\alpha\frac{\partial}{\partial x}}f(x)=f(x+\alpha)$$

finally gives the generating function for $H_n(x,y,z;\tau_1,\tau_2)$ as

$$\begin{aligned}&\sum_{l=0}^{\infty} c^{k-l} {L_l}^{k-l}(-bc) H_l(x,y,z;\tau_1,\tau_2)\\ &= \exp\left(bx+b^2y\tau_1+b^3z\tau_2\right)\\ &\quad H_k(x+2by\tau_1+3b^2z\tau_2+c, y\tau_1+3bz\tau_2, z;\tau_1,\tau_2).\end{aligned} \tag{2.12}$$

In the next section, we discuss how the method we have just outlined can be exploited to the theory of mixed generating functions and will show that they provide a new point of view to the theory of generalized special functions of many variables. These special functions are useful in electromagnetism, wave propagation, beam life-time in storage ring [16] and fluid mechanics etc.

3. Operational formulism and mixed generating functions

The theory of mixed generating functions has been pioneered by Carlitz [3] and Dattoli et al. [6], who employed the Lagrange expansion as the essential tool to develop a unifying point of view on the problem and to drive families of mixed generating functions in a fairly direct way [14].

The two variable extension of Laguerre polynomials $L_n(x,y)$ are defined by [9]

$$L_n(x,y) = n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2}, \tag{3.1}$$

satisfying the property

$$L_n(x,1) = L_n(x),$$

with $L_n(x)$ being ordinary Laguerre polynomials and

$$L_n(x,0) = \frac{(-x)^n}{n!}.$$

The Laguerre polynomials $L_n(x,y)$ behave as quasi-monomials [5] under the action of the operators

$$\hat{M} = y - D_x^{-1},\ \hat{P} = -\partial_x x \partial_x, \tag{3.2}$$

where D_x^{-1} denotes the inverse of the derivative operator, will be characterized by the operational rule

$$D_x^{-n}(1) = \frac{x^n}{n!}.$$

Furthermore, the generating function for the $L_n(x,y)$ is given by [5]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\ L_n(x,y) = e^{t(y-D_x^{-1})}(1) = \exp(yt)\ C_0(xt), \tag{3.3}$$

where $C_0(x)$ denotes the zeroth-order Tricomi function. The ordinary Tricomi functions $C_n(x)$ are defined as [1]:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(n+r)!} = x^{-n/2} J_n(2\sqrt{x}), \tag{3.4}$$

and $J_n(x)$ denotes the ordinary Bessel function.

The polynomials $L_n(x,y)$ can explicitly written in the form [5]

$$\begin{aligned} L_n(x,y) = \left(y - D_x^{-1}\right)^n &= \sum_{r=0}^{n} \binom{n}{r} (-1)^r y^{n-r} D_x^{-r}(1) \\ &= n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)! (r!)^2}, \end{aligned} \tag{3.5}$$

and are a fairly direct consequence of the properties of the operator D_x^{-1} .

Further

$$\begin{aligned} \exp\left(-y \frac{\partial}{\partial x}\, x\, \frac{\partial}{\partial x}\right) \frac{(-1)^n x^n}{n!} &= L_n(x,y), \\ \exp\left(-y \frac{\partial}{\partial x}\, x\, \frac{\partial}{\partial x}\right) L_n(x,y) &= L_n(x,y+1). \end{aligned} \tag{3.6}$$

The generating function of ordinary cylinderical Bessel functions is

$$\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{\frac{x}{2}(t - \frac{1}{t})}, \tag{3.7}$$

which are linked with the Tricomi function

$$J_n(x) = \left(\frac{x}{2}\right)^n C_n\left(\frac{x^2}{4}\right). \tag{3.8}$$

To generate Hermite-Bessel function associated with the polynomials $H_n(x,y,z;\tau_1,\tau_2)$, we introduce the generating function

$$\begin{aligned} G(x,y,z;\tau_1,\tau_2;t) &= \exp \frac{\hat{M}}{2}\left(t - \frac{1}{t}\right)(1) \\ &= \exp\left(\frac{x + 2y\tau_1 \frac{\partial}{\partial x} + 3z\tau_2 \frac{\partial^2}{\partial x^2}}{2}\right)\left(t - \frac{1}{t}\right)(1). \end{aligned} \tag{3.9}$$

