Commun. Korean Math. Soc. **30** (2015), No. 4, pp. 415–427
http://dx.doi.org/10.4134/CKMS.2015.30.4.415

BOUNDEDNESS IN THE NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Sang Il Choi and Yoon Hoe Goo

Abstract. In this paper, we investigate bounds for solutions of the nonlinear functional differential systems.

1. Introduction

Pachpatte [14, 15] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the operator T. Goo [10] studied the stability and boundedness of the solutions of perturbed nonlinear systems under some suitable conditions. In this paper, under conditions stronger than Goo [10] we examined the bounded result of the solutions of perturbed nonlinear systems.

Pinto [16, 17] introduced the notion of h-stability (hS) which is an important extension of exponential asymptotic stability. He introduced hS with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptyotic stability to a variety of reasonable systems called h-systems. Choi, Ryu [5] and Choi, Koo, and Ryu [4] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [7, 8, 9, 10] and Goo et al. [2, 3, 11] investigated boundedness of solutions for nonlinear perturbed systems.

The aim of this paper is to obtain some results on boundedness of the nonlinear functional differential systems under suitable conditions on perturbed term. To do this, we need some integral inequalities.

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

Received December 10, 2014.
2010 *Mathematics Subject Classification.* 34D10.
Key words and phrases. h-stability, t_∞-similarity, functional differential system.

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t,0) = 0$. Also, we consider the perturbed functional differential systems of (2.1)

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s))ds + h(t,y(t),Ty(t)),\ y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $g(t,0) = 0$, $h(t,0,0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$.

Let $x(t,t_0,x_0)$ denote the unique solution of (2.1) with $x(t_0,t_0,x_0) = x_0$, existing on $[t_0,\infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t,0)v(t),\ v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t,x(t,t_0,x_0))z(t),\ z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t,t_0,x_0)$ of (2.4) is given by

$$\Phi(t,t_0,x_0) = \frac{\partial}{\partial x_0} x(t,t_0,x_0),$$

and $\Phi(t,t_0,0)$ is the fundamental matrix of (2.3).

We recall some notions of h-stability [16].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called an *h-system* if there exist a constant $c \geq 1$ and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called (hS) *h-stable* if there exists $\delta > 0$ such that (2.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [6, 12].

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞-similarity.

We give some related properties that we need in the sequel.

Lemma 2.4 ([17])**.** *The linear system*

$$x' = A(t)x, \;\; x(t_0) = x_0, \tag{2.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (*respectively h-stable*) *if and only if there exist $c \geq 1$ and a positive continuous* (*respectively bounded*) *function h defined on $\mathbb{R}^+$ such that*

$$|\phi(t,t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t,t_0)$ is a fundamental matrix of (2.6)*.*

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t,y) + g(t,y), \;\; y(t_0) = y_0, \tag{2.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t,0) = 0$. Let $y(t) = y(t,t_0,y_0)$ denote the solution of (2.8) passing through the point (t_0,y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.5. *Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of* (2.1) *and* (2.8)*, respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t,t_0,y_0) \in \mathbb{R}^n$,*

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^t \Phi(t,s,y(s))\, g(s,y(s))\, ds.$$

Theorem 2.6 ([5])**.** *If the zero solution of* (2.1) *is hS, then the zero solution of* (2.3) *is hS.*

Theorem 2.7 ([4])**.** *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of* (2.3) *is hS, then the solution $z = 0$ of* (2.4) *is hS.*

Lemma 2.8 (Bihari-type inequality)**.** *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda(s) w(u(s)) ds, \;\; t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t \lambda(s) ds\Big], \;\; t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.9 ([3])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u *and* $u \leq w(u)$. *If for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau ds, \;\; t \geq t_0 \geq 0,$$

then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau)ds\Big], \; t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.10 ([2])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau ds \\ &+ \int_{t_0}^{t} \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)w(u(\tau))d\tau ds, \;\; 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds\Big], \\ t_0 \leq t < b_1, \end{aligned}$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau \\ + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

We obtain the following corollary from Lemma 2.10.

Corollary 2.11. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau ds, \; 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau)ds\Big], \; t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

3. Main results

In this section, we investigate bounds for the nonlinear functional differential systems following [10].

We need the following lemma to prove Theorem 3.2.

