Commun. Korean Math. Soc. **30** (2015), No. 4, pp. 385–402
http://dx.doi.org/10.4134/CKMS.2015.30.4.385

SEMI-PRIME CLOSURE OPERATIONS ON BCK-ALGEBRA

Hashem Bordbar and Mohammad Mehdi Zahedi

Abstract. In this paper we study the (good) semi-prime closure operations on ideals of a BCK-algebra, lower BCK-semilattice, Noetherian BCK-algebra and meet quotient ideal and then we give several theorems that make different (good) semi-prime closure operations. Moreover by given some examples we show that the given different notions are independent together, for instance there is a semi-prime closure operation, which is not a good semi-prime. Finally by given the notion of "c_f-Max X", we prove that every member of "c_f-Max X" is a prime ideal. Also we conclude some more related results.

1. Introduction

BCK-algebras and BCI-algebras are abbreviated to two B-algebras. BCK-algebra was introduced in 1966 by Y. Imai and K. Iseki [6], and BCI-algebra was put forward in the same year due to K. Iseki. The BCK-algebra is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional calcului. Also for the first time, E. H. Moore [9] in 1910 introduced closure operation on a set. After that many researchers have worked on closure operation, see for example [4, 7, 10, 11]. Finally in 2012 , N. Epestein published a paper [5] about closure operation on commutative algebra. After that we introduced the notion of closure operation in [3]. Now in this paper we introduce the notion of semi-prime closure operation and obtain some results as mentioned in the abstract.

2. Preliminaries

In this section, we give some basic notions relevant to closure operations and semi-prime closure operations on ideals of a BCK-algebra. We will use them in the next sections.

Definition 2.1 ([12, Definition 1.1.1]). An algebra $(X;*,0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions: for any $x,y,z \in X$,

Received June 5, 2015; Revised August 10, 2015.
2010 *Mathematics Subject Classification.* Primary 06F35, 03G25.
Key words and phrases. BCI-algebra, BCK-algebra, closure operation, finite type closure operation, semi-prime closure operation.

BCI-1: $((x * y) * (x * z)) * (z * y) = 0$.
BCI-2: $x * 0 = x$.
BCI-3: $x * y = 0$ and $y * x = 0$ imply $x = y$.

We call the binary operation $*$ on X as the multiplication on X, and the constant 0 of X the zero element of X. We often write X instead of $(X; *, 0)$ for a BCI-algebra in briefly.

Proposition 2.2 ([12, Definition 1.1.2])**.** *Let $(X; *, 0)$ be a BCI-algebra. Define a binary relation $\leq$ on X by which $x \leq y$ if and only if $x*y = 0$ for any $x, y \in X$. Then $(X; \leq)$ is a partially ordered set.*

Definition 2.3 ([12, Definition 1.2.1])**.** For a given BCI-algebra X, if it satisfies the condition:

BCK-1: $0 * x = 0$ for all $x \in X$ (which means that for each $x \in X$, $0 \leq x$), then we call this algebra a BCK-algebra.

Definition 2.4 ([8, Definition 8.1])**.** A BCK-algebra X is called bounded if there exists the greatest element of X, with respect to the ordered relation $\leq$. It means that there exists an element like 1 such that for each $x \in X$, $x \leq 1$.

Definition 2.5 ([12, Proposition 1.1.2])**.** A partially ordered set $(X; \leq)$ is called a lower semilattice if any two elements in X have the greatest lower bound (glb). It is called an upper semilattice if each pair of elements in X has its least upper bound (lub).

For a given BCK-algebra X, if it forms a lower semilattice with respect to its BCI-ordering $\leq$, then the algebra X is called a lower BCK-semilattice. Similarly, we can define an upper BCK-semilattice.

In a lower BCK-semilattice we denote $x \wedge y = glb\{x, y\}$.

Definition 2.6 ([12, Definition 1.4.1])**.** A subset A of a BCI-algebra X is called an ideal of X if (i) $0 \in A$, (ii) $x \in A$ and $y*x \in A$ imply $y \in A$ for any $x, y \in X$.

Note that X and $\{0\}$ are ideals of X, and they are called the trivial ideals of X.

Theorem 2.7 ([8, Theorem 1.2])**.** *Let A be an ideal of a BCK-algebra X. Then for any $x, y \in X$, $x \in A$ and $y \leq x$ we have $y \in A$.*

Definition 2.8 ([12, Definition 1.4.2])**.** An ideal A of a BCI-algebra X is called closed if A is closed under the multiplication on X.

Definition 2.9 ([12, Definition 1.4.3])**.** Suppose that S is a subset of a BCI-algebra X. The least ideal of X, containing S, is called the generated ideal of X by S and denoted by $\langle S \rangle$ or $(S]$. An ideal A of a BCK-algebra X is said to be finitely generated if there is a finite subset S of X such that $A = \langle S \rangle$. The ideal $\langle a \rangle$ generated by one generator a is also called a *principal ideal* of X.

Theorem 2.10 ([8, Theorem 5.5])**.** *A BCK-algebra X is commutative if and only if $(X; \leq)$ is a lower semilattice with $x \wedge y = y * (y * x)$ for any x, $y \in X$.*

Definition 2.11 ([8, Definition 7.8]). A BCK-algebra X is said to be Noetherian if each ideal of X is finitely generated.

Definition 2.12 ([8, Definition 7.5]). For a given BCK-algebra X, we say that X satisfies the ascending chain condition, abbreviated by ACC, if there does not exists an infinite properly ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ in $\mathcal{I}_X$.

Theorem 2.13 ([8, Theorem 7.9]). *Given a BCK-algebra X, the following are equivalent*:
(i) *X is Noetherian*;
(ii) *X satisfies ACC.*

Definition 2.14 ([12, Definition 1.6.1]). Suppose $(X;*,0)$ and $(X';*',0')$ are two BCK-algebras. A mapping $f : X \longrightarrow X'$ is called a homomorphism from X into X' if, for any x, y $\in X$

$$f(x * y) = f(x) *' f(y).$$

In addition, if the mapping f is onto, then f is called an epimorphism and the mapping is called an isomorphism if it is both an epimorphism and one-to-one.