By exploiting (3.9) and using already quoted decoupling procedure, we get Hermite-Bessel function of three variables.

$$\sum_{n=-\infty}^{\infty} t^n\, {}_H J_n(x,y,z;\tau_1,\tau_2) = e^{\frac{x}{2}(t-\frac{1}{t}) + \frac{y\tau_1}{4}(t-\frac{1}{t})^2 + \frac{z\tau_2}{8}(t-\frac{1}{t})^3}, \tag{3.10}$$

which satisfy the following properties

$${}_H J_n(x,0,0;1,1) = J_n(x),$$

and

$$(3.11)\qquad \begin{aligned} &\frac{\partial}{\partial x}\ {}_HJ_n(x,y,z;\tau_1,\tau_2) = n\,{}_HJ_{n-1}(x,y,z;\tau_1,\tau_2),\\ &\frac{\partial}{\partial y}\ {}_HJ_n(x,y,z;\tau_1,\tau_2) = n(n-1)\,{}_HJ_{n-2}(x,y,z;\tau_1,\tau_2),\\ &\frac{\partial}{\partial z}\ {}_HJ_n(x,y,z;\tau_1,\tau_2) = n(n-1)(n-2)\,{}_HJ_{n-3}(x,y,z;\tau_1,\tau_2). \end{aligned}$$

Last three relations can be combined to get

$$\frac{\partial}{\partial y}\ {}_HJ_n(x,y,z;\tau_1,\tau_2) = \frac{\partial^2}{\partial x^2}\ {}_HJ_n(x,y,z;\tau_1,\tau_2)$$

and

$$\frac{\partial}{\partial z}\ {}_HJ_n(x,y,z;\tau_1,\tau_2) = \frac{\partial^3}{\partial x^3}\ {}_HJ_n(x,y,z;\tau_1,\tau_2),$$

which leads to the operational rule

$$(3.12)\qquad {}_HJ_n(x,y,z;\tau_1,\tau_2) = e^{y\tau_1\frac{\partial^2}{\partial x^2}+z\tau_2\frac{\partial^3}{\partial x^3}}J_n(x),$$

clearly for $\tau_1 = \tau_2 = 1$, reduce to the operational rule for ${}_HJ_n(x,y,z)$ as

$$(3.13)\qquad {}_HJ_n(x,y,z) = e^{y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}}J_n(x).$$

Laguerre-Bessel functions are also defined as

$$(3.14)\qquad \sum_{n=-\infty}^{\infty} t^n\ {}_LJ_n(x,y) = \exp\frac{y}{2}\left(t-\frac{1}{t}\right)C_0\left(\frac{x}{2}(t-\frac{1}{t})\right).$$

To obtain the generating relations for Hermite-Bessel and Lagurre Hermite polynomials, we consider the relation [1]

$$(3.15)\qquad \sum_{n=0}^{\infty} J_n(t)\frac{x^n}{n!} = J_0\left(\sqrt{t^2-2xt}\right),$$

and operating $e^{y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}}$ on both sides and using (3.13) and (2.8), we obtain the following bilateral generating function

$$(3.16)\qquad \sum_{n=0}^{\infty}\frac{H_n(x,y,z)}{n!}J_n(t) = {}_H\ J_0\left(\sqrt{t^2-2xt},y,z\right),$$

which according to (3.8) can be recast as

$$(3.17)\qquad {}_HJ_0\left(\sqrt{t^2-2xt},y,z\right) = \sum_{n=0}^{\infty}\frac{H_n(x,y,z)t^n}{n!}2^{-n}C_n\left(\frac{t^2}{4}\right).$$

Another example is provided by the relation [1]

$$(3.18)\qquad x\ \sin x = 2\sum_{n=1}^{\infty}(2n)^2J_{2n}(x),$$

by operating $e^{y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}}$ on both sides and using (3.13) and (2.8), we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+2}(x,y,z)}{(2n+1)!}=2\sum_{n=1}^{\infty}(2n)^2\,{}_HJ_{2n}(x,y,z). \tag{3.19}$$

