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COREGULARITY OF ORDER-PRESERVING SELF-MAPPING SEMIGROUPS OF FENCES

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Abstract. A fence is an ordered set that the order forms a path with alternating orientation. Let $\mathbf{F} = (F; \leq)$ be a fence and let $OT(\mathbf{F})$ be the semigroup of all order-preserving self-mappings of $\mathbf{F}$. We prove that $OT(\mathbf{F})$ is coregular if and only if $|F| \leq 2$. We characterize all coregular elements in $OT(\mathbf{F})$ when $\mathbf{F}$ is finite. For any subfence $\mathbf{S}$ of $\mathbf{F}$, we show that the set $COT_S(\mathbf{F})$ of all order-preserving self-mappings in $OT(\mathbf{F})$ having S as their range forms a coregular subsemigroup of $OT(\mathbf{F})$. Under some conditions, we show that a union of $COT_S(\mathbf{F})$'s forms a coregular subsemigroup of $OT(\mathbf{F})$.

1. Introduction and preliminaries

Let X be an arbitrary set and let $T(X)$ be the semigroup of self-mappings of X. The semigroup $T(X)$ is well-studied.

Consider X as the base set of an algebraic or relational structure $\mathbf{X}$ such as a vector space, a topological space, a unary algebra, an ordered set, a graph, etc.

A mapping $\alpha : X \to X$ is called an *endomorphism* of $\mathbf{X}$ if α preserves the structure of $\mathbf{X}$. For example, every endomorphism of $\mathbf{X}$ is a continuous mapping if $\mathbf{X}$ is a topological space. If $\mathbf{X}$ is an ordered set, then an endomorphism of $\mathbf{X}$ is an order-preserving self-mapping, that is, if $x \leq y$, then $\alpha(x) \leq \alpha(y)$. For an ordered set $\mathbf{X}$, we denote by $OT(\mathbf{X})$ the semigroup of all order-preserving self-mappings of $\mathbf{X}$.

The semigroup of all endomorphisms of $\mathbf{X}$ have been widely investigated. In [7], Gluskin showed that if $OT(\mathbf{X})$ is isomorphic to $OT(\mathbf{Y})$, then the ordered sets $\mathbf{X}$ and $\mathbf{Y}$ are isomorphic or anti-isomorphic. Lyapin [10] characterized the semigroups of endomorphisms of a relation structure. Cezus, Magill and Subbiah [2] proved that the semigroup of all linear transformations of a vector space $\mathbf{V}$ over a division ring can be used to determine that $\mathbf{V}$ is finite dimensional. In [14], Radaleczki showed the connection between the congruence lattice of a unary algebra $\mathbf{X}$ and the automorphism group of $\mathbf{X}$. Higgins,

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Mitchell and Ruškuc [8] found that the rank of the semigroup $T(X)$ is related to the semigroups $OT(\mathbf{X})$ for some chains $\mathbf{X} = (X; \leq)$. Later, Pozdnyakova [13] showed that the infinite monounary algebras satisfying some certain conditions are determined by their semigroups of endomorphisms. These results show that the endomorphism semigroup can be used to determine the structure of the initial algebraic system. Such semigroups play an important role in the study of algebraic systems.

In Semigroup Theory the concept of regularity is one of the most-studied topics. There have been many research works studying regularity of semigroups, especially endomorphism semigroups of algebraic structures (see [12], [9], [11] and [16]).

A special case of a regular element is a coregular element. An element a in a semigroup S is called *coregular* if there is an element $b \in S$ with $aba = a = bab$ and S is called *coregular* if every element of S is coregular. Coregular semigroup was first introduced and studied in [1] by Bijev and Todorov. They proved that a semigroup S is coregular if and only if $a^3 = a$ for all $a \in S$. It can prove that an element a in a semigroup S is coregular if and only if $a^3 = a$. Chvalina and Matoušková [3] gave a necessary and sufficient condition for endomorphisms of a unary algebra to be coregular. In [5] Dimitrova and Koppitz showed the description of coregular subsemigroups of the symmetric semigroup of self-mappings on an n-element set.

A *fence* $\mathbf{F}$ is an ordered set $(F; \leq)$ in which either

$$a_1 < a_2 > a_3, \ldots, a_{2m-1} > a_{2m} < a_{2m+1}, \ldots$$

or

$$a_1 > a_2 < a_3, \ldots, a_{2m-1} < a_{2m} > a_{2m+1}, \ldots$$

are the only comparability relations where $F = \{a_1, a_2, \ldots, a_n, \ldots\}$. Every element in $\mathbf{F}$ is minimal or maximal. If $a_1 < a_2$, then $\mathbf{F}$ is called an *up fence* and it is called a *down fence* if $a_1 > a_2$. For $x, y \in F$, we define the distance $d(x, y)$ from x to y in $\mathbf{F}$ by

$$d(x, y) = \inf\{|S| - 1 \mid \mathbf{S} \text{ is a subfence of } \mathbf{F} \text{ and } x, y \in S\}.$$

Algebraic properties of order-preserving self-mappings of fences have been long considered. Demetrovics and Rónyai [4] studied the clones of all order-preserving operations for fences. In [15], Rutkowski gave the formula for the number of order-preserving self-mappings of a fence. Later, Farley [6] computed the number of order-preserving self-mappings of a fence.

In this paper, our main purpose is to investigate the coregularity of the semigroup of order-preserving self-mappings of a fence. Throughout we use $\operatorname{ran}\alpha$ to denote the range of a mapping α and the n^{th} composition of α is denoted by α^n.

It is known that for an ordered set $\mathbf{X}$, the identity mapping id_X and a constant mapping c_a which maps all elements in X into a are order-preserving.

Because of $(id_X)^3(x) = x = id_X(x)$ and $(c_a)^3(x) = a = c_a(x)$ for all $x \in X$, we get that id_X and c_a are coregular elements in $OT(\mathbf{X})$.