Proposition 2.15 ([12, Proposition 1.6.3]). *Let f be a BCI-homomorphism from X to X'. Then f is isomorphic if and only if the inverse mapping f^{-1} is isomorphic.*

Definition 2.16 ([12, Definition 1.5.1]). An equivalence relation θ on a BCI-algebra X is called a congruence on X if it has the substitution property:

$$x \sim y(\theta), u \sim v(\theta) \text{ imply } x * u \sim y * v(\theta)$$

for any $x, y, u, v \in X$.

If θ is an equivalence relation on a BCI-algebra X, we denote θ_x for the equivalence class containing x. It means that $\theta_x = \{y \in X \mid y \sim x(\theta)\}$. Also we denote $\frac{X}{\theta}$ for the quotient set $\{\theta_x \mid x \in X\}$. If θ is a congruence on X, the operation $*$ on $\frac{X}{\theta}$ given by $\theta_x * \theta_y = \theta_{x*y}$ is well-defined. Then $(\frac{X}{\theta}, *, \theta_0)$ is an algebra which is called the quotient algebra of X induced by θ.

Definition 2.17 ([12, Definition 5.1]). A BCK-algebra X is called commutative if

$$x * (x * y) = y * (y * x)$$

for any $x, y \in X$.

Definition 2.18 ([12, Definition 2.4.2]). A BCK-algebra X is called n-fold commutative if there exists a fixed natural number n such that the following identity holds:

$$x * y = x * (y * (y * x^n)).$$

Definition 2.19 ([12, Definition 2.4.3]). A BCK-algebra X is called multiply commutative if for any $x, y \in X$, there exists a fixed natural number $n = n(x, y)$ such that

$$x * y = x * (y * (y * x^n)).$$

Definition 2.20 ([12, Definition 2.5.3])**.** An ideal A of a BCI-algebra X is called commutative if $x * y \in A$ implies

$$x * ((y * (y * x)) * (0 * (y * x))) \in A$$

for any $x, y \in X$.

Theorem 2.21 ([12, Theorem 1.6.5])**.** *Let $f : X \longrightarrow X'$ be an epimorphism and A be an ideal of X. Then $f(A)$ is an ideal of X'.*

Theorem 2.22 ([12])**.** *Let $f : X \longrightarrow X'$ be an BCI-homomorphism and A be a subset of X, and A' a subset of X'. Then*

(i) *If A is a subalgebra of X, then $f(A)$ is a subalgebra of X'.*
(ii) *If A' is a subalgebra of X', then $f^{-1}(A')$ is a subalgebra of X.*
(iii) *If A' is an ideal of X', then $f^{-1}(A')$ is an ideal of X.*
(iv) *If A' is a closed ideal of X', then $f^{-1}(A')$ is a closed ideal of X.*

Theorem 2.23 ([12, Theorem 2.5.6])**.** *An ideal A of a BCK-algebra X is commutative if and only if the quotient algebra $\frac{X}{A}$ is a commutative BCK-algebra.*

Definition 2.24 ([3, Definition 3.1])**.** By an operation "*d*" on "$\mathcal{I}_X$" the set of all ideals of a BCK-algebra X, we mean a function $d : \mathcal{I}_X \longrightarrow \mathcal{I}_X$, and for simplicity of notation for any $A \in I_X$ we write $d(A) = A^d$.

Definition 2.25 ([3, Definition 3.2])**.** Let X be a BCK-algebra. A closure operation "*cl*" on the set $\mathcal{I}_X$ of all ideals of X, is an operation $cl : \mathcal{I}_X \longrightarrow \mathcal{I}_X$ such that $A \longmapsto A^{cl}$ satisfying the following conditions:

(i) $A \subseteq A^{cl}$ for all $A \in \mathcal{I}_X$ (Extension).
(ii) $A^{cl} = (A^{cl})^{cl}$ for all $A \in \mathcal{I}_X$ (Idempotence).
(iii) If A and B are in $\mathcal{I}_X$ and $B \subseteq A$, then $B^{cl} \subseteq A^{cl}$ (Order-preservation).

Definition 2.26 ([3, Definition 3.3])**.** We say that an ideal A in $\mathcal{I}_X$ is "*cl-closed*" if $A = A^{cl}$. Therefore for any ideal A of X, A^{cl} is "*cl-closed*".

Definition 2.27 ([3, Definition 3.5])**.** Let A and B be ideals of a lower BCK-semilattice X. Define

$$A \wedge B = \langle \{x \wedge y \,|\, x \in A, y \in B\} \rangle.$$

Also, if $x \in X$, then $x \wedge B = \{x\} \wedge B = \langle \{x \wedge y \,|\, y \in B\} \rangle$.

Definition 2.28 ([3, Definition 3.6])**.** Let X be a lower BCK-semilattice and $\sum \subseteq \mathcal{I}_X$. Then we say that $\sum$ is $\wedge$-closed if $A \wedge B \in \sum$ for any two ideals $A, B \in \sum$.

Remark 2.29 ([3, Definition 3.7]). (i) From the above definition we get that:

$$A^2 = A \wedge A = \langle \{x \wedge y \,|\, x, y \in A\} \rangle \text{ and } A^3 = A^2 \wedge A, \ldots .$$

(ii) In a lower BCK-semilattice (specially, in a commutative BCK-algebra), we have

$$\cdots \subseteq A^3 \subseteq A^2 \subseteq A,$$

because a commutative BCK-algebra is a lower BCK-semilattice and $x \wedge y = x * (x * y) \leq x$ for any $x, y \in X$.

Definition 2.30 ([3, Definition 3.22]). Let cl_1 and cl_2 be two closure operations on a BCK-algebra X. Then we write $cl_1 \leq cl_2$ if for every ideal A, $A^{cl_1} \subseteq A^{cl_2}$.

Lemma 2.31 ([3, Definition 4.1]). *Let "c" be a closure operation. Consider "c_f" by setting $A^{c_f} = \bigcup\{B^c \mid B$ is a finitely generated ideal such that $B \subseteq A\}$. Then "c_f" is a closure operation.*

Definition 2.32 ([3, Definition 4.2]). If $c = c_f$, we say that "c" is a closure operation of finite type.

3. Semi-prime closure operations

In this section, we define some types of semi-prime closure operations on ideals and we give several theorems that make different (good) semi-prime closure operations, especially on a Noetherian BCK-algebra.