Lemma 3.1. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t}\lambda_1(s)w(u(s))ds + \int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)u(\tau) \\ &+ \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t}\lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)w(u(\tau))d\tau ds. \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[&W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau \\ &+ \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds\Big],\ t_0 \leq t < b_1, \end{aligned} \tag{3.2}$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \geq t_0 : W(c) &+ \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau \\ &+ \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Proof. Define a function $z(t)$ by the right member of (3.1). Then, we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= \lambda_1(t)w(u(t)) + \lambda_2(t)\int_{t_0}^{t}(\lambda_3(s)u(s) + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)w(u(\tau))d\tau)ds \\ &\quad + \lambda_6(t)\int_{t_0}^{t}\lambda_7(s)w(u(s))ds \\ &\leq (\lambda_1(t) + \lambda_2(t)\int_{t_0}^{t}(\lambda_3(s) + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)d\tau)ds \\ &\quad + \lambda_6(t)\int_{t_0}^{t}\lambda_7(s)ds)w(z(t)),\ t \geq t_0. \end{aligned}$$

Since $z(t)$ and $w(u)$ are nondecreasing, $u \le w(u)$, and $u(t) \le z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$z(t) \le c + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \tag{3.3}$$
$$+ \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)d\tau) w(z(s)))ds.$$

It follows from Lemma 2.8 that (3.3) yields the estimate (3.2). □

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.

(H3) $w(u)$ is nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.

Theorem 3.2. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that* (H1), (H2), *and* (H3) *hold, and g in* (2.2) *satisfies*

$$|g(t, y(t))| \le a(t)|y(t)| + b(t) \int_{t_0}^{t} k(s) w(|y(s)|)ds \tag{3.4}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)(w(|y(t)|) + |Ty(t)|),$$
$$|Ty(t)| \le \int_{t_0}^{t} q(s) w(|y(s)|)ds, \; t \ge t_0 \ge 0, \tag{3.5}$$

where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) *is bounded on $[t_0, \infty)$ and*

$$|y(t)| \le h(t) W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau)$$
$$+ b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau + c(s) \int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup \Big\{ t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau$$
$$+ c(s) \int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW}^{-1} \Big\}.$$

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solutions of (2.1) and (2.2) with the same initial values are represented by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) (\int_{t_0}^{s} g(\tau, y(\tau))d\tau + h(s, y(s), Ty(s)))ds. \tag{3.6}$$

By Theorem 2.6, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.3) is hS. Therefore from (H1) and by Theorem 2.7, the solution $z = 0$ of (2.4) is hS. By Lemma 2.4, the hS condition of $x = 0$ of (2.1), (3.4), (3.5), and (3.6), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))|\Big(\int_{t_0}^{s} |g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|\Big)ds \\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s}(a(\tau)|y(\tau)| \\ &\quad + b(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|)dr)d\tau + c(s)(w(|y(s)|) \\ &\quad + \int_{t_0}^{s} q(\tau)w(|y(\tau)|)d\tau)\Big)ds.\end{aligned}$$

It follows from (H2) and (H3) that

$$\begin{aligned}&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(c(s)w(\frac{|y(s)|}{h(s)}) \\ &\quad + \int_{t_0}^{s}(a(\tau)\frac{|y(\tau)|}{h(\tau)} + b(\tau)\int_{t_0}^{\tau} k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau \\ &\quad + c(s)\int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, from Lemma 3.1, we have

$$\begin{aligned}|y(t)| \le h(t)W^{-1}\Big[&W(c) + c_2\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau) \\ &+ b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],\end{aligned}$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Remark 3.3. Letting $c(t) = 0$ in Theorem 3.2, we obtain the similar result as that of Theorem 3.4 in [7].

Theorem 3.4. *Let* $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. *Suppose that* (H1), (H2), *and* (H3) *hold, and* g *in* (2.2) *satisfies*

$$\int_{t_0}^{s} |g(\tau,y(\tau))|d\tau \le a(s)w(|y(s))| + b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau \tag{3.7}$$

and

$$\begin{aligned}&|h(s,y(s),Ty(s))| \le c(s)(w(|y(s)|) + |Ty(s)|), \\ &|Ty(s)| \le \int_{t_0}^{s} q(\tau)w(|y(\tau)|)d\tau,\ s \ge t_0 \ge 0,\end{aligned} \tag{3.8}$$

where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of* (2.2) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\Big[W(c)+c_2\int_{t_0}^{t}(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau+c(s)\int_{t_0}^{s}q(\tau)d\tau)ds\Big],$$
$$t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.8 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau + c(s)\int_{t_0}^{s}q(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.2, the solution $z = 0$ of (2.4) is hS. Applying Lemma 2.4, (3.6), (3.7), and (3.8), we have