Consider the 4-element fence $\mathbf{F} = (F; \leq)$ as shown in Figure 1. Define

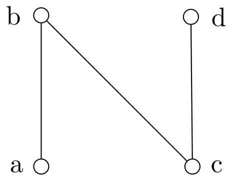

FIGURE 1. The 4-element fence $\mathbf{F}$

$\alpha : F \to F$ by $\alpha(a) = b = \alpha(b)$, $\alpha(c) = a$ and $\alpha(d) = b$. It is easy to verify that α is order-preserving and so, $\alpha \in OT(\mathbf{F})$. By the definition of α,

$$\alpha^3(c) = \alpha(\alpha(\alpha(c))) = \alpha(\alpha(a)) = \alpha(b) = b \neq a = \alpha(c).$$

Hence, α is not coregular and therefore, $OT(\mathbf{F})$ is not a coregular semigroup.

If $\mathbf{F}$ is an 1-element fence, then $OT(\mathbf{F})$ is the set of a constant mapping. Hence, $OT(\mathbf{F})$ is a coregular semigroup. It is natural to ask when the semigroup $OT(\mathbf{F})$ is coregular. In Section 2 we give a necessary and sufficient condition for $OT(\mathbf{F})$ to be coregular. We also study properties of coregular elements of $OT(\mathbf{F})$. Because every α in $OT(\mathbf{F})$ need not be coregular, in Section 3 we completely describe the coregular elements of $OT(\mathbf{F})$. Finally, in Section 4 is devoted to the study of coregular subsemigroups of $OT(\mathbf{F})$.

2. Coregular elements in $OT(\mathbf{F})$ and their properties

We now investigate properties of coregular elements of $OT(\mathbf{F})$. To do so we need a result concerning preserving subfences of an order-preserving self-mapping of a fence $\mathbf{F}$. An ordered set $\mathbf{P}$ is called *connected* if for all $a, b \in P$ there is a fence $\mathbf{F} \subseteq \mathbf{P}$ with endpoints a and b. It is well known that if $\mathbf{P}$ is connected and $\alpha : P \to Q$ is order-preserving, then $\alpha(P)$ is connected. Consequently, every order-preserving mapping maps order-connected sets to order-connected set. Because order-connected subsets of a fence $\mathbf{F}$ are precisely the subfences, an order-preserving mapping $\alpha : F \to F$ maps subfences to subfences.

For a subordered set $\mathbf{S}$ of a fence $\mathbf{F} = (F; \leq)$, let $\alpha(\mathbf{S}) := (\alpha(S); \leq_S)$ where $\leq_S = \leq \cap S^2$.

As we mentioned in Section 1, the semigroup $OT(\mathbf{F})$ of order-preserving self-mappings of a fence need not be coregular. The following theorem shows a characterization of fences $\mathbf{F}$ having a coregular semigroup $OT(\mathbf{F})$.

Theorem 2.1. *Let $\mathbf{F}$ be a finite fence. Then $OT(\mathbf{F})$ is a coregular semigroup if and only if $|F| \leq 2$.*

Proof. Let $OT(\mathbf{F})$ be a coregular semigroup. Suppose that $|F| > 2$. Then there are distinct elements a, b and $c \in F$ with $a > b < c$ or $a < b > c$. We may assume that $a < b > c$. Then b and a, c are maximal and minimal, respectively, Define the extension $\alpha : F \to F$ of the mapping given in Figure 1 by

$$\alpha(x) = \begin{cases} a, & x \text{ is minimal and } x \neq a, \\ b, & \text{otherwise.} \end{cases}$$

Then α is order-preserving and not coregualr, a contradiction. Therefore,

$$|F| \leq 2.$$

Conversely, assume that $|F| < 2$. If $|F| = 1$, then $OT(\mathbf{F})$ is the set of a constant mapping. Hence, $OT(\mathbf{F})$ is a coregular semigroup. Assume that $|F| = 2$ and $F = \{a, b\}$ with $a < b$. Let $\alpha \in OT(\mathbf{F})$. Then $|\operatorname{ran}\alpha| = 1$ or $|\operatorname{ran}\alpha| = 2$. If $|\operatorname{ran}\alpha| = 1$, then α is a constant mapping and hence, α is coregular. Let $|\operatorname{ran}\alpha| = 2$. Then $OT(\mathbf{F})$ is the set of the identity mapping id_F and two constant mappings. Therefore $OT(\mathbf{F})$ is coregular. □

For a fence $\mathbf{F}$, we denote by $COT(\mathbf{F})$ the set of all coregular order-preserving self-mappings of $\mathbf{F}$. The following proposition gives useful properties of coregular elements $\alpha \in OT(\mathbf{F})$ that are used later.

Proposition 2.2. *Let $\alpha \in COT(\mathbf{F})$. Then the following conditions hold*:

(i) *If $a \in \operatorname{ran}\alpha$ with $\alpha(a) = b$, then $\alpha(b) = a$.*
(ii) *$\alpha|_{\operatorname{ran}\alpha}$ is injective and $\alpha^2|_{\operatorname{ran}\alpha} = id_{\operatorname{ran}\alpha}$.*
(iii) *If $a, b \in \operatorname{ran}\alpha$ with $a < b$, then $\alpha(a) < \alpha(b)$.*
(iv) *If $a \in \operatorname{ran}\alpha$ with $\alpha(a) = b$ and a is minimal* (*maximal*) *in $\mathbf{F}$, then b is also minimal* (*maximal*) *in $\mathbf{F}$.*

Proof. (i) Let $a \in \operatorname{ran}\alpha$ with $\alpha(a) = b$. Suppose that $\alpha(b) \neq a$. Then $a \neq b$ since otherwise, $\alpha(b) = a$. From $a \in \operatorname{ran}\alpha$, there is an $x \in F$ with $x \neq a$ and $\alpha(x) = a$. It follows that

$$\alpha^3(x) = \alpha(\alpha(\alpha(x))) = \alpha(\alpha(a)) = \alpha(b) \neq a = \alpha(x).$$

Hence, α is not coregular which is a contradiction.

(ii) Let $a, b \in \operatorname{ran}\alpha$ with $\alpha(a) = \alpha(b)$. Then there are $x, y \in F$ with $a = \alpha(x)$ and $b = \alpha(y)$. So, $\alpha(\alpha(x)) = \alpha(\alpha(y))$ implies that $\alpha^3(x) = \alpha^3(y)$. By coregularity of α, we have $a = \alpha(x) = \alpha(y) = b$.