Definition 3.1. Let "cl" be a closure operation on a lower BCK-semilattice X. We say that "cl" is;

(i) A semi-prime closure operation, if for every two ideals A and B of X, we have

$$A \wedge B^{cl} \subseteq (A \wedge B)^{cl},$$

$$A^{cl} \wedge B \subseteq (A \wedge B)^{cl}.$$

(ii) A good semi-prime closure operation, if for every two ideals A and B of X, we have

$$A \wedge B^{cl} = A^{cl} \wedge B = (A \wedge B)^{cl}.$$

Remark 3.2. Remember that for ideals A and B,

$$A \wedge B = \langle \{x \wedge y \mid x \in A, y \in B\} \rangle.$$

Proposition 3.3. *Let A and B be two arbitrary ideals of a lower BCK-semilattice X. Then we have $A \wedge B = B \wedge A$. It means that "$\wedge$" is commutative on ideals in every lower BCK-semilattice.*

Proof. Suppose that $x \wedge y$ is a generator of $A \wedge B$. Then since we have $x \wedge y \leq x$, $x \wedge y \leq y$, $x \in A$, $y \in B$ and A, B are ideals of X, we conclude that $x \wedge y \in B$ and $x \wedge y \in A$, by Theorem 2.7. Thus

$$x \wedge y = (x \wedge y) \wedge (x \wedge y) \in B \wedge A.$$

Therefore $A \wedge B \subseteq B \wedge A$. Similarly we have $B \wedge A \subseteq A \wedge B$ and the proof is complete. □

Remark 3.4. An important point is involved in Definition 3.1 that, by using Proposition 3.3 if for every two ideals A and B in a lower BCK-semilattice X, $A \wedge B^{cl} \subseteq (A \wedge B)^{cl}$ or $A^{cl} \wedge B \subseteq (A \wedge B)^{cl}$, then "*cl*" is a semi-prime closure operation. Because if $A \wedge B^{cl} \subseteq (A \wedge B)^{cl}$, then $A^{cl} \wedge B = B \wedge A^{cl} \subseteq (B \wedge A)^{cl} = (A \wedge B)^{cl}$.

Therefore "*cl*" is a semi-prime closure operation, precisely when for every two ideals A and B of X, we have

$$A \wedge B^{cl} \subseteq (A \wedge B)^{cl}.$$

Example 3.5. Suppose that X is the set $\{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

Then X is a lower BCK-semillatice with 4 as the greatest element and it has 5 ideals $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 2, 3\}$ and $A_4 = \{0, 1, 2, 3, 4\} = X$ as we can see below:

Define "*cl*" on ideals as follows:

$$(A_0)^{cl} = A_0,\ (A_1)^{cl} = A_2,\ (A_2)^{cl} = A_2,\ (A_3)^{cl} = A_4 \text{ and } (A_4)^{cl} = A_4.$$

One can easily see that "*cl*" is a semi-prime closure operation which is not good, because $(A_1 \wedge A_3)^{cl} = (A_1)^{cl} = A_2$, $A_1^{cl} \wedge A_3 = A_2 \wedge A_3 = A_2$ and $A_1 \wedge A_3^{cl} = A_1 \wedge A_4 = A_1$. Hence

$$(A_1 \wedge A_3)^{cl} = A_1^{cl} \wedge A_3 \neq A_1 \wedge A_3^{cl}.$$

Therefore it is not a good semi-prime closure operation.

Example 3.6. In Example 3.5, if we define "*cl*" as an identity closure operation, which means that

$$(A_0)^{cl} = A_0,\ (A_1)^{cl} = A_1,\ (A_2)^{cl} = A_2,\ (A_3)^{cl} = A_3 \text{ and } (A_4)^{cl} = A_4,$$

then "*cl*" is a good semi-prime closure operation.

Proposition 3.7. *Let "cl" be a closure operation on a lower BCK-semilattice X. Then the following statements are equivalent*:
(i) *"cl" is a semi-prime closure operation.*
(ii) *For all ideals A and B, $(A^{cl} \wedge B^{cl})^{cl} = (A \wedge B)^{cl}$.*

Proof. Suppose that (i) holds. By extension property of closure operation, $A \wedge B \subseteq A^{cl} \wedge B^{cl}$. Hence by order-preservation property of "*cl*", $(A \wedge B)^{cl} \subseteq (A^{cl} \wedge B^{cl})^{cl}$. For the converse, since "*cl*" is a semi-prime closure operation, $A^{cl} \wedge B^{cl} \subseteq (A^{cl} \wedge B)^{cl}$ and $A^{cl} \wedge B \subseteq (A \wedge B)^{cl}$. Therefore

$$(A^{cl} \wedge B^{cl})^{cl} \subseteq ((A^{cl} \wedge B)^{cl})^{cl} = (A^{cl} \wedge B) \subseteq (A \wedge B)^{cl}.$$

Now if (ii) holds, then by extension property of closure operation

$$A \wedge B^{cl} \subseteq A^{cl} \wedge B^{cl} \subseteq (A^{cl} \wedge B^{cl})^{cl} = (A \wedge B)^{cl}.$$

Similarly

$$A^{cl} \wedge B \subseteq (A \wedge B)^{cl}.$$

Therefore "*cl*" is a semi-prime closure operation. □

Remark 3.8. We can check that Proposition 3.7 holds for the BCK-algebra X in Example 3.5. It means that for all ideals A and B, since "*cl*" is a semi-prime closure operation we should have $(A^{cl} \wedge B^{cl})^{cl} = (A \wedge B)^{cl}$. For instance, $(A_2^{cl} \wedge A_3^{cl})^{cl} = (A_2 \wedge A_4)^{cl} = (A_2)^{cl} = A_2$ and $(A_2 \wedge A_3)^{cl} = (A_2)^{cl} = A_2$. Thus $(A_2^{cl} \wedge A_3^{cl})^{cl} = (A_2 \wedge A_3)^{cl}$.

