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BOUNDEDNESS IN NONLINEAR PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate bounds for solutions of the nonlinear perturbed functional differential systems using the notion of t_∞-similarity.

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1. Introduction and preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the nonlinear perturbed functional differential systems of (1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)),\ y(t_0) = y_0, \tag{2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t),\ v(t_0) = v_0 \tag{3}$$

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and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (3).

We recall some notions of h-stability [15].

Definition 1.1. The system (1) (the zero solution $x = 0$ of (1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1) (the zero solution $x = 0$ of (1)) is called *h-stable* (hS) if there exists $\delta > 0$ such that (1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Integral inequalities play a vital role in the study of boundedness and other qualitative properties of solutions of differential equations. In particular, Bihari's integral inequality continues to be an effective tool to study sophisticated problems such as stability, boundedness, and uniqueness of solutions.

The notion of h-stability (hS) was introduced by Pinto [14, 15] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems[15]. Choi and Koo [2], Choi and Ryu [3], and Choi et al. [4] investigated h-stability and bounds of solutions for the perturbed functional differential systems. Also, Goo [6, 7, 8, 9] and Goo et al. [10] studied h-stability and boundedness of solutions for the perturbed functional differential systems.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞*-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 11].

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞-similarity.

We give some related properties that we need in the sequal.

Lemma 1.1 ([15])**.** *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (6).

We need Alekseev formula to compare between the solutions of (1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.2. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 1.3 ([3])**.** *If the zero solution of (1) is hS, then the zero solution of (3) is hS.*

Theorem 1.4 ([4])**.** *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (3) is hS, then the solution $z = 0$ of (4) is hS.*

Lemma 1.5. $(Bihari-type\ inequality)$ *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 1.6 ([2, 7])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)u(\tau)d\tau ds, \ \ 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau)ds\Big], \ t_0 \leq t < b_1,$$

where W *and* W^{-1} *are the same functions as in Lemma 1.5, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 1.7 ([6])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)u(\tau)d\tau ds + \int_{t_0}^{t} \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)d\tau)ds\Big], \ t_0 \leq t < b_1$$

where W *and* W^{-1} *are the same functions as in Lemma 1.5, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

2. Main results

In this section, we investigate boundedness for solutions of the nonlinear perturbed functional differential systems via t_∞-similarity.

Lemma 2.1. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds \\ &+ \int_{t_0}^{t} \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)u(\tau)d\tau ds, \ \ 0 \leq t_0 \leq t. \end{aligned} \tag{9}$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau)ds\Big], \tag{10}$$

$t_0 \le t < b_1$, *where* W *and* W^{-1} *are the same functions as in Lemma 1.5, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau$$
$$+\lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Define a function $v(t)$ by the right member of (9) . Then

$$v'(t) = \lambda_1(t)u(t) + \lambda_2(t)w(u(t)) + \lambda_3(t)\int_{t_0}^{t} \lambda_4(s)u(s)ds + \lambda_5(t)\int_{t_0}^{t} \lambda_6(s)u(s)ds,$$

which implies

$$v'(t) \quad \le \Big[\lambda_1(t) + \lambda_2(t) + \lambda_3(t)\textstyle\int_{t_0}^{t} \lambda_4(s)ds + \lambda_5(t)\int_{t_0}^{t} \lambda_6(s)ds\Big]w(v(t)),$$

since v and w are nondecreasing, $u \le w(u)$ and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \le c + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)w(v(s))ds. \quad (11)$$

Then, by the well-known Bihari-type inequality, (11) yields the estimate (10). □

Theorem 2.2. *Let* $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$ *and* $w(u)$ *be nondecreasing in* u *such that* $u \le w(u)$ *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$. *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \ge t_0 \ge 0$ *and* $|x_0| \le \delta$ *for some constant* $\delta > 0$, *the solution* $x = 0$ *of (1) is hS with the increasing function* h, *and* g *in (2) satisfies*

$$\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau \le a(s)|y(s)| + b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau,\ t \ge t_0 \ge 0, \quad (12)$$

and

$$|h(t, y(t), Ty(t))| \le c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds \quad (13)$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, $\int_{t_0}^{\infty} k(s)ds < \infty$, *and* $\int_{t_0}^{\infty} q(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c)+c_2\int_{t_0}^{t}(a(s)+c(s)+b(s)\int_{t_0}^{s} k(\tau)d\tau+c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

$t_0 \le t < b_1$, *where* W *and* W^{-1} *are the same functions as in Lemma 1.5, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau$$
$$+c(s)\int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solution $y(t) = y(t, t_0, y_0)$ of (2) passing through (t_0, y_0) is given by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))(\int_{t_0}^{s} g(\tau, y(\tau))d\tau + h(s, y(s), Ty(s)))ds. \quad (14)$$

By Theorem 1.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 1.4, the solution $z = 0$ of (4) is hS. By Lemma 1.1 the hS condition of $x = 0$ of (1), (12), (13), and (14), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(a(s)|y(s)| + c(s)w(|y(s)|) \\
&+b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau + c(s)\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\Big)ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(a(s)\frac{|y(s)|}{h(s)} + c(s)w(\frac{|y(s)|}{h(s)} \\
&+b(s)\int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau + c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.
\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, an application of Lemma 2.1 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\, h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (2) is bounded on $[t_0, \infty)$, and so the proof is complete. $\square$

Remark 2.1. *Letting $w(u) = u$ and $c(t) = 0$ in Theorem 2.2, we obtain the same result as that of Theorem 3.1 in [9].*