(iii) Let $a, b \in \operatorname{ran}\alpha$ with $a < b$. From α is order-preserving and $a < b$, we have $\alpha(a) \leq \alpha(b)$. Since $\alpha|_{\operatorname{ran}\alpha}$ is injective and $a \neq b$, so $\alpha(a) \neq \alpha(b)$ implies that $\alpha(a) < \alpha(b)$.

(iv) Let $a \in \operatorname{ran}\alpha$ with $\alpha(a) = b$. If $|\operatorname{ran}\alpha| = 1$, then $a = b$ and hence (iv) is satisfied. Let $|\operatorname{ran}\alpha| > 1$. Suppose that a and b are minimal and maximal in $\mathbf{F}$, respectively. Then from $\operatorname{ran}\alpha$ is a subfence of $\mathbf{F}$, there is a $c \in \operatorname{ran}\alpha$ with $a < c$. By (iii), we have $b = \alpha(a) < \alpha(c)$. Thus b is not maximal which is a contradiction. □

As we known, the range of a coregular order-preserving self-mapping of a fence $\mathbf{F}$ is always a subfence of $\mathbf{F}$. It is natural to ask whether a subfence of $\mathbf{F}$ can be the range of such a mapping. A subfence $\mathbf{S}$ of a fence $\mathbf{F}$ is an *order retract* if there is an order-preserving mapping $\alpha : F \to S$ such that $\alpha|_S = id_S$. Such a mapping α is called a *retraction* of $\mathbf{F}$ onto $\mathbf{S}$. It is clear that $\alpha^2 = \alpha$ and hence $\alpha^3 = \alpha$. So, a retraction is always coregular.

Example 2.3. Consider a subfence $\mathbf{S}$ of a fence $\mathbf{F} = (F; \leq)$ where $S = \{a_1, a_2, \dots, a_n\}$. Define $\alpha : F \to F$ by

$$\alpha(x) = \begin{cases} a_1, d(a,x) < d(a,a_1) \\ x, x \in S \\ a_n, d(a,x) > d(a,a_n), \end{cases}$$

where a is the initial point of $\mathbf{F}$. Clearly, α is order-preserving and $\alpha|_S = id_S$. Hence, α is a retraction and therefore, α is coregular.

Example 2.3 gives us the following proposition.

Proposition 2.4. *Every subfence is the rage of a retraction and therefore of a coregular mapping.*

Consider a subfence $\mathbf{S}$ of a fence $\mathbf{F}$ for which $|S|$ is odd, it can be the range of a coregular mapping which is not a retraction as shown in the following proposition.

Proposition 2.5. *Let n be an odd number with $n > 1$ and let $\mathbf{S}$ be a subfence of a fence $\mathbf{F}$ with $|S| = n$. Then there exists an $\alpha \in COT(\mathbf{F})$ with $\operatorname{ran}\alpha = S$ and $\alpha|_S \neq id_S$.*

Proof. Since $\mathbf{S}$ is an odd sized fence, there is a non-identity automorphism $\sigma : S \to S$ of order 2. Let α be a retraction of $\mathbf{F}$ onto $\mathbf{S}$. Then the composition $\sigma\alpha$ is a non-identity order-preserving mapping from $\mathbf{F}$ onto $\mathbf{S}$. Since $\alpha|_S = id_S$, so $\alpha\sigma\alpha = \sigma\alpha$. If follows that

$$(\sigma\alpha)^3 = \sigma\alpha\sigma(\alpha\sigma\alpha) = \sigma\alpha\sigma\sigma\alpha = \sigma\alpha(id_S)\alpha = \sigma\alpha\alpha = \sigma\alpha.$$

So, $\sigma\alpha$ is coregular. □

3. Characterizations of coregular elements in $OT(\mathbf{F})$

In this section, our aim is to describe coregular elements in $OT(\mathbf{F})$. We start by proving a lemma.

Lemma 3.1. *Let $\mathbf{S}$ be a subfence of a finite fence $\mathbf{F}$ and let α be a bijection with $\operatorname{ran}\alpha = S$. Assume that $S = \{a_1, a_2, \dots, a_n\}$ and $\alpha(a_k) = a_l$. Let $w \in \mathbb{N}$ with $w \geq 2$. Then the following conditions hold:*

(i) *Assume that $\alpha(a_{k-1}) = a_{l+1}$. If $a_{k\pm w} \in S$, then $\alpha(a_{k\pm w}) = a_{l\mp w}$.*
(ii) *Assume that $\alpha(a_{k-1}) = a_{l-1}$. If $a_{k\pm w} \in S$, then $\alpha(a_{k\pm w}) = a_{l\pm w}$.*

Proof. (i) Assume that $\alpha(a_{k-1}) = a_{l+1}$. If $a_{k-1} < a_k > a_{k+1}$, then by Proposition 2.2(iii), $\alpha(a_{k-1}) < \alpha(a_k) > \alpha(a_{k+1})$. From $\alpha(a_k) = a_l$, we have $\alpha(a_{k+1}) = a_{l-1}$. Similarly, $\alpha(a_{k+1}) = a_{l-1}$ if $a_{k-1} > a_k < a_{k+1}$.