Lemma 3.9. *Suppose that $(X_1, *), (X_2, *')$ are two lower BCK-semilattices with $x \wedge y = y * (y * x)$ and $f : X_1 \longrightarrow X_2$ is a BCK-homomorphism from X_1 into X_2. If A and B are ideals of X_1, then*

$$f(A \wedge B) = f(A) \wedge f(B).$$

Proof. Since $A \wedge B = \langle \{x \wedge y \,|\, x \in A, y \in B\} \rangle$ and $x \wedge y = y * (y * x)$, we have

$$f(x \wedge y) = f(y * (y * x)) = f(y) *' (f(y * x)) = f(y) *' (f(y) *' f(x)) = f(x) \wedge f(y).$$

Therefore $f(A \wedge B) = f(A) \wedge f(B)$. □

Theorem 3.10. *Let $\varphi : X_1 \longrightarrow X_2$ be a BCK-epimorphism between two lower BCK-semilattices X_1 and X_2 with $x \wedge y = y * (y * x)$ and "cl" be a semi-prime closure operation on X_2. For an ideal A of X_1, define $A^c = \varphi^{-1}((\varphi(A))^{cl})$. Then "c" is a semi-prime closure operation on X_1.*

Proof. Since "*cl*" is a semi-prime closure operation, so for any two arbitrary ideals A and B of X_1 we have

$$(\varphi(A)) \wedge (\varphi(B))^{cl} \subseteq (\varphi(A) \wedge \varphi(B))^{cl} = (\varphi(A \wedge B))^{cl}, \text{ by Lemma 3.9.}$$

Hence, $\varphi^{-1}(\varphi(A) \wedge \varphi(B)^{cl}) \subseteq \varphi^{-1}(\varphi(A \wedge B)^{cl})$. Also we have

$$A \wedge (\varphi^{-1}(\varphi(B)^{cl})) \subseteq \varphi^{-1}((\varphi(A)) \wedge (\varphi^{-1}(\varphi(B)^{cl}))) \subseteq \varphi^{-1}(\varphi(A) \wedge \varphi(B)^{cl}).$$

Therefore $A \wedge B^c \subseteq (A \wedge B)^c$ and "c" is a semi-prime closure operation. □

Theorem 3.11. *Let $\varphi : X_1 \longrightarrow X_2$ be a BCK-epimorphism between lower BCK-semilattices X_1 and X_2 with $x \wedge y = y*(y*x)$ and let "cl" be a semi-prime closure operation on X_1. For each ideal A of X_2, define $A^c = \varphi((\varphi^{-1}(A))^{cl})$. Then "c" is a semi-prime closure operation on X_2.*

Proof. The proof is similar to the proof of Theorem 3.10 by imposing the suitable modification. □

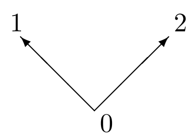
Remark 3.12. In Theorems 3.10 and 3.11, if φ is an isomorphism and "*cl*" is a good semi-prime closure operation, then "*c*" is a good semi-prime closure operation too. The proof is straightforward and left to the reader.

Now we give an example to show that it is necessary to have the isomorphism condition in Remark 3.12.

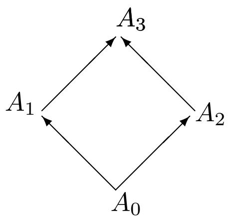
Example 3.13. Suppose that X_1 is the set $\{0, 1, 2\}$. Define a binary operation $*$ on X_1 by the following Cayley table:

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

It is easy to seen that X_1 is a lower BCK-semilattice with $x \wedge y = y * (y * x)$ (because it is commutative). Also we have $0 \leq 1$, $0 \leq 2$ and $1 \wedge 2 = 0$, that is:



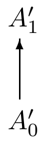
The BCK-semilattice X_1 has 4 ideals, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$ and $A_3 = \{0, 1, 2\} = X_1$:



Also suppose that X_2 is the set $\{0, 1\}$. Define a binary operation $*'$ on X_2 by the following Cayley table:

$*$	0	1
0	0	0
1	1	0

It is clear that X_2 is a lower BCK-semilattice with $x \wedge y = y * (y * x)$. The lower BCK-semilattice X_2 has only 2 ideals, $A'_0 = \{0\} \subseteq A'_1 = \{0, 1\} = X_2$:



Let $\varphi : X_1 \longrightarrow X_2$ be the mapping such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(2) = 0$. Then $\text{Im}(\varphi) = X_2$ and routine verification gives that φ is an epimorphism, but it is not isomorphism. Now define a closure operation "*cl*" on ideals of X_2 as follows:

$$(A'_0)^{cl} = A'_0,\ (A'_1)^{cl} = A'_1.$$

It is easy to check that "*cl*" is a good semi-prime closure operation. Also we have $\varphi(A_0) = \varphi(0) = \{0\} = A'_0$. So

$$\varphi(A_0)^{cl} = (A'_0)^{cl} = A'_0 = \{0\} \Rightarrow \varphi^{-1}(\varphi(A_0)^{cl}) = \varphi^{-1}(0) = \{0, 2\} = A_2.$$

Therefore $A_0^c = A_2$. A similar way show that $A_1^c = A_3, A_2^c = A_2$ and $A_3^c = A_3$.

Since "*cl*" is a semi-prime closure operation on X_2 and φ is an epimorphism, by Theorem 3.10 "*c*" is a semi-prime closure operation too. But "*c*" is not a good semi-prime closure operation because,

$$(A_0 \wedge A_1)^c = A_0^c = A_2$$

and

$$A_0 \wedge A_1^c = A_0 \wedge A_3 = A_0,\ A_0^c \wedge A_1 = A_2 \wedge A_1 = A_0$$

therefore

$$(A_0 \wedge A_1)^c \neq A_0^c \wedge A_1 = A_0 \wedge A_1^c.$$

Lemma 3.14. *Suppose that* $\{c_\lambda\}_{\lambda\in\Lambda}$ *is a set of closure operations on a lower BCK-semilattice* X *such that for each* λ*, the operation "*c_λ*" be a (good) semi-prime closure operation. Then the operation "*c*" defined by* $A^c = \bigcap_{\lambda\in\Lambda} A^{c_\lambda}$ *is a (good) semi-prime closure operation.*

Proof. By using Lemma 3.21 of [3], "c" is a closure operation. Since for each λ and for any two arbitrary ideals A and B of X, $A \wedge B^{c_\lambda} \subseteq (A\wedge B)^{c_\lambda}$. Hence

$$\bigcap_{\lambda\in\Lambda}(A\wedge B^{c_\lambda}) \subseteq \bigcap_{\lambda\in\Lambda}(A\wedge B)^{c_\lambda}.$$

So $A\wedge(\bigcap_{\lambda\in\Lambda}(B^{c_\lambda})) \subseteq \bigcap_{\lambda\in\Lambda}(A\wedge B)^{c_\lambda}$. It means that $A\wedge B^c \subseteq (A\wedge B)^c$. Therefore "$c$" is a semi-prime closure operation.