Further applying the above stated procedure for the relation [1]

$$\cos\ x=J_0(x)\ +\ 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x), \tag{3.20}$$

we obtain the following relation

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n}(x,y,z)}{(2n)!}={}_H\ J_0(x,y,z)\ +\ 2\sum_{n=1}^{\infty}(-1)^n\,{}_HJ_{2n}(x,y,z). \tag{3.21}$$

Consider the well known identity of Laguerre polynomial [1]

$$\exp(-ax)=\frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^n L_n(x), \tag{3.22}$$

now replacing x by $x+2y\frac{\partial}{\partial x}+3z\frac{\partial^2}{\partial x^2}$ and using the already quoted decoupling formula we obtain

$$\exp(-ax+a^2y-a^3z)=\frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^n\ {}_HL_n(x,y,z). \tag{3.23}$$

Again consider (3.22) and replace x by x^2 and operating $e^{y\frac{\partial^2}{\partial x^2}}$ on both sides and using identity

$$e^{y\frac{\partial^2}{\partial x^2}}\exp(-ax^2)=\frac{1}{\sqrt{1+4ay}}\ e^{(\frac{-ax^2}{1+4ay})}, \tag{3.24}$$

we obtain

$$\frac{1}{\sqrt{1+4ay}}\ e^{(\frac{-ax^2}{1+4ay})}=\frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^n\ e^{y\frac{\partial^2}{\partial x^2}}C_0(x^2t), \tag{3.25}$$

and

$$\frac{1}{\sqrt{1+4ay}}\ e^{(\frac{-ax^2}{1+4ay})}=\frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^n\ E_0(x,y;t), \tag{3.26}$$

where left hand side is a generalization of Gleiser identity which can be put in the following form also

$$\frac{1}{2\sqrt{\pi y}}\ \int_{-\infty}^{\infty}e^{-\frac{(ax-t)^2}{4y}}=\frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^n\ E_0(x,y;t). \tag{3.27}$$

Again consider the identity (3.22) and operating $\exp(-y\frac{\partial}{\partial x}\ x\ \frac{\partial}{\partial x})$ both sides and using identity (3.6) we obtain the following generalization

$$\exp(-ax+ay) = \frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^n L_n(x,1+y). \tag{3.28}$$

By recalling the polynomials [10]

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\ {}_LH_n(x,y;z) = \exp[(y-\hat{D}_x^{-1})t+zt^2], \tag{3.29}$$

where

$$ {}_LH_n(x,y;z) = \sum_{l=0}^{[n/2]}\frac{n!z^s L_{n-2s}(x,y)}{(n-2s)!s!}$$

can be viewed as a Laguerre-Hermite polynomials and are quasi-monomials under the action of

$$\hat{M} = (y-\hat{D}_x^{-1})-2z(\partial_x x\partial_x),\ \hat{P} = -\partial_x x\partial_x, \tag{3.30}$$

which can be used to generate Burchnall identity [2] as

$$ {}_LH_n(x,y;z) = [(y-\hat{D}_x^{-1})-2z(\partial_x x\partial_x)]^n, \tag{3.31}$$

or

$$ {}_LH_n(x,y;z) = \sum_{s=0}^{n}\binom{n}{s}(-1)^s(\partial_x x\partial_x)^s(2z)^s\ {}_LH_{n-s}(x,y;z). \tag{3.32}$$

These polynomials can be exploited to derive bilateral generating relation by replacing x by $[(y-\hat{D}_x^{-1})-2z(\partial_x x\partial_x)]$ in (2.8) and using (3.32), we obtain the following new identity relating to Laguerre-Hermite polynomials and Tricomi function

$$\exp(by-b^2z)\sum_{s=0}^{n}\binom{n}{s}(-1)^s(\partial_x x\partial_x)^s(2z)^s\ {}_LH_{n-s}(x,y+c;z)C_s(bx) \tag{3.33}$$

$$= \sum_{l=0}^{\infty}c^{k-l}{L_l}^{k-l}(-bc)\ {}_LH_n(x,y;z).$$

Thus we conclude that the methods based on operational identities may provide powerful tools to deal with the possibilities offered by generalized forms of ordinary polynomials.

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