Let $w \in \mathbb{N}$ with $w \geq 2$ and $a_{k-w} \in S$. We show that $\alpha(a_{k-w}) = a_{l+w}$ by Strong Induction. If $w = 2$, then either $a_{k-2} < a_{k-1} > a_k$ or $a_{k-2} > a_{k-1} < a_k$. If $a_{k-2} < a_{k-1} > a_k$, then $\alpha(a_{k-2}) < \alpha(a_{k-1}) > \alpha(a_k)$ and from $\alpha(a_{k-1}) = a_{l+1}$ and $\alpha(a_k) = a_l$, we have $\alpha(a_{k-2}) = a_{l+2}$. Assume that for each $m \in \mathbb{N}$ with $2 \leq m < w$, if $a_{k-m} \in S$, then $\alpha(a_{k-m}) = a_{l+m}$. From $w \geq 2$, we have $w-1, w-2 \geq 0$. Since $a_{k-w} \in S$, so $1 \leq k-w \leq n$ implies that $1 < k-(w-1)$, $k-(w-2) \leq n$. Thus $a_{k-(w-1)}, a_{k-(w-2)} \in S$. By the assumptions, $\alpha(a_{k-(w-1)}) = a_{l+(w-1)}$ and $\alpha(a_{k-(w-2)}) = a_{l+(w-2)}$. Since either $a_{k-(w-2)} > a_{k-(w-1)} < a_{k-w}$ or $a_{k-(w-2)} < a_{k-(w-1)} > a_{k-w}$, either $\alpha(a_{k-(w-2)}) > \alpha(a_{k-(w-1)}) < \alpha(a_{k-w})$ or $\alpha(a_{k-(w-2)}) < \alpha(a_{k-(w-1)}) > \alpha(a_{k-w})$ by Proposition 2.2(iii). It follows that $\alpha(a_{k-w}) = a_{l+w}$. From $\alpha(a_{k-1}) = a_{l+1}$ and $\alpha(a_k) = a_l$, we have $\alpha(a_{k+1}) = a_{l-1}$. Using the similar method as above, we can prove that if $a_{k+w} \in S$, then $\alpha(a_{k+w}) = a_{l-w}$.

(ii) The proof is similar to that of (i). □

We now prove a technical theorem which will be the main tool for describing coregular α in $OT(\mathbf{F})$.

Theorem 3.2. *Let* $\mathbf{S}$ *be a subfence of a finite fence* $\mathbf{F}$ *and let* $\alpha \in COT(\mathbf{F})$ *with* $\operatorname{ran}\alpha = S$ *and* $\alpha|_S \neq id_S$. *If* $x \in S$ *with* $\alpha(x) = y$, *then* $d(a, x) = d(y, b)$ *where* a *and* b *are the initial point and the endpoint of* $\mathbf{S}$, *respectively.*

Proof. Let $S = \{a_1, \dots, a_n\}$ with $a = a_1 < a_2 > a_3 < \cdots > (<) a_n = b$ and let $\alpha(a_k) = a_l$. Suppose that $d(a_1, a_k) < d(a_l, a_n)$. Then $0 \leq k-1 < n-l$ implies that $l \leq n-1$.

First we consider $k \neq l$. Then $d(a_k, a_l) \neq 0$. Let $k < l$. From $1 \leq k < l$, we have $l \geq 2$. Thus $2 \leq l \leq n-1$ implies that $1 \leq l-1, l+1 \leq n$. Consequently, $a_{l-1}, a_{l+1} \in S$. If $a_k = a_1$, then $a_{k-1} \notin S$ and $a_{k+1} \in S$. Because of either $a_{l-1} < a_l > a_{l+1}$ or $a_{l-1} > a_l < a_{l+1}$ and $\alpha \in COT(\mathbf{F})$, we have $\alpha(a_{l-1}) = a_{k+1} = \alpha(a_{l+1})$ since $\alpha(a_l) = a_k$. Thus $\alpha|_S$ is not injective, a contradiction. Therefore, $k > 1$ and so, $a_{k-1}, a_{k+1} \in S$. It follows that $\alpha(a_{l-1}), \alpha(a_{l+1}) \in \{a_{k-1}, a_{k+1}\}$.

We consider $\alpha(a_{l+1})$ in the following 2 cases.

<u>Case 1</u>: $\alpha(a_{l+1}) = a_{k-1}$. Then $\alpha(a_{k-1}) = a_{l+1}$ by Proposition 2.2(i). Since $a_{k-(k-1)} = a_1 \in S$ and $a_{k-(k-2)} = a_2 \in S$, so $\alpha(a_1) = \alpha(a_{k-(k-1)}) = a_{l+(k-1)}$ and $\alpha(a_2) = \alpha(a_{k-(k-2)}) = a_{l+(k-2)}$ by Lemma 3.1(i). Again by Proposition 2.2(i), $\alpha(a_{l+(k-1)}) = a_1$ and $\alpha(a_{l+(k-2)}) = a_2$. From $1 \leq k-1 < n-l$, we have $1 \leq l+k \leq n$. Thus $a_{l+k} \in S$. Since a_1 is minimal, $a_{l+(k-1)}$ is minimal by Proposition 2.2(iv). Because $a_{l+(k-1)}$ and a_{l+k} are comparable, $a_{l+(k-1)} < a_{l+k}$. It follows that $a_1 = \alpha(a_{l+(k-1)}) < \alpha(a_{l+k})$. Because there is only one element in S that is comparable to a_1, namely, a_2, so $\alpha(a_{l+k}) = a_2 = \alpha(a_{l+(k-2)})$, a contradiction.

Case 2: $\alpha(a_{l+1}) = a_{k+1}$. Then $\alpha(a_{l-1}) = a_{k-1}$. Now we consider $d(a_l, a_n)$ in the following 2 cases.

Case 2.1: $d(a_k, a_l) \leq d(a_l, a_n)$. Then $1 < l - k \leq n - l$ implies that $1 < l + (l - k) \leq n$. Thus $a_{l+(l-k)} \in S$. By Lemma 3.1(ii), $\alpha(a_{l+(l-k)}) = a_{k+(l-k)} = a_l$. But $\alpha(a_k) = a_l$, so $\alpha|_S$ is not injective since $a_{l+(l-k)} \neq a_k$, a contradiction.

Case 2.2: $d(a_k, a_l) > d(a_l, a_n)$. Then from $d(a_l, a_n) > d(a_1, a_k)$, we have $d(a_k, a_l) > d(a_1, a_k)$ implying $l - k > k - 1 \geq 1$. Hence, $k \leq l - (k - 1) < l$. It follows that there is an $m \in \{1, 2, \ldots, (l - k) - 1\}$ with $l - (k - 1) = k + m$. Thus $a_{k+m} \in S$. From $a_{l+m} = \alpha(a_{k+m}) = \alpha(a_{l-(k-1)}) = a_{k-(k-1)} = a_1$, we have $l + m = 1$, a contradiction since $l + m > 1$.