The proof of the good case is similar to above. □

Definition 3.15. Suppose that $\{c_\lambda\}_{\lambda\in\Lambda}$ is a set of closure operations. We say that it is a direct set if for any $\lambda_1, \lambda_2 \in \Lambda$, there exists some $\mu \in \Lambda$ such that $c_{\lambda_i} \leq c_\mu$ for $i = 1, 2$.

Theorem 3.16. *Suppose that* $\{c_\lambda\}_{\lambda\in\Lambda}$ *is a set of closure operations on a lower BCK-semilattice* X *such that for each* λ*, the operation "*c_λ*" is a (good) semi-prime closure operation. Let* $\{c_\lambda\}_{\lambda\in\Lambda}$ *be a direct set of closure operation and assume that every ideal of* X *is finitely generated. Then the operation "*c*" defined by* $A^c = \bigcup_{\lambda\in\Lambda} A^{c_\lambda}$ *is a (good) semi-prime closure operation.*

Proof. The proof is similar to the proof of Lemma 3.14, by using Lemma 3.24 of [3]. □

Theorem 3.17. *Let "*c*" be a semi-prime closure operation and* X *be a lower BCK-semilattice. Consider "*c_f*",* $A^{c_f} = \bigcup\{B^c \mid B$ *is a finitely generated ideal such that* $B \subseteq A\}$*. Then "*c_f*" is a semi-prime closure operation.*

Proof. It is enough to show that for arbitrary ideals A and B, $A \wedge B^{c_f} \subseteq (A\wedge B)^{c_f}$. Suppose that $a \in A$ and $b \in B^{c_f}$. By definition of "c_f" there exists a finitely generated ideal K such that $K \subseteq B$ and $b \in K^c$. Since "c" is a semi-prime closure operation, so

$$a \wedge b \in (a] \wedge K^c \subseteq ((a] \wedge K)^c.$$

Also, $(a] \wedge K$ is a finitely generated ideal and $(a] \wedge K \subseteq A \wedge B$. Thus, $a \wedge b \in (A\wedge B)^{c_f}$. Therefore $A \wedge B^{c_f} \subseteq (A\wedge B)^{c_f}$. □

The next proposition shows that Theorem 3.17 is true for a good semi-prime closure operation on finitely generated ideals.

Proposition 3.18. *Suppose that "*c*" is a good semi-prime closure operation and* X *be a lower BCK-semilattice. Then "*c_f*" is a good semi-prime closure operation on finitely generated ideals of* X*.*

Proof. From the proof of Theorem 3.17 we can conclude that $A \wedge B^{c_f} \subseteq (A \wedge B)^{c_f}$ and $A^{c_f} \wedge B \subseteq (A \wedge B)^{c_f}$. If $x \in (A \wedge B)^{c_f}$, then there is a finitely generated ideal K such that $K \subseteq (A \wedge B)$ and $x \in K^c$. Since "c" is a good semi-prime closure operation, $K^c \subseteq (A \wedge B)^c = A \wedge B^c$. Hence $x \in A \wedge B^c$. So there exist $a \in A$ and $b \in B^c$ such that $x = a \wedge b$. Also we have $b \in B^{c_f}$, because B is a finitely generated ideal and $b \in B^c$. Therefore $x = a \wedge b \in A \wedge B^{c_f}$ and $(A \wedge B)^{c_f} \subseteq A \wedge B^{c_f}$. Similarly, we have $(A \wedge B)^{c_f} \subseteq A^{c_f} \wedge B$. Thus "$c_f$" is a good semi-prime closure operation. □

Remark 3.19. Since in a Noetherian BCK-algebra every ideal is finitely generated, Lemma 3.14, Theorems 3.16 and 3.17 are true for every Noetherian lower BCK-semilattice.

Definition 3.20. Let X be a lower BCK-semilattice and A, B be two ideals of X. Then the notion of meet quotient ideal $(A :_\wedge B)$ is defined by

$$(A :_\wedge B) = \langle \{x \in X : x \wedge B \subseteq A\} \rangle.$$

Clearly this is another ideal of X and $A \subseteq (A :_\wedge B)$.

In the special case in which $A = 0$, the ideal meet quotient

$$(0 :_\wedge B) = \langle \{x \in X : x \wedge B = 0\} \rangle = \langle \{x \in X : x \wedge b = 0 \text{ for all } b \in B\} \rangle$$

is called the annihilator of B and is also denoted by B^*.

Lemma 3.21. *Let A, B and C be ideals of a commutative BCK-algebra X, and let $(A_\lambda)_{\lambda \in \Lambda}$ be a family of ideals of X. Then*

(i) $((A :_\wedge B) :_\wedge C) = (A :_\wedge B \wedge C) = ((A :_\wedge C) :_\wedge B)$;

(ii) $(\cap_{\lambda \in \Lambda} A_\lambda :_\wedge C) = \cap_{\lambda \in \Lambda}(A_\lambda :_\wedge C)$.

Proof. (i) Suppose that $x \in ((A :_\wedge B) :_\wedge C)$, then $x \wedge C \subseteq (A :_\wedge B)$. Hence $(x \wedge C) \wedge B \subseteq A$. Since X is a commutative BCK-algebra, $(x \wedge C) \wedge B = x \wedge (C \wedge B)$. Therefore $x \wedge (C \wedge B) \subseteq A$ and $x \in (A :_\wedge B \wedge C)$. The inverse is clear.