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)h(s)^{-1}\Big((a(s)w(|y(s)|) + b(s)\int_{t_0}^{s}k(\tau)w(|y(\tau)|)d\tau + c(s)(w(|y(s)|) + \int_{t_0}^{s}q(\tau)w(|y(\tau)|)d\tau)\Big)ds.$$

By the assumptions (H2) and (H3), we obtain

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)\Big((a(s) + c(s))w(\frac{|y(s)|}{h(s)}) + b(s)\int_{t_0}^{s}k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + c(s)\int_{t_0}^{s}q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Define $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 2.10, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau + c(s)\int_{t_0}^{s}q(\tau)d\tau\Big],\ t_0 \leq t < b_1,$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded. Hence, the proof is complete. □

We obtain the following corollary using Corollary 2.11.

Corollary 3.5. *Let* $a, b, k, q, u, w \in C(\mathbb{R}^+)$. *Suppose that* (H1), (H2), *and* (H3) *hold, and* g *in* (2.2) *satisfies the condition* (3.7) *of Theorem 3.4 and*

$$|h(s, y(s), Ty(s))| \leq b(s)(w(|y(s)|) + |Ty(s)|),$$
$$|Ty(s)| \leq \int_{t_0}^{s}q(\tau)w(|y(\tau)|)d\tau,\ s \geq t_0 \geq 0,$$

where $a, b, k, q \in L_1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of* (2.2) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) + b(s) \int_{t_0}^{s} (k(\tau) + q(\tau)d\tau)ds\Big],$$
$$t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) + b(s) \int_{t_0}^{s} (k(\tau) + q(\tau))d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Remark 3.6. Letting $c(s) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 3.2 in [7].

We need the following lemma for the proof of Theorem 3.8.

Lemma 3.7. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that, for some* $c \ge 0$, *we have*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau)w(u(\tau)) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(s)u(r)dr)d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)u(\tau)d\tau\Big)ds,\ t \ge t_0. \tag{3.9}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)]ds\Big],\ t \ge t_0. \tag{3.10}$$

Proof. Define a function $v(t)$ by the right member of (3.9). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)u(t) + \lambda_2(t)\Big(\int_{t_0}^{t} (\lambda_3(s)w(u(s)) + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)u(\tau)d\tau)ds \\ &\quad + \lambda_6(t) \int_{t_0}^{t} \lambda_7(s)u(s)ds\Big) \\ &\le \Big[\lambda_1(t) + \lambda_2(t)\Big(\int_{t_0}^{t} (\lambda_3(s) + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds \\ &\quad + \lambda_6(t) \int_{t_0}^{t} \lambda_7(s)ds\Big)\Big]w(v(t)),\ t \ge t_0. \end{aligned}$$

Since $v(t)$ is nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \le c + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)d\tau\Big) w(z(s))ds. \tag{3.11}$$

Thus, (3.11) yields the estimate (3.10). □

Theorem 3.8. *Let* $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. *Suppose that* (H1), (H2), *and* (H3) *hold, and* g *in* (2.2) *satisfies*

$$|g(t, y(t))| \le a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)|y(s)|ds \tag{3.12}$$

and

$$\begin{aligned} &|h(t, y(t), Ty(t))| \le c(t)(|y(t)| + |Ty(t)|), \\ &|Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds, \ t \ge t_0 \ge 0, \end{aligned} \tag{3.13}$$

where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of* (2.2) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} [c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau + c(s) \int_{t_0}^{s} q(\tau)d\tau]ds\Big],$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} [c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau + c(s) \int_{t_0}^{s} q(\tau)d\tau]ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.2, the solution $z = 0$ of (2.4) is hS. Applying the nonlinear variation of constants formula (3.6), the hS condition of $x = 0$ of (2.1), (3.12), and (3.13), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s} (a(\tau)w(|y(\tau)|) + b(\tau) \int_{t_0}^{\tau} k(r)|y(r)|dr)d\tau + c(s)(|y(s)| + \int_{t_0}^{s} q(\tau)|y(\tau)|d\tau)\Big)ds.$$

Using (H2) and (H3), we obtain

$$\begin{aligned}|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(c(s)\frac{|y(s)|}{h(s)} \\ + \int_{t_0}^{s}(a(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{h(r)}dr)d\tau \\ + c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 3.7, we have

$$\begin{aligned}|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}[c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau \\ + c(s)\int_{t_0}^{s} q(\tau)d\tau]ds\Big],\end{aligned}$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. □

Remark 3.9. Letting $c(t) = 0$ in Theorem 3.8, we obtain the similar result as that of Theorem 3.7 in [9].