Next we assume that $k = l$ and consider k in the following 3 cases.

Case 1: $k = 1$. Then $a_{k-1} \notin S$ and $a_{k+1} \in S$. From $a_k = a_1 < a_2 = a_{k+1}$, we have $a_1 = a_l = \alpha(a_k) = \alpha(a_1) < \alpha(a_2) = \alpha(a_{k+1})$ implying $\alpha(a_{k+1}) = a_2 = a_{l+1}$. By Lemma 3.1(ii), $\alpha(a_{1+w}) = \alpha(a_{k+w}) = a_{l+w} = a_{1+w}$ for all $w \in \{1, 2, \ldots, n - 1\}$. Hence, $\alpha|_S = id_S$, a contradiction.

Case 2: $k = n$. The proof is dually the same as Case 1.

Case 3: $2 \leq k \leq n - 1$. Then $1 \leq k - 1, k + 1 \leq n$ and $0 \leq k - 2 < n$ implies that $a_{k-1}, a_{k+1} \in S$. Since $\alpha(a_k) = a_k$, so $\alpha(a_{k-1}), \alpha(a_{k+1}) \in \{a_{k-1}, a_{k+1}\}$.

Case 3.1: $\alpha(a_{k-1}) = a_{k+1}$. Then $\alpha(a_{k+1}) = a_{k-1}$. Since $k - 1 = d(a_1, a_k) < d(a_l, a_n) = n - l = n - k$, so $1 \leq k + (k - 2) < k + (k - 1) < n$ implies that $1 \leq k + k \leq n$. Thus $a_{k+(k-2)}, a_{k+(k-1)}, a_{k+k} \in S$. By Lemma 3.1(i) and the assumptions, $\alpha(a_{k+(k-2)}) = a_{k-(k-2)} = a_2$ and $\alpha(a_{k+(k-1)}) = a_{k-(k-1)} = a_1$. Because a_1 is minimal, $a_{k+(k-1)}$ is minimal. Hence, $a_{k+(k-1)} < a_{k-k}$ implies that $\alpha(a_{k+(k-1)}) < \alpha(a_{k+k})$. But $\alpha(a_{k+(k-1)}) = a_1$, so $\alpha(a_{k+k}) = a_2$ implies that $\alpha(a_{k+(k-2)}) = \alpha(a_{k+k})$. Because of $a_{k+(k-2)} \neq a_{k+k}$, we get that $\alpha|_S$ is not injective, a contradiction.

Case 3.2: $\alpha(a_{k-1}) = a_{k-1}$. Then $\alpha(a_{k+1}) = a_{k+1}$. It follows that for each $u \in \{1, 2, \ldots, n - k\}$ and $w \in \{1, 2, \ldots, k - 1\}$, $\alpha(a_{k+u}) = a_{k+u}$ and $\alpha(a_{k-w}) = a_{k-w}$ by Lemma 3.1(ii) and the assumptions. Hence, $\alpha|_S = id_S$, a contradiction.

Altogether, we can prove that $d(a_1, a_k) = d(a_l, a_n)$. $\square$

Proposition 3.3. *Let* $\mathbf{S}$ *be a subfence of a finite fence* $\mathbf{F}$ *and let* $\alpha \in COT(\mathbf{F})$ *with* $\operatorname{ran} \alpha = S$. *Then* $\alpha|_S = id_S$ *if and only if* $\alpha(a) = a$ *where* a *is the initial point of* $\mathbf{S}$.

Proof. It is clear that $\alpha(a) = a$ if $\alpha|_S = id_S$. Conversely, we assume that $\alpha(a) = a$. If α is constant, then $S = \operatorname{ran} \alpha = \{a\}$ implies that $\alpha|_S = id_S$. Now assume that α is non-constant. Let b be the endpoint of $\mathbf{S}$. Then by Theorem 3.2, $d(a, a) = d(a, b)$ which is a contradiction since $d(a, a) = 0$ and $d(a, b) \neq 0$. $\square$

Note that $d(x, y)$ is even if x, y are minimal (maximal) in a fence $\mathbf{F}$ and $d(x, y)$ is odd if x is minimal (maximal) and y is maximal (minimal) in $\mathbf{F}$.

In what follows, we restrict our study to the case of a mapping α in $OT(\mathbf{F})$ for which the cardinal number of $\operatorname{ran}\alpha$ is even. Theorem 3.4 shows that it has only one possibility for such a mapping to be coregular. Namely, the restriction to its range has to be an identity mapping.

Theorem 3.4. *Let* $\mathbf{S}$ *be a subfence of a finite fence* $\mathbf{F}$ *for which* $|S|$ *is even and let* $\alpha \in OT(\mathbf{F})$ *with* $\operatorname{ran}\alpha = S$. *Then* α *is coregular if and only if* $\alpha|_S = id_S$.

Proof. Clearly, α is coregular if $\alpha|_S = id_S$. Conversely, we assume that α is coregular and $\mathbf{S}$ is an up-fence. Let a, b be the initial point and the endpoint of $\mathbf{S}$, respectively. Then from $|S|$ is even, a is minimal and b is maximal.

Suppose that $\alpha(a) \neq a$. Then $\alpha|_S \neq id_S$ and there is an element $x \in S \setminus \{a\}$ with $\alpha(a) = x$. By Theorem 3.2, $0 = d(a,a) = d(x,b)$. Because a is minimal, x is minimal. It follows that $d(x,b)$ is odd since b is maximal and hence $d(x,b) \neq 0$, a contradiction. Therefore, $\alpha(a) = a$ and by Proposition 3.3, $\alpha|_S = id_S$. $\square$

To finish this section, it remains to describe coregular mappings α in $OT(\mathbf{F})$ for which the cardinal numbers of their ranges are odd.