A similar way with commutativity of $\wedge$ show that

$$((A :_\wedge B) :_\wedge C) = ((A :_\wedge C) :_\wedge B).$$

(ii) Let $x \in (\cap_{\lambda \in \Lambda} A_\lambda :_\wedge C)$. Then $x \wedge C \subseteq (\cap_{\lambda \in \Lambda} A_\lambda)$. It means that for each $c \in C$, $x \wedge c \in (\cap_{\lambda \in \Lambda} A_\lambda)$. So we have $x \wedge c \in A_\lambda$ for each $\lambda \in \Lambda$. Thus $x \wedge C \subseteq A_\lambda$ and $x \in (A_\lambda :_\wedge C)$ for each $\lambda \in \Lambda$. Therefore $x \in \cap_{\lambda \in \Lambda}(A_\lambda :_\wedge C)$. □

Theorem 3.22. *Let A and B be two arbitrary ideals of a BCK-algebra X. Then*

(i) *if A is a commutative ideal, then $(A :_\wedge B)$ is a commutative ideal,*

(ii) *if A is an n-fold (multiply) commutative ideal, then $(A :_\wedge B)$ is an n-fold (multiply) commutative ideal,*

(iii) *if A is an implicative ideal, then $(A :_\wedge B)$ is an implicative ideal,*

(iv) *if A is a positive implicative ideal, then $(A :_\wedge B)$ is a positive implicative ideal.*

Proof. Since $A \subseteq (A :_{\wedge} B)$, by using Theorems 2.6, 3.6 and 4.5 of [8] the proof is straightforward. □

Theorem 3.23. *Let X be a commutative BCK-algebra, A be a prime ideal of X and $B \neq 0$ be an ideal of X. Then $(A :_{\wedge} B)$ is a prime ideal.*

Proof. Suppose that $x \wedge y \in (A :_{\wedge} B)$ and $x \notin (A :_{\wedge} B)$. Then $(x \wedge y) \wedge B \subseteq A$ and there exists $b \in B$ such that $x \wedge b \notin A$. Since X is a commutative BCK-algebra, $(x \wedge y) \wedge b = x \wedge (y \wedge b) = x \wedge (b \wedge y) = (x \wedge b) \wedge y \in A$. Hence $y \in A$ because A is a prime ideal.

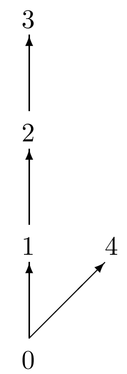
For each $b \in B$, $y \wedge b \leq y$. So $y \wedge b \in A$ because A is an ideal and $y \in A$. Therefore $y \in (A :_{\wedge} B)$ and $(A :_{\wedge} B)$ is a prime ideal. □

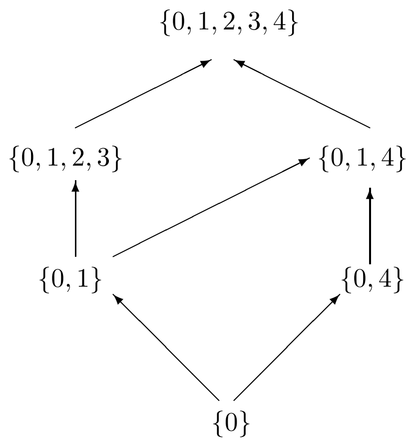
In Theorem 3.23, if X is a lower BCK-semilattice and A is a prime ideal, then $(A :_{\wedge} B)$ is not a prime ideal necessary. It means that X should be a commutative BCK-algebra. Let us illustrate the point with an example.

Example 3.24. Suppose that X is the set $\{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

X is a lower BCK-semillatice and it has 6 ideals $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 4\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = \{0, 1, 2, 3, 4\} = X$ as we can see below:





X is not a commutative BCK-algebra because $2\wedge 1 = 1\wedge 2 = 1$ but $2*(2*1) = 2*2 = 0$. Hence $1\wedge 2 \neq 2*(2*1)$. It is easy to check that $P = A_3 = \{0, 1, 2, 3\}$ is a prime ideal.

By routine verification we can check that

$$(P :_\wedge A) = \{x \in X; x \wedge A \subseteq P\} = X.$$

Therefore $(P :_\wedge A)$ can not be a prime ideal because it is not proper.

Theorem 3.25. *Let "cl" be a (good) semi-prime closure operation on a lower BCK-semilattice X and A, B be two ideals of X. Then*

(i) $(A :_\wedge B)^{cl} \subseteq (A^{cl} :_\wedge B)$.

(ii) *If A is a cl-closed ideal, then $(A :_\wedge B)$ is a cl-closed ideal.*

(iii) *$(A^{cl} :_\wedge B)$ is a cl-closed ideal.*

Proof. (i) Suppose that $x \in (A :_\wedge B)^{cl}$, then $B \wedge x \subseteq B \wedge (A :_\wedge B)^{cl}$. Since "cl" is a semi-prime closure operation,

$$B \wedge (A :_\wedge B)^{cl} \subseteq (B \wedge (A :_\wedge B))^{cl}.$$

Also, $B \wedge (A :_\wedge B) \subseteq A$. Hence $(B \wedge (A :_\wedge B))^{cl} \subseteq A^{cl}$. Therefore $B \wedge x \subseteq A^{cl}$ and $x \in (A^{cl} :_\wedge B)$.

(ii) If A is a "cl-closed" ideal, then

$$(A :_\wedge B)^{cl} \subseteq (A^{cl} :_\wedge B) = (A :_\wedge B).$$

Therefore $(A :_\wedge B)$ is a "cl-closed" ideal.

(iii) Since A^{cl} is a "cl-closed" ideal, the proof is clear by (ii). □

Theorem 3.26. *Suppose that "cl" is a (good) semi-prime closure operation on a lower BCK-semilattice X, A is an ideal and S is a $\wedge$-close subset of X.*

If X is a Noetherian lower BCK-semilattice and A is a "cl-closed" ideal, then $B = \langle\{x \in X \mid$ there exists $s \in S, x \wedge s \in A\}\rangle$ is a "cl-closed" ideal.

Proof. Since X is a Noetherian BCK-algebra, we have $B = (b_1, b_2, \ldots, b_n]$ where $b_i \in X$, $i = 1, 2, \ldots, n$. So there exists $s_i \in S$ for each $i = 1, 2, \ldots, n$, such that $b_i \wedge s_i \in A$. Put $P = s_1 \wedge s_2 \wedge \cdots \wedge s_n$, we have $P \subseteq S$ and $B \wedge P \subseteq A$. Hence $B \subseteq (A :_\wedge P)$. Now suppose that $x \in (A :_\wedge P)$, then $x \wedge P \subseteq A$ which means that $x \in B$. So $(A :_\wedge P) \subseteq B$. Therefore $(A :_\wedge P) = B$.

