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WEIERSTRASS POINTS ON HYPERELLIPTIC MODULAR CURVES

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Abstract. In this paper, we find all Weierstrass points on the hyperelliptic modular curves $X_1(N)$.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. At an arbitrary point P of X there is in general no function on X that has a pole of order less than or equal to g at P and is regular elsewhere. Those points P for which such a function exists are said to be *Weierstrass points*. There are only finitely many such points, and if $w(X)$ is their number, then $2g+2 \leq w(X) \leq g^3 - g$. As an immediate application, the set of Weierstrass points is an invariant of X which is useful in the study of the automorphism group of X and the fixed points of automorphisms.

Let $\mathbb{H}$ be the complex upper half plane and Γ be a congruence subgroup of the full modular group $\mathrm{SL}_2(\mathbb{Z})$. We consider the modular curve $X(\Gamma)$ obtained from compactification of the quotient space $\Gamma\backslash\mathbb{H}$ by adding finitely many points called *cusps*. For any integer $N \geq 1$, we have subgroups $\Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ defined by matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo N to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, respectively. We let $X(N)$, $X_1(N)$, $X_0(N)$ be the modular curves defined over $\mathbb{Q}$ associated to $\Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0(N)$, respectively. The X's are compact Riemann surfaces.

The Weierstrass points of $X_1(N)$ and $X_0(N)$ have been investigated by Atkin [1], Choi [2], Kilger [6], Kohnen [7, 8], Lehner and Newman [9], Ogg [12], Ono [13], and Rohrlich [14].

On the other hand, Mestre [10] proved that $X_1(N)$ is hyperelliptic if and only if $N = 13, 16, 18$. A curve is said to be *hyperelliptic* if its genus is greater than or equal to 2 and it admits a map of degree 2 to $\mathbb{P}^1$.

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In this paper, we find all Weierstrass points on the hyperelliptic modular curves $X_1(N)$.

2. Preliminaries

Let X be a non-singular algebraic curve of genus $g \geq 2$ and let P be a point on X. Consider the following vector spaces:

$$\mathcal{L}(0), \mathcal{L}(P), \mathcal{L}(2P), \mathcal{L}(3P), \ldots,$$

where $\mathcal{L}(kP)$ is the space of rational functions on X whose order at P is at least $-k$ and with no other poles. Then the dimension $\ell(kP)$ of $\mathcal{L}(kP)$ satisfies the following two conditions:

(1) $1 = \ell(0) \leq \ell(P) \leq \ell(2P) \leq \ell(3P) \leq \cdots$.
(2) $\ell(kP) = k - g + 1$ for $k \geq 2g - 1$.

Thus for a non-Weierstrass point P the sequence of $\ell(kP)$ is as follows:

$$1, 1, \ldots, 1, 2, 3, \ldots, g-1, g, g+1, \ldots.$$

A *Weierstrass gap* for P is a value of k such that no function on X has a pole only at P of order k. Thus the gap sequence is

$$1, 2, \ldots, g$$

for a non-Weierstrass point, and for a Weierstrass point its gap sequence contains at least one number greater than g. If X is a hyperelliptic curve, then there exist a function f and a point P such that f has a double pole only at P. Thus such a P has the gap sequence is

$$1, 3, 5, \ldots, 2g - 1. \tag{2.1}$$

If $1 = w_1, w_2, w_3, \ldots$ is a gap sequence of a point P, then we define the *Weierstrass weight* of P by

$$\operatorname{wt}(P) = \sum_{i=1}^{g} (w_i - i).$$

By definition, we know that P is a Weierstrass point if and only if $\operatorname{wt}(P) > 0$. It is known that $\sum_{P \in X} \operatorname{wt}(P) = g^3 - g$.

Note that $X_1(N) \to X_0(N)$ is a Galois covering with Galois group $\Gamma_0(N)/\pm\Gamma_1(N) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\} \right\}$ which gives automorphisms on $X_1(N)$. For an integer a prime to N, let $[a]$ denote the automorphism of $X_1(N)$ represented by $\gamma \in \Gamma_0(N)$ such that $\gamma \equiv \left(\begin{smallmatrix} a & * \\ 0 & * \end{smallmatrix}\right) \mod N$. Sometimes we regard $[a]$ as a matrix.

For each divisor $d \mid N$ with $(d, N/d) = 1$, consider the matrices of the form $\left(\begin{smallmatrix} dx & y \\ Nz & dw \end{smallmatrix}\right)$ with $x, y, z, w \in \mathbb{Z}$ and determinant d. Then these matrices define a unique involution on $X_0(N)$ which is called the *Atkin-Lehner involution* and denoted by W_d. In particular, if $d = N$, then W_N is called the *full Atkin-Lehner involution.* We also denote by W_d a matrix of the above form. In general, the W_d do not define a unique automorphism on $X_1(N)$.