Theorem 3.5. *Let* $\mathbf{S}$ *be a subfence of a finite fence* $\mathbf{F}$ *for which* $|S|$ *is odd and let* $\alpha \in OT(\mathbf{F})$ *with* $\operatorname{ran}\alpha = S$. *Assume that* $S = \{a_1, a_2, \ldots, a_n\}$. *Then* α *is coregular if and only if one of the following conditions satisfies*:

(i) $\alpha|_S = id_S$,

(ii) $\alpha(a_k) = a_{n-(k-1)}$ *for all* $k \in \{1, 2, \ldots, n\}$.

Proof. Assume that α is coregular and $\alpha|_S \neq id_S$. Let $a_k \in \operatorname{ran}\alpha$ with $\alpha(a_k) = a_l$. Then by Theorem 3.1, $d(a_1, a_k) = d(a_l, a_n)$ implies that $k - 1 = n - l$ and hence, $l = n - (k-1)$.

Conversely, assume that one of the conditions is satisfied. Let $x \in F$. Then there is an $a_k \in S$ with $\alpha(x) = a_k$. If $\alpha|_S = id_S$, then

$$\alpha^3(x) = \alpha(\alpha(\alpha(x))) = \alpha(\alpha(a_k)) = \alpha(a_k) = a_k = \alpha(x).$$

Let $\alpha|_S \neq id_S$. Then Theorem 3.5 implies that

$$\alpha^3(x) = \alpha(\alpha(\alpha(x))) = \alpha(\alpha(a_k)) = \alpha(a_{n-(k-1)}) = a_{n-[n-(k-1)-1]} = a_k = \alpha(x).$$

Altogether, we can prove that α is coregular. $\square$

4. Coregular subsemigroup of $OT(F)$

As we proved in Theorem 2.1, $OT(\mathbf{F})$ is not coregular if $|F| > 2$. In this section, our goal is to find coregular subsemigroups of $OT(\mathbf{F})$, that is, subsemigroups of $OT(\mathbf{F})$ that all elements are coregular. Because $COT(\mathbf{F})$ is the set of all coregular elements in $OT(\mathbf{F})$. It is natural to ask whether it is a subsemigroup of $OT(\mathbf{F})$. To answer this question, we need the following propositions.

Lemma 4.1. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be subfences of a finite fence* $\mathbf{F}$ *with* $S \subseteq T$ *and let* $\alpha, \beta \in COT(\mathbf{F})$ *with* $\operatorname{ran}\alpha = S$ *and* $\operatorname{ran}\beta = T$. *The following conditions hold*:

(i) $\operatorname{ran}(\alpha\beta) = S$.

(ii) *If* $\beta|_T = id_T$*, then* $\alpha\beta$ *is coregular.*

Proof. (i) Because of $T \subseteq F$ and $\operatorname{ran}\alpha = S$, we have $\alpha(T) \subseteq S$. But $T = \operatorname{ran}\beta$, we have $\operatorname{ran}(\alpha\beta) = \alpha\beta(F) = \alpha(T) \subseteq S$.

Conversely, let $y \in S$. Then from α is coregular, $\alpha|_S : S \to S$ is injective and from S is finite, $\alpha|_S : S \to S$ is bijective. So, there is an $x \in S$ with $\alpha(x) = y$. But $S \subseteq T$, so $y \in \alpha(T) = \operatorname{ran}(\alpha\beta)$. Hence, $S \subseteq \operatorname{ran}(\alpha\beta)$ and therefore, $\operatorname{ran}(\alpha\beta) = S$.

(ii) We show that $(\alpha\beta)|_S = \alpha|_S$. Let $a \in S \subseteq T$. Then from $\beta|_S$ is an identity mapping, $(\alpha\beta)(a) = \alpha(\beta(a)) = \alpha(a)$. Therefore, $\alpha\beta$ is coregular by Theorem 3.4 and Theorem 3.5. □

Proposition 4.2. *Let* $\mathbf{S}, \mathbf{T}$ *be subfences of a finite fence* $\mathbf{F}$ *with* $|S| \geq 2$ *and* $S \subset T$ *and let* $\alpha, \beta \in COT(\mathbf{F})$ *with* $\operatorname{ran}\alpha = S$ *and* $\operatorname{ran}\beta = T$*. Assume that* $T = \{a_1, \ldots, a_n\}$ *and* $S = \{a_k, \ldots, a_l\}$ *with* $d(a_1, a_k) \neq d(a_l, a_n)$*. If* $\beta|_T \neq id_T$ *and* $\alpha|_S = id_S$ *with* $\alpha(x) \neq \beta(x)$ *for all* $x \in T \setminus S$*, then* $\alpha\beta \notin COT(\mathbf{F})$*.*

Proof. By Lemma 4.1(i), $\operatorname{ran}(\alpha\beta) = S$. First we show that there is a $t \in \{k, k+1, \ldots, l\}$ with $a_{n-(t-1)} \notin S$. Suppose that $a_{n-(t-1)} \in S$ for all $t \in \{k, k+1, \ldots, l\}$. Then $a_{n-(k-1)}, a_{n-(l-1)} \in S$ implies that $k \leq n-(k-1)$, $n-(l-1) \leq l$. Next we show that $l = n-(k-1)$. Suppose that $l \neq n-(k-1)$. Then $n-(k-1) < l$ implies that $n-(l-1) < k$, a contradiction. So, $l = n-(k-1)$ implies that $d(a_1, a_k) = k-1 = n-l = d(a_l, a_n)$ which contradict to the assumption. Therefore, $a_{n-(t-1)} \notin S$ for some $t \in \{k, k+1, \ldots, l\}$.

Consider $\alpha\beta(a_t)$. From $a_{n-(t-1)} \notin S = \operatorname{ran}(\alpha\beta)$, we have $a_{n-(t-1)} \neq \alpha\beta(a_t) = \alpha(a_{n-(t-1)}) \neq \beta(a_{n-(t-1)}) = a_t$. Theorem 3.4 and Theorem 3.5 imply that $\alpha\beta$ is not coregular. □

Proposition 4.2 tells that the composition of coregular order-preserving self-mappings of a fence need not be coregular. It follows that $COT(\mathbf{F})$ need not be a subsemigroup of $OT(\mathbf{F})$.

