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APPROXIMATELY QUINTIC MAPPINGS IN NON-ARCHIMEDEAN 2-NORMED SPACES BY FIXED POINT THEOREM

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Abstract. In this paper, using the fixed point method, we investigate the generalized Hyers-Ulam stability of the system of quintic functional equation

$$\begin{cases} f(x_1+x_2,y)+f(x_1-x_2,y)=2f(x_1,y)+2f(x_2,y) \\ f(x,2y_1+y_2)+f(x,2y_1-y_2)=f(x,y_1-2y_2)+f(x,y_1+y_2) \\ -f(x,y_1-y_2)+15f(x,y_1)+6f(x,y_2). \end{cases}$$

in non-Archimedean 2-Banach spaces.

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1. Introduction and preliminaries

In 1940, Ulam [22] posed the following problem concerning the stability of functional equations:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

Hyers [8] solved the Ulam's problem for the case of approximately additive functions in Banach spaces. Since then, the stability of several functional equations has been extensively investigated by several mathematicians [3, 5, 9, 10, 11, 14, 17]. The Hyers-Ulam stability for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

was proved by Skof [21] for a function $f : E_1 \longrightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space and later by Jung [13] on unbounded domains.

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Rassias [20] investigated the stability for the following cubic functional equation

$$f(2x+y)-3f(x+y)+3f(x)-f(x-y)=6f(y)$$

and Jun and Kim [12] investigated the stability for the following cubic funtional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1}$$

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that for any $r,s\in\mathbb{K}$, the following conditions hold: (i) $|r|=0$ if and only if $r=0$, (ii) $|rs|=|r||s|$, and (iii) $|r+s|\leq|r|+|s|$. A field $\mathbb{K}$ is called *a valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r+s|\leq max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$, then the valuation $|\cdot|$ is called a *non-Archimedean valuation* and the field with a non-Archimedean valuation is called a *non-Archimedean field.* If $|\cdot|$ is a non-Archimedean valuation on $\mathbb{K}$, then clearly, $|1|=|-1|$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Definition 1.1. Let X be a vector space over a non-Archimedean field $\mathbb{K}$. A function $\|\cdot\|:X\longrightarrow\mathbb{R}$ is called *a non-Archimedean norm* if it satisfies the following conditions:
(a) $\|x\|=0$ if and only if $x=0$,
(b) $\|rx\|=|r|\|x\|$, and
(c) $||x+y||\leq max\{\|x\|,\|y\|\}$ for all $x,y\in X$ and all $r\in\mathbb{K}$.
If $\|\cdot\|$ is a non-Archimedean norm, then $(X,\|\cdot\|)$ is called *a non-Archimedean normed space.*

Let $(X,\|\cdot\|)$ be a non-Archimedean normed space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be *convergent* in $(X,\|\cdot\|)$ if there exists an $x\in X$ such that $\lim_{n\to\infty}\|x_n-x\|=0$. In case, x is called *the limit of the sequence* $\{x_n\}$, and one denotes it by $\lim_{n\to\infty}x_n=x$. A sequence $\{x_n\}$ is said to be *Cauchy* in $(X,\|\cdot\|)$ if $\lim_{n\to\infty}\|x_{n+p}-x_n\|=0$ for all $p\in\mathbb{N}$. By (c) in Definition 1.1,

$$\|x_n-x_m\|\leq max\{\|x_{j+1}-x_j\|\mid m\leq j\leq n-1\}\ \ (n>m),$$

a sequence $\{x_n\}$ is Cauchy in $(X,\|\cdot\|)$ if and only if $\{x_{n+1}-x_n\}$ converges to zero in $(X,\|\cdot\|)$. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

Gähler [6, 7] has introduced the concept of 2-normed spaces and White [23] introduced the concept of 2-Banach spaces. In 1999 to 2003, Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces [15, 16].

Definition 1.2. Let X be a linear space over a non-Archimedean field $\mathbb{K}$ with $\dim X>1$ and $\|\cdot,\cdot\|:X\times X\to\mathbb{R}$ a function satisfying the following properties
$(NA1)$ $\|x,y\|=0$ if and only if x and y are linearly dependent,
$(NA2)$ $\|x,y\|=\|y,x\|$,

$(NA3)$ $\|x, ay\| = |a|\|x, y\|$, and
$(NA4)$ $\|x, y+z\| \ \leq max\{\|x, y\|, \|x, z\|\}$
for all $x, y, z \in X$ and all $a \in \mathbb{K}$. Then $\|\cdot, \cdot\|$ is called *a non-Archimedean 2-norm* and $(X, \|\cdot, \cdot\|)$ is called a *non-Archimedean 2-normed spaces.*

Definition 1.3. A sequence $\{x_n\}$ in a non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$ is called *a Cauchy sequence* if

$$\lim_{n,m\to\infty} \|x_n - x_m,\ x\| = 0$$

for all $x \in X$.