Since A is a "cl-closed" ideal, B is a "cl-closed" ideal too, by Theorem 3.25(ii). □

Theorem 3.27. *Let "cl" be a (good) semi-prime closure operation on a lower BCK-semilattice X with 1 as the greatest element and A be an ideal of X. Then the maximal elements of the set $H = \{A : A^{cl} = A \neq X\}$ are prime ideals.*

Proof. Suppose that B is a maximal element of H and $x, y \in X$. If $x \wedge y \in B$ and $x \notin B$, then $x \in (B :_\wedge y)$. So $B \subset (B :_\wedge y)$. By Theorem 3.25(ii), $(B :_\wedge y)$ is a "cl-closed" ideal. The maximality of B shows that $(B :_\wedge y) = X$. Since $1 \in X = (B :_\wedge y)$, so $1 \wedge y = y \in B$. Therefore B is a prime ideal. □

Notation 3.28. Let "cl" be a (good) semi-prime closure operation on a lower BCK-semilattice X. Then we denote the set of maximal "c_f-closed" ideals by "c_f-Max X".

Corollary 3.29. *Suppose that "cl" be a semi-prime closure operation on a lower BCK-semilattice X with 1 as the greatest element. Then*

(i) *every member of "c_f-Max X" is a prime ideal.*

(ii) *every "c_f-closed" ideal is contained in one of the member of "c_f-Max X".*

Proof. (i) By using Theorem 3.17, "c_f" is a semi-prime closure operation too. Now Theorem 3.27, indicates that every member of "c_f-Max X" is a prime ideal.

(ii) Theorem 4.4 of [3], shows that, "c_f" is a finite-type closure operation. By using Theorem 4.6 of [3], we have every "c_f-closed" ideal is contained in one of the member of "c_f-Max X". □

Remark 3.30. By using Proposition 3.18, we can conclude that

(i) If "*cl*" is a good semi-prime closure operation on a lower BCK-semilattice X with 1 as the greatest element, then Corollary 3.29 holds for finitely generated ideals of X.

(ii) Corollary 3.29 holds for every Noetherian lower BCK-semilattice.

Lemma 3.31. *Let X be a lower BCK-semilattice and P be a prime ideal. Then $X - P$ is a "$\wedge$-closed" subset of X.*

Proof. Suppose that $x, y \in X - P$. On the contrary suppose that $x \wedge y \notin X - P$. Then $x \wedge y \in P$. Since P is a prime ideal, we have $x \in P$ or $y \in P$ which is a contradiction. □

Theorem 3.32. *Suppose that A is an ideal of a lower BCK-semilattice X with 1 as the greatest element and P is a prime ideal. Define "cl_P" as follows:*

$$A^{cl_P} = \bigcup_{d\in X-P} (A :_\wedge (d]).$$

Then "cl_P" is a semi-prime closure operation.

Proof. It is clear that $A \subseteq A^{cl_P}$. Now suppose that A and B are two ideals of X such that $A \subseteq B$. Then $\bigcup_{d\in X-P}(A :_\wedge (d]) \subseteq \bigcup_{d\in X-P}(B :_\wedge (d])$. Hence $A^{cl_P} \subseteq B^{cl_P}$. Put $H = A^{cl_P} = \bigcup_{d\in X-P}(A :_\wedge (d])$ and let $\alpha \in (A^{cl_P})^{cl_P} = H^{cl_P} = \bigcup_{d\in X-P}(H :_\wedge (d])$. Then there exists $s \in X - P$ such that $\alpha \wedge s \in H$. Hence there exists $t \in X - P$ such that $(\alpha \wedge s) \wedge t \in A$. By using Lemma 3.31, $(s \wedge t) \in X - P$ and $\alpha \wedge (s \wedge t) \in A$ which means that $\alpha \in A^{cl_P}$. Therefore "cl_P" is a closure operation.

Now suppose that A and B are two ideals of X. Then it is clear that $A \wedge (B :_\wedge (d]) \subseteq (A \wedge B :_\wedge (d])$. Hence

$$A \wedge (\bigcup_{d\in X-P} (B :_\wedge (d]) \subseteq \bigcup_{d\in X-P} (A \wedge B :_\wedge (d])$$

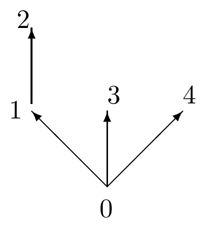
and $A \wedge B^{cl_P} \subseteq (A \wedge B)^{cl_P}$. Therefore "$cl_P$" is a semi-prime closure operation. □

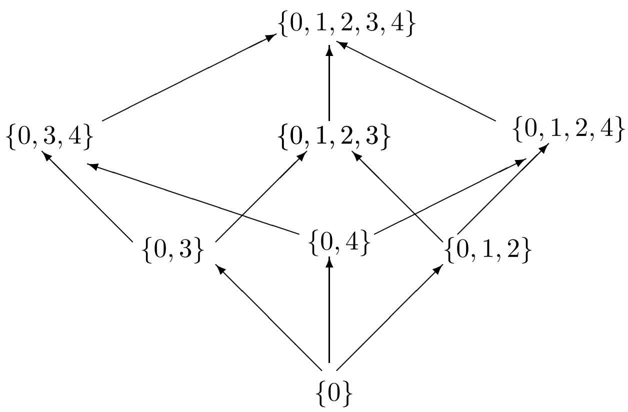
The next example shows that Theorem 3.32 is not true for a good semi-prime closure operation.