Let $\mathbf{S}$ be a subfence of a fence $\mathbf{F}$. We denote by $COT_S(\mathbf{F})$ the set of all $\alpha \in COT(\mathbf{F})$ having S as their ranges.

Theorem 4.3. *Let* $\mathbf{S}$ *be a subfence of a finite fence* $\mathbf{F}$*. Then* $COT_S(\mathbf{F})$ *is a coregular subsemigroup of* $OT(\mathbf{F})$*.*

Proof. Let α and $\beta \in OT(\mathbf{F})$. It is known that $\alpha\beta \in OT(\mathbf{F})$. By Lemma 4.1(i), $\operatorname{ran}(\alpha\beta) = S$. We consider S in the following cases.

<u>Case 1</u>: $|S|$ is even. By Theorem 3.3, $\alpha|_S = id_S = \beta|_S$ implies that $(\alpha\beta)|_S = id_S$. Hence, $\alpha\beta$ is coregular.

<u>Case 2</u>: $|S|$ is odd. Let $S = \{a_1, \ldots, a_n\}$. If $\beta|_S$ is an identity mapping, then from Lemma 4.1(ii), $\alpha\beta$ is coregular.

Next, assume that $\beta|_S$ is not an identity mapping. Then by Theorem 4.4, $\beta(a_k) = a_{n-(k-1)}$ for all $a_k \in \{a_1, \ldots, a_n\}$. We consider again α in the following 2 cases.

Case 2.1: $\alpha|_S$ is an identity mapping. Then

$$(\alpha\beta)(a_k) = \alpha(\beta(a_k)) = \alpha(a_{n-(k-1)}) = a_{n-(k-1)}$$

for all $a_k \in \{a_1, \ldots, a_n\}$. Hence, $\alpha\beta$ is coregular.

Case 2.2: $\alpha|_S$ is not an identity mapping. Then $\alpha(a_k) = a_{n-(k-1)}$ for all $a_k \in \{a_1, \ldots, a_n\}$. It follows that $(\alpha\beta)(a_k) = \alpha(\beta(a_k)) = \alpha(a_{n-(k-1)}) = a_{n-[n-(k-1)-1]} = a_k$ for all $a_k \in \{a_1, \ldots, a_n\}$, that is, $(\alpha\beta)|_S = id_S$. Hence, $\alpha\beta$ is coregular.

Altogether, we can prove that $\alpha\beta \in COT_S(\mathbf{F})$ and therefore $COT_S(\mathbf{F})$ is a subsemigroup of $OT(\mathbf{F})$. Because every elements in $COT_S(\mathbf{F})$ is coregular, $COT_S(\mathbf{F})$ is a coregular subsemigroup of $OT(\mathbf{F})$. □

Next we are looking for other coregular subsemigroups of $OT(\mathbf{F})$. Before doing so, we need the following proposition that gives a sufficient condition for the composition of coregualr elements in $OT(\mathbf{F})$ to be coregular.

Proposition 4.4. *Let $\mathbf{S}, \mathbf{T}$ be subfences of a finite fence $\mathbf{F}$ and $S \subseteq T$. Assume that $T = \{a_1, \ldots, a_n\}$ and $S = \{a_k, \ldots, a_l\}$. Let $\alpha, \beta \in COT(\mathbf{F})$ with $\operatorname{ran}\alpha = S$ and $\operatorname{ran}\beta = T$. If $d(a_1, a_k) = d(a_l, a_n)$, then $\alpha\beta$ and $\beta\alpha$ are coregular such that $\operatorname{ran}\alpha\beta = S = \operatorname{ran}\beta\alpha$.*

Proof. We consider $|T|$ in the following 2 cases. First, we assume that $|T|$ is odd. From $d(a_1, a_k) = d(a_l, a_n)$, we have $|S|$ is odd implying that

$\beta|_T$ is an identity mapping or $\beta(a_m) = a_{n-(m-1)}$ for all $m \in \{1, 2, \ldots, n\}$ and

$\alpha|_S$ is an identity mapping or $\alpha(a_m) = a_{l-(m-k)}$ for all $m \in \{k, k+1, \ldots, l\}$.

By Lemma 4.1(i), $\operatorname{ran}(\alpha\beta) = S$. We consider β in the following 2 cases.

Case 1: $\beta|_T$ is an identity mapping. By Lemma 4.1(ii), $\alpha\beta$ is coregular. Next we show that $\beta\alpha$ is coregular. From $\beta(F) = T$ and $\alpha(F) = S$, we have $\beta\alpha(F) = \beta(S)$. Since $S \subseteq T$ and $\beta|_T$ is an identity mapping, $\beta(S) = S$ implies that $\operatorname{ran}(\beta\alpha) = S$. If $\alpha|_S$ is an identity mapping, then $(\beta\alpha)|_S$ is an identity mapping. Hence, $\beta\alpha$ is coregular. Assume that $\alpha|_S$ is not an identity mapping. Let $a_i \in S$. Then $a_i \in T = \operatorname{ran}\beta$ and

$$(\beta\alpha)(a_i) = \beta(\alpha(a_i)) = \beta(a_{l-(i-k)}) = a_{l-(i-k)}.$$

By Theorem 3.5, $\beta\alpha$ is coregular.

Case 2: $\beta|_T$ is not an identity mapping. From $d(a_1, a_k) = d(a_l, a_n)$, we have

$$\begin{aligned} n-(i-1) &= n - d(a_1, a_i) = n - [d(a_1, a_k) + d(a_k, a_i)] \\ &= n - d(a_1, a_k) - d(a_k, a_i) = [n - d(a_1, a_k)] - d(a_k, a_i) \\ &= [n - d(a_l, a_n)] - d(a_k, a_i) = [n - (n-l)] - (i-k) \\ &= l - (i-k) \end{aligned}$$

for all $i \in \{k, k+1, \ldots, l\}$. Since $\beta|_T$ is not an identity mapping, $\beta(a_i) = a_{n-(i-1)} = a_{l-(i-k)} \in S$ for all $a_i \in \{a_k, \ldots, a_l\} = S$ implies that $\beta(S) \subseteq S$.