We recall that if a curve X is hyperelliptic, then there exists a unique involution ν, called a *hyperelliptic involution*, such that $X/\langle\nu\rangle$ is a rational curve.

According to Ishii-Momose [4], the hyperelliptic involutions on the $X_1(N)$ for $N = 13, 16, 18$ are [5], [7], and W_2, respectively.

For the number of fixed points of an involution, we need the following result:

Proposition 2.1. *Let v be any involution on the compact Riemann surface X and $\#$ denote the number of the fixed points of v. Then we have the following genus formula*:

$$g(v\backslash X) = \frac{1}{4}(2g(X) + 2 - \#).$$

Proof. It follows from the Hurwitz formula. □

In virtue of [11] we have the following description of cusps. The cusps of $X(N)$ can be regarded as pairs $\pm\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, where $x, y \in \mathbb{Z}/N\mathbb{Z}$, and are relatively prime, and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -x \\ -y \end{smallmatrix}\right)$ are identified. Since $\Gamma_1(N)/\Gamma(N) = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{Z}/N\mathbb{Z}\}$ operates naturally on the left, and so a cusp of $X_1(N)$ can be regarded as an orbit $\left\{\pm\left(\begin{smallmatrix} x+by \\ y \end{smallmatrix}\right)\right\}$. Note that we can choose a representative $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ with x reduced modulo $d = \gcd(y, N)$ and $\gcd(x, d) = 1$. Thus for each $d \mid N$ with $N = 11$ or $N > 12$, we have $\frac{1}{2}\varphi(d)\varphi(N/d)$ cusps of $X_1(N)$. Given a cusp $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ of $X_1(N)$ it is fixed by $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in \Gamma_0(N)/\pm\Gamma_1(N)$ if and only if $ax \equiv \pm x \pmod{d}$ and $a^{-1}y \equiv \pm y \pmod{N}$, i.e., $a \equiv \pm 1 \pmod{d}$ and $a \equiv \pm 1 \pmod{N/d}$.

3. Weierstrass points

In this section, we determine all Weierstrass points on $X_1(N)$ for $N = 13, 16, 18$. For the purpose we state a well-known lemma as follows:

Lemma 3.1. *If X is a hyperelliptic curve of genus g, then X has $2g + 2$ Weierstrass points. Moreover, the Weierstrass points are the fixed points of a hyperelliptic involution.*

Proof. Suppose ν is a hyperelliptic involution of X, then we know that ν has $2g + 2$ fixed points on X from Proposition 2.1. We can regard the map $f : X \to X/\langle\nu\rangle$ as a function on X of degree 2. Let P be a fixed point on X by ν. Then the function $\frac{1}{f-f(P)}$ has a pole at P of order 2 and it is regular elsewhere. Thus P is a Weierstrass point of X. From (2.1), we have $\mathrm{wt}(P) = \frac{g(g-1)}{2}$. Since $\sum_{Q \in X} \mathrm{wt}(Q) = g^3 - g$, there exist at most $2g+2$ Weierstrass points. Therefore X has exactly $2g + 2$ Weierstrass points. □

Remark 3.2. It can be shown also by using Schöneberg's Theorem [15] that the fixed points of a hyperelliptic involution are Weierstrass points.

Note that each $X_1(N)$ for $N = 13, 16, 18$ has genus 2. Thus, by Lemma 3.1 there exist $2g + 2 = 6$ Weierstrass points of $X_1(N)$ which are fixed points of a hyperelliptic involution. First, consider $N = 13$. Take

$$[5] = \begin{pmatrix} 5 & -2 \\ 13 & -5 \end{pmatrix}.$$

Then it is an elliptic element of $\Gamma_0(13)$, and it fixes $\omega := \frac{5+i}{13}$. Each point $\tau \in \mathbb{H}$ gives a point on $X_1(N)$ and $X_0(N)$, and let $\{\tau\}_1$ and $\{\tau\}_0$ denote its corresponding points on $X_1(N)$ and $X_0(N)$, respectively. The Galois covering $\phi : X_1(13) \to X_0(13)$ is of degree 6 and its Galois group $(\mathbb{Z}/13\mathbb{Z})^*/\{\pm 1\}$ is a cyclic group generated by [2]. Since $\Gamma_1(13)$ doesn't contain an elliptic element, ϕ is ramified at $[\omega]_0$ with ramification index 2. Thus there exist 3 points