Theorem 3.10. *Let* $a, b, c, q \in C(\mathbb{R}^+)$. *Suppose that* (H1), (H2), *and* (H3) *hold, and* g *in* (2.2) *satisfies*

$$\begin{aligned}\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)|y(t)|, \\ |h(t, y(t), Ty(t))| \le b(t)|y(t)| + c(t)|Ty(t)|\end{aligned} \tag{3.14}$$

and

$$|Ty(t)| \le \int_{t_0}^{t} q(s)w(|y(s)|)ds, \tag{3.15}$$

where $a, b, c, q \in L_1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of* (2.2) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. It is well known that the solution of (1.2) is represented by the integral equation (3.6). By the same argument as in the proof in Theorem 3.2, the solution $z = 0$ of (2.4) is hS. Applying Lemma 2.4, the hS condition of $x = 0$ of (2.1), (3.7), (3.14), and (3.15), we have

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big((a(s)+b(s))|y(s)| + c(s)\int_{t_0}^{s} q(\tau)w(|y(\tau)|)d\tau\Big)ds.$$

Using the assumptions (H2) and (H3), we obtain

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big((a(s)+b(s))\frac{|y(s)|}{h(s)} + c(s)\int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 2.9, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s)+b(s)+c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$. This completes the proof. □

Remark 3.11. Letting $w(u) = u$ and $b(t) = c(t) = 0$ in Theorem 3.10, we obtain the similar result as that of Theorem 3.3 in [11].

Acknowledgment. The authors are very grateful for the referee's valuable comments.

References

[1] V. M. Alexseev, *An estimate for the perturbations of the solutions of ordinary differential equations*, Vestnik Moskov. Univ. Ser. I. Math. Mekh. **2** (1961), no. 2, 28–36.

[2] S. I. Choi and Y . H. Goo, *Boundedness in the functional nonlinear differential systems*, Far East J. Math. Sci. **96** (2015), no. 7, 801–819.

[3] ______, *h-stability and boundedness in nonlinear functional perturbed differential systems*, submitted.

[4] S. K. Choi, N. J. Koo, and H. S. Ryu, *h-stability of differential systems via t_∞-similarity*, Bull. Korean. Math. Soc. **34** (1997), no. 3, 371–383.

[5] S. K. Choi and H. S. Ryu, *h-stability in differential systems*, Bull. Inst. Math. Acad. Sinica **21** (1993), no. 3, 245–262.

[6] R. Conti, *Sulla t_∞-similitudine tra matricie l'equivalenza asintotica dei sistemi differenziali lineari*, Riv. Mat. Univ. Parma **8** (1957), 43–47.

[7] Y. H. Goo, *Boundedness in perturbed nonlinear differential systems*, J. Chungcheong Math. Soc. **26** (2013), 605–613.

[8] ______, *Boundedness in the perturbed nonlinear differential systems*, Far East J. Math. Sci. **79** (2013), 205–217.

[9] ______, *Boundedness in the perturbed differential systems*, J. Korean Soc. Math. Edu. Ser. B Pure Appl. Math. **20** (2013), no. 3, 223–232.

[10] ______, *Boundedness in nonlinear perturbed differential systems*, J. Appl. Math. Inform. **32** (2014), no. 1-2, 247–254.

[11] Y . H. Goo and D. H. Ry, *h-stability of the nonlinear perturbed differential systems*, J. Chungcheong Math. Soc. **23** (2010), 827–834.
[12] G. A. Hewer, *Stability properties of the equation of first variation by t_∞-similarity*, J. Math. Anal. Appl. **41** (1973), 336–344.
[13] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications Vol. I*, Academic Press, New York and London, 1969.
[14] B. G. Pachpatte, *Stability and asymptotic behavior of perturbed nonlinear systems*, J. Differential Equations **16** (1974), 14–25.
[15] ———, *Perturbations of nonlinear systems of differential equations*, J. Math. Anal. Appl. **51** (1975), no. 3, 550–556.
[16] M. Pinto, *Perturbations of asymptotically stable differential systems*, Analysis **4** (1984), no. 1-2, 161–175.
[17] ———, *Stability of nonlinear differential systems*, Appl. Anal. **43** (1992), no. 1-2, 1–20.

Sang Il Choi
Department of Mathematics
Hanseo University
Seosan Chungnam 356-706, Korea
E-mail address: schoi@hanseo.ac.kr

Yoon Hoe Goo
Department of Mathematics
Hanseo University
Seosan Chungnam 356-706, Korea
E-mail address: yhgoo@hanseo.ac.kr