But $\beta|_T$ is injective and $S \subseteq T$, so $\beta(S) = S$ and hence, $\mathrm{ran}(\beta\alpha) = \beta\alpha(F) = \beta(S) = S$.

We consider α in the following 2 cases.

Case 2.1: $\alpha|_S$ is an identity mapping. Let $a_i \in S$. Then $k \leq i \leq l$ and $(\alpha\beta)(a_i) = \alpha(\beta(a_i)) = \alpha(a_{n-(i-1)}) = \alpha(a_{l-(i-k)}) = a_{l-(i-k)}$ and $(\beta\alpha)(a_i) = \beta(\alpha(a_i)) = \beta(a_i) = a_{n-(i-1)} = a_{l-(i-k)}$. Theorem 3.5 implies that $\alpha\beta$ and $\beta\alpha$ are coregular.

Case 2.2: $\alpha|_S$ is not an identity mapping. Let $a_i \in S$. Then we have

$$\begin{aligned}(\alpha\beta)(a_i) &= \alpha(\beta(a_i)) \\ &= \alpha(a_{n-(i-1)}) \\ &= \alpha(a_{l-(i-k)}) \\ &= a_{l-[l-(i-k)-k]} \\ &= a_{l-[l-(i-k)]+k} \\ &= a_{l-l+(i-k)+k} \\ &= a_i\end{aligned}$$

and $(\beta\alpha)(a_i) = \beta(a_{l-(i-k)}) = \beta(a_{n-(i-1)}) = a_{n-[n-(i-1)-1]} = a_i$. Hence, $(\alpha\beta)|_S = id_S = (\beta\alpha)|_S$ and therefore, $\alpha\beta$ and $\beta\alpha$ are coregular.

Next we assume that $|T|$ is even. From $S \subseteq T$ and $d(a_1, a_k) = d(a_l, a_n)$, we get that $|S|$ is even. By Theorem 3.4, $\alpha|_S$ and $\beta|_T$ are identity mappings. Similarly as Case 1, we can prove that $\alpha\beta$ and $\beta\alpha$ are coregular. $\square$

As a consequent of Proposition 4.4, we obtain a new coregular subsemigroup of $OT(\mathbf{F})$ which contains some $COT_S(\mathbf{F})$.

Theorem 4.5. *Let a and b be the initial point and the endpoint of a finite fence $\mathbf{F}$, respectively. Assume that $Sub = \{S \subseteq F \mid d(a, x) = d(y, b)\}$ where x and y are the initial point and the endpoint of S, respectively. Then $\bigcup_{S \in Sub} COT_S(\mathbf{F})$ is a coregular subsemigroup of $OT(\mathbf{F})$.*

Proof. Let $\alpha, \beta \in \bigcup_{S \in Sub} COT_S(\mathbf{F})$. Then there are $S, T \in Sub$ with $\mathrm{ran}\,\alpha = S$ and $\mathrm{ran}\,\beta = T$, respectively. Assume that x, x' and y, y' are initial points and endpoints of S and T, respectively. We may assume that $d(a, x) \geq d(a, x')$. Then from $S, T \in Sub$, we have $d(y, b) = d(a, x) > d(a, x') = d(y', b)$. It follows that $S \subseteq T$ and $d(a, x) - d(a, x') = d(y, b) - d(y', b)$. Since $d(a, x) - d(a, x') = d(x', x)$ and $d(y, b) - d(y', b) = d(y, y')$, so $d(x', x) = d(y, y')$. Proposition 4.4 implies that $\alpha\beta, \beta\alpha \in COT_S(\mathbf{F}) \subseteq \bigcup_{S \in Sub} COT_S(\mathbf{F})$. Hence, $\bigcup_{S \in Sub} COT_S(\mathbf{F})$ is a subsemigroup of $OT(\mathbf{F})$. Therefore $\bigcup_{S \in Sub} COT_S(\mathbf{F})$ is a coregular subsemigroup since every element in $\bigcup_{S \in Sub} COT_S(\mathbf{F})$ is coregular. $\square$

It seem to be that $\bigcup_{S\in Sub} COT_S(\mathbf{F})$ is a maximal coregular subsemigroup of $OT(\mathbf{F})$. Unfortunately, it may not be the case since there are elements $\alpha \in \bigcup_{S\in Sub} COT_S(\mathbf{F})$ and $\beta \in COT(\mathbf{F}) \setminus \bigcup_{S\in Sub} COT_S(\mathbf{F})$ for which $\alpha\beta$ is coregular as shown in Proposition 4.6.

Proposition 4.6. *Let* $\mathbf{S}, \mathbf{T}$ *be subfences of a finite fence* $\mathbf{F} = (F; \leq)$ *with* $|S| \geq 2$ *and* $S \subset T$ *and let* $\alpha, \beta \in COT(\mathbf{F})$ *with* $\operatorname{ran}\alpha = S$ *and* $\operatorname{ran}\beta = T$. *Assume that* $T = \{a_1, \ldots, a_n\}$ *and* $S = \{a_k, \ldots, a_l\}$ *with* $d(a_1, a_k) \neq d(a_l, a_n)$. *Let* $\beta|_T \neq id_T$ *and* $\alpha|_S = id_S$ *with* $\alpha(x) = \beta(x)$ *for all* $x \in T\backslash S$. *If* $a_{n-(u-1)} \notin S$ *or* $a_u = a_{n-(u-1)}$ *for all* $u \in \{k, \ldots, l\}$, *then* $\alpha\beta \in COT(\mathbf{F})$.

Proof. By Lemma 4.1(i), $\operatorname{ran}(\alpha\beta) = S$. Let $a_u \in \{a_k, \ldots, a_l\} = S$. If $a_{n-(u-1)} \notin S$, then $\alpha\beta(a_u) = \alpha(a_{n-(u-1)}) = \beta(a_{n-(u-1)}) = a_u$. If $a_u = a_{n-(u-1)}$, then $\alpha\beta(a_u) = \alpha(a_{n-(u-1)}) = \alpha(a_u) = a_u$. Hence, $(\alpha\beta)|_S = id_S$ and therefore, $\alpha\beta$ is coregular. □

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