Example 3.33. Suppose that X is the set $\{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

The lower BCK-semillatice X is commutative and it has 8 ideals:





Let $P = \{0,3,4\}$. Then P is a prime ideal (one can easily check it). So $X - P = \{1,2\}$. Also $(1] = \{x \in X : x * 1^n = 0\} = \{0,1,2\}$ and $(2] = \{x \in X : x * 2^n = 0\} = \{0,1,2\}$. Put $A_0 = \{0\}$, $A_1 = \{0,3\}$ and $A_2 = \{0,1,2\}$. Then A_0, A_1 and A_2 are ideals and

$$(A_0 :_\wedge (1]) = \{x \in X : x \wedge (1] \subseteq \{0\}\} = \{0,3,4\},$$
$$(A_0 :_\wedge (2]) = \{x \in X : x \wedge (2] \subseteq \{0\}\} = \{0,3,4\},$$
$$(A_1 :_\wedge (1]) = \{x \in X : x \wedge (1] \subseteq \{0,3\}\} = \{0,3,4\},$$
$$(A_1 :_\wedge (2]) = \{x \in X : x \wedge (2] \subseteq \{0,3\}\} = \{0,3,4\}.$$

Also

$$(A_2 :_\wedge (1]) = \{x \in X : x \wedge (1] \subseteq \{0,1,2\}\} = \{0,1,2,3,4\},$$
$$(A_2 :_\wedge (2]) = \{x \in X : x \wedge (2] \subseteq \{0,1,2\}\} = \{0,1,2,3,4\}.$$

Hence $A_0^{cl_P} = (A_0 :_\wedge (1]) \cup (A_0 :_\wedge (2]) = \{0,3,4\}$, $A_1^{cl_P} = (A_1 :_\wedge (1]) \cup (A_1 :_\wedge (2]) = \{0,3,4\}$ and $A_2^{cl_P} = (A_2 :_\wedge (1]) \cup (A_2 :_\wedge (2]) = \{0,1,2,3,4\}$.

By using Theorem 3.32, cl_P is a semi-prime closure operation but it is not good, because $A_1^{cl_P} \wedge A_2 = \{0,3,4\} \wedge \{0,1,2\} = \{0\}$, $A_1 \wedge A_2^{cl_P} = \{0,3\} \wedge \{0,1,2,3,4\} = \{0,3\}$ and $(A_1 \wedge A_2)^{cl_P} = (\{0,3\} \wedge \{0,1,2\})^{cl_P} = (\{0\})^{cl_P} = \{0,3,4\}$. Therefore

$$A_1^{cl_P} \wedge A_2 \neq A_1 \wedge A_2^{cl_P} \neq (A_1 \wedge A_2)^{cl_P}.$$

Theorem 3.34. *Let "cl" be a semi-prime closure operation on ideals of a lower BCK-semilattice X with 1 as the greatest element. For each ideal A of X, define "cl_S" as follows*:

$$A^{cl_S} = \langle \{x \in X \mid \forall P \in c_f\text{-}Max\ X, \exists d \in X - P, d \wedge x \in A\}\rangle.$$

Then

(i) *"cl_S" is a semi-prime closure operation,*
(ii) $cl_S \leq c_f$.

Proof. (i) It is clear that $A^{cl_S} = \bigcap_{P \in c_f\text{-Max } X} A^{cl_P}$. Thus Theorem 3.32 and Lemma 3.14, show that "cl_S" is a semi-prime closure operation.

(ii) Let $x \in A^{cl_S}$. Then for each prime ideal $P \in c_f$-Max X, there exists $d_P \in X - P$ such that $d_P \wedge x \in A$. Suppose that $B = \langle \{d_P : P \in c_f\text{-Max } X\} \rangle$. Then $B \wedge x \subseteq A$. It is enough to prove that $B^{c_f} = X$.

Now on the contrary let B^{c_f} be a proper ideal of X. Then by Corollary 3.29, there exists an ideal P' in "c_f-Max X" such that $B^{c_f} \subseteq P'$. But $d_{P'} \in B \subseteq B^{c_f}$. Hence $d_{P'} \in P'$ and it is a contradiction. Therefore $B^{c_f} = X$.

So we have:

$$x = 1 \wedge x \in X \wedge x = B^{c_f} \wedge x \subseteq (B \wedge x)^{c_f} \subseteq A^{c_f}.$$

Therefore $x \in A^{c_f}$. □

Remark 3.35. Since every commutative BCK-algebra is a lower BCK-semilattice with $x \wedge y = y * (y * x)$, Lemma 3.9, Theorems 3.10, 3.11 hold for a commutative BCK-algebra too.

4. Conclusions

As we mentioned in the abstract, in this article we give the notions of semi-prime closure operation and good semi-prime closure operation. After that, we obtain some different semi-prime closure operations together with some more related results on Noetherian BCK-algebras, lower BCK-semilattices and commutative BCK-algebras. Also we define notions of meet quotient and "c_f-Max X" and obtain some results.

Now how can we define a prime closure operation on ideals of a BCK-algebra or some other types of closure operation and the relationship between them. Also if the ideal A has a especial property, then what can be conclude about the properties of closure operation of A.

References

[1] G. Birkhoff, *Lattice Theory*, third ed., American Mathematical Society Colloquium Publications, vol. XXV, Amer. Math. Soc., Providence, RI, 1967.

[2] H. Bordbar, M. M. Zahedi, and R. A. Borzooei, *Some properties of closure operation on (Noetherian) BCK-algebra*, 12th AHA congress, Greece, 2014.

[3] ______, *(A finite type) Closure operations on BCK-algebra*, Submitted.

[4] N. Epstein, *Reductions and special parts of closure*, J. Algebra **323** (2010), no. 8, 2209–2225.

[5] ______, *A Guide to Closure Operation in Commutative Algebra*, Progress in commutative algebra 2, Walter de Gruyter, Berlin, 2012.

[6] Y. Imai and K. Iseki, *On axiom systems of propositional calculi. XIV*, Proc. Japan Acad. **42** (1966), 19–22.

[7] D. Kirby, *Closure Operations on Ideals and Submodules*, J. London Math. Soc. **44** (1969), 283–291.

[8] J. Meng and Y. B. Jun, *BCK-Algebra*, Kyung Moon SA CO., 1994.

[9] E. H. Moore, *Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium, pp. 1–150, Yale University Press, New Haven, 1910.
[10] J. C. Vassilev, *Structure on the set of closure operations of a commutative ring*, J. Algebra **321** (2009), no. 10, 2737–2753.
[11] L. J. Ratliff, Δ-*closures of ideals and rings*, Trans. Amer. Math. Soc. **313** (1989), no. 1, 221–247.
[12] H. Yisheng, *BCI-Algebra*, Published by Science Press, 2006.

Hashem Bordbar
Faculty of Mathematics
Statistics and Computer Science
Shahid Bahonar University
Kerman, Iran
E-mail address: Bordbar.amirh@gmail.com

Mohammad Mehdi Zahedi
Department of Mathematics
Graduate University of Advanced Technology
Mahan-Kerman, Iran
E-mail address: zahedi_mm@kgut.ac.ir