Theorem 2.3. *Let $a, b, c, q, u, w \in C(\mathbb{R}^+)$ and $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with the increasing function h, and g in (2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)w(|y(t))|, \quad (15)$$

and

$$|h(t, y(t), Ty(t))| \le b(t)w(|y(t)|) + c(t)|Ty(t)|, |Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds,\ t \ge t_0 \ge 0, \quad (16)$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, *and* $\int_{t_0}^{\infty} q(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where W *and* W^{-1} *are the same functions as in Lemma 1.5, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. It is known that the solution of (2) is represented by the integral equation (14). By Theorem 1.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 1.4, the solution $z = 0$ of (4) is hS. Using the nonlinear variation of constants formula (14), the hS condition of $x = 0$ of (1), (15), and (16), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(a(s)w(|y(s)|) + b(s)w(|y(s)|) \\
&+c(s)\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\Big)ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)(a(s) + b(s))w(\frac{|y(s)|}{h(s)})ds \\
&+ \int_{t_0}^{t} c_2 h(t)c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds.
\end{aligned}$$

Put $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.6, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\, h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Remark 2.2. *Letting* $b(t) = c(t) = 0$ *in Theorem 2.3, we obtain the same result as that of Theorem 3.2 in [10].*

Lemma 2.4. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C[\mathbb{R}^+, \mathbb{R}^+]$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that, for some* $c \ge 0$, *we have*

$$\begin{aligned}
u(t) \le c + &\int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\Big(\int_{t_0}^{s}(\lambda_3(\tau)u(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(s)u(r)dr)d\tau \\
&+ \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)u(\tau)d\tau\Big)ds,\ t \ge t_0.
\end{aligned} \tag{17}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^t [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^s \lambda_7(\tau)d\tau\Big)]ds\Big],\ t \ge t_0. \tag{18}$$

Proof. Define a function $v(t)$ by the right member of (17). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)w(u(t)) + \lambda_2(t)\Big(\int_{t_0}^t (\lambda_3(s)u(s) + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)u(\tau)d\tau)ds \\ &+\lambda_6(t)\int_{t_0}^t \lambda_7(s)u(s)ds\Big) \\ &\le \Big[\lambda_1(t) + \lambda_2(t)\Big(\int_{t_0}^t (\lambda_3(s) + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)d\tau)ds \\ &+\lambda_6(t)\int_{t_0}^t \lambda_7(s)ds\Big)\Big]w(v(t)), \end{aligned}$$

$t \ge t_0$, since $v(t)$ is nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \le c + \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^s \lambda_7(\tau)d\tau\Big)w(z(s))ds. \tag{19}$$

Thus, (19) yields the estimate (18). □

Theorem 2.5. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$ and $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with the increasing function h, and g in (2) satisfies*

$$|g(t, y(t))| \le a(t)|y(t)| + b(t)\int_{t_0}^t k(s)|y(s)|ds \tag{20}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \le \int_{t_0}^t q(s)|y(s)|ds, \tag{21}$$

$t \ge t_0 \ge 0$, where $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$, $\int_{t_0}^\infty c(s)ds < \infty$, $\int_{t_0}^\infty k(s)ds < \infty$, and $\int_{t_0}^\infty q(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2) is bounded on on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau$$

$$+c(s)\int_{t_0}^{s} q(\tau)d\tau]ds\Big],$$

where W and W^{-1} are the same functions as in Lemma 1.5, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}[c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau$$
$$+c(s)\int_{t_0}^{s} q(\tau)d\tau]ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. By Theorem 1.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 1.4, the solution $z = 0$ of (4) is hS. Applying the nonlinear variation of constants formula (14), the hS condition of $x = 0$ of (1), (20), and (21), we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds$$
$$\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(\int_{t_0}^{s}(a(\tau)|y(\tau)|$$
$$+b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr)d\tau + c(s)(w(|y(s)|) + \int_{t_0}^{s} q(\tau)|y(\tau)|d\tau)\Big)ds$$
$$\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)\Big(c(s)w(\frac{|y(s)|}{h(s)})$$
$$+\int_{t_0}^{s}(a(\tau)\frac{|y(\tau)|}{h(\tau)} + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{h(r)}dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 2.4, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}[c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau$$
$$+c(s)\int_{t_0}^{s} q(\tau)d\tau]ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. □

Remark 2.3. Letting $w(u) = u$ and $c(t) = 0$ in Theorem 2.5, we obtain the similar result as that of Theorem 3.1 in [9].

Theorem 2.6. *Let $a, b, c, q, u, w \in C(\mathbb{R}^+)$ and $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant*

$\delta > 0$, the solution $x = 0$ of (1) is hS with the increasing function h, and g in (2) satisfies

$$|g(t, y(t))| \le a(t)w(|y(t)|) \tag{22}$$

and

$$|h(t, y(t), Ty(t))| \le b(t)|y(t)| + c(t)\int_{t_0}^{t} q(\tau)|y(\tau)|d\tau, \tag{23}$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, and $\int_{t_0}^{\infty} q(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2) is bounded on $[t_0, \infty)$ and

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau + \int_{t_0}^{s} a(\tau)d\tau)ds\Big]$$

where W and W^{-1} are the same functions as in Lemma 1.5, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau + \int_{t_0}^{s} a(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. By Theorem 1.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 1.4, the solution $z = 0$ of (4) is hS. By Lemma 2.1, the hS condition of $x = 0$ of (1), (22), and (23), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s} a(\tau)w(|y(\tau)|)d\tau \\
&\quad + b(s)|y(s)| + c(s)\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\Big)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(b(s)\frac{|y(s)|}{h(s)} \\
&\quad + c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau + \int_{t_0}^{s} a(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.
\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.7, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau + \int_{t_0}^{s} a(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (2) is bounded on $[t_0, \infty)$. This completes the proof. □

Remark 2.4. Letting $b(t) = c(t) = 0$ in Theorem 2.6, we obtain the similar result as that of Theorem 3.5 in [10].

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