$$\{\omega\}_1, \{[2]\omega\}_1, \{[2^2]\omega\}_1$$

lying above $[\omega]_0$. One can easily check that $[5][2^k] = \gamma[2^k][5]$ for some $\gamma \in \Gamma_1(13)$ where $k = 0, 1, 2$. Thus $[5]\{[2^k]\omega\}_1 = \{[5][2^k]\omega\}_1 = \{\gamma[2^k][5]\omega\}_1 = \{\gamma[2^k]\omega\}_1 = \{[2^k]\omega\}_1$ for all $k = 0, 1, 2$, and hence the hyperelliptic involution [5] fixes all three points as above. According to Corollary 3.7.2 of [3], $X_0(13)$ has two elliptic points. Take

$$[8] = \begin{pmatrix} 8 & -5 \\ 13 & -8 \end{pmatrix}.$$

Then it is an elliptic element of $\Gamma_0(13)$, and it fixes $\omega' := \frac{8+i}{13}$. Note that for any element $\alpha \in \Gamma_0(13)$, the isotropy group $\Gamma_0(13)_{\alpha(\omega)} = \alpha\Gamma_0(13)_\omega\alpha^{-1} = \alpha\langle[5]\rangle\alpha^{-1}$ of $\alpha(\omega)$ gives a conjugacy class of a cyclic group of order 4 in $\Gamma_0(13)$. However $\Gamma_0(13)_{\omega'} = \langle[8]\rangle$, and hence $\{\omega\}_0 \neq \{\omega'\}_0$. Since $-5 \equiv 8 \pmod{13}$, $[5]\{\omega'\}_1 = \{[5]\omega'\}_1 = \{[8]\omega'\}_1 = \{\omega'\}_1$. Thus [5] fixes $\{\omega'\}_1$ and we can check that it fixes the other two points lying above $\{\omega'\}_0$. Therefore the six points lying above $\{\omega\}_0$ and $\{\omega'\}_0$ are all Weierstrass points of $X_1(13)$.

Next, consider $N = 16$. In this case there exists no elliptic element of the form [7], and hence it has no fixed non-cusp point. For any $d = 2, 4, 8$, the following two congruences holds:

$$7 \equiv \pm 1 \pmod{d}, \quad 7 \equiv \pm 1 \pmod{16/d}.$$

Thus we know that six cusps $\binom{x}{y}$ with $\gcd(y, N) = d$ are fixed points of [7] from the arguments in Section 2, and hence they are all Weierstrass points of $X_1(16)$.

Last, let us consider $N = 18$. Take

$$W_2 = \begin{pmatrix} 4 & -1 \\ 18 & -4 \end{pmatrix}.$$

Then it fixes $\upsilon := \frac{4+i\sqrt{2}}{18}$. By Proposition 2.1, W_2 has two fixed points in $X_0(18)$. The author, Kim and Schweizer [5] proposed an systematic way to find $\Gamma_0(N)$-inequivalent fixed points of W_d. Using their method, we can find the other fixed point $\{\upsilon'\}_0$ where $\upsilon' := \frac{-4+i\sqrt{2}}{18}$. The Galois covering $X_1(18) \to$

$X_0(18)$ is of degree 3 and its Galois group $(\mathbb{Z}/18\mathbb{Z})^*/\{\pm 1\}$ is a cyclic group generated by [5]. One can easily check that

$$W_2[a] \equiv [a]W_2 \bmod \Gamma_1(N).$$

By the exact same reason as in the case $N = 13$, the six points lying above $\{\upsilon\}_0$ and $\{\upsilon'\}_0$ are all Weierstrass points of $X_1(18)$.

Consequently, we have the following our main result:

Theorem 3.3. *All Weierstrass points of $X_1(N)$ for $N = 13, 16, 18$ are listed in the following*:

(1) $N = 13$;

$$\left\{\frac{5+i}{13}\right\}_1, \left\{\frac{5+i}{26}\right\}_1, \left\{\frac{18+i}{65}\right\}_1, \left\{\frac{8+i}{13}\right\}_1, \left\{\frac{8+i}{65}\right\}_1, \left\{\frac{99+i}{338}\right\}_1.$$

(2) $N = 16$;

$$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\6\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}1\\8\end{pmatrix}, \begin{pmatrix}3\\8\end{pmatrix}.$$

(3) $N = 18$;

$$\left\{\frac{4+i\sqrt{2}}{18}\right\}_1, \left\{\frac{58+i\sqrt{2}}{198}\right\}_1, \left\{\frac{22+i\sqrt{2}}{54}\right\}_1,$$
$$\left\{\frac{-4+i\sqrt{2}}{18}\right\}_1, \left\{\frac{590+i\sqrt{2}}{1494}\right\}_1, \left\{\frac{248+i\sqrt{2}}{918}\right\}_1.$$

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