Definition 1.4. A sequence $\{x_n\}$ in a non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$ is called *convergent* if

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in X$ and for some $x \in X$. In case, x is called *the limit of the sequence* $\{x_n\}$, and we denoted by $x_n \to x$ *as* $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Let $\{x_n\}$ be a sequence in a non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$. It follows from (NA4) that

$$\|x_m - x_n, y\| \ \leq max\{\|x_{j+1} - x_j, y\| \mid n \leq j \leq m-1\} \qquad (n < m),$$

for all $y \in X$ and so a sequence $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot, \cdot\|)$ if and only if $\{x_{m+1} - x_m\}$ converges to zero in $(X, \|\cdot, \cdot\|)$.

A non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$ is called a *non-Archimedean 2-Banach space* if every Cauchy sequence in $(X, \|\cdot, \cdot\|)$ is convergent. Now, we state the following results as lemma [18].

Lemma 1.5. *Let* $(X, \|\cdot, \cdot\|)$ *be a non-Archimedean 2-normed space. Then we have the following* :
(1) $\big|\|x, z\| - \|y, z\|\big| \leq \ \|x - y, z\|$ *for all* $x, y, z \in X$,
(2) $\|x, z\| = 0$ *for all* $z \in X$ *if and only if* $x = 0$, *and*
(3) *for any convergent sequence* $\{x_n\}$ *in* $(X, \|\cdot, \cdot\|)$,

$$\lim_{n\to\infty} \|x_n, z\| \ = \ \|\lim_{n\to\infty} x_n, z\|$$

for all $z \in X$.

In 2003, Radu [19] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative (see also [1, 2]).

We recall the following theorem by Margolis and Diaz.

Theorem 1.6 ([4])**.** *Let* (X, d) *be a complete generalized metric space and let* $J : X \longrightarrow X$ *be a strictly contractive mapping with some Lipschitz constant L with* $0 < L < 1$. *Then for each given element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers n or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$

(4) $d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

In this paper, we investigate the following cubic functional equation

$$f(2x+y) + f(2x-y) = f(x-2y) + f(x+y) - f(x-y) + 15f(x) + 6f(y) \tag{2}$$

and using fixed point method, we inverstigate the generalized Hyers-Ulam stability for the system of the quintic functional equation

$$\begin{cases} f(x_1+x_2, y) + f(x_1-x_2, y) = 2f(x_1, y) + 2f(x_2, y) \\ f(x, 2y_1+y_2) + f(x, 2y_1-y_2) = f(x, y_1-2y_2) + f(x, y_1+y_2) \\ -f(x, y_1-y_2) + 15f(x, y_1) + 6f(x, y_2), \end{cases} \tag{3}$$

and prove the generalized Hyers-Ulam stability for (3) in non-Archimedean 2-Banach spaces. In this paper, we will assume that $(X, \|\cdot\|)$ is a non-Archimedean normed space and $(Y, \|\cdot, \cdot\|)$ is a non-Archimedean 2-Banach space.

2. Stability of quintic mappings

In this section, using the fixed point method, we investigate the generalized Hyers-Ulam stability for the system of quintic functional equation (3) in non-Archimedean 2-Banach spaces. We start the following lemma.

Lemma 2.1. *Let* $f : X \longrightarrow Y$ *be a mapping with* (2). *Then* f *is a cubic mapping.*

Proof. Suppose that f satisfies (2). Letting $x = y = 0$ in (2), we have $f(0) = 0$ and letting $y = 0$ in (2), we have

$$f(2x) = 8f(x) \tag{4}$$

for all $x \in X$. Letting $x = 0$ in (2), by (4), we have $f(y) = -f(-y)$ for all $y \in X$ and so f is odd. Letting $y = -y$ in (2), we have

$$f(2x-y) + f(2x+y) - f(x+2y) - f(x-y) + f(x+y) - 15f(x) + 6f(y) = 0 \tag{5}$$

for all $x, y \in X$ and by (2) and (5), we have

$$f(x+2y) - f(x-2y) - 2f(x+y) + 2f(x-y) - 12f(y) = 0 \tag{6}$$

for all $x, y \in X$. Interching x and y in (6), since f is odd, f satisfies (1) and hence f is cubic. □

The function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x, y) = cx^2y^3$ is a solution of (3). In partcular, letting $y = x$ in (3), we get a quintic function $g : \mathbb{R} \longrightarrow \mathbb{R}$ in one variable given by $g(x) = f(x, x) = cx^5$.

Proposition 2.2. *If a mapping* $f : X^2 \longrightarrow Y$ *satisfies* (3), *then* $f(\lambda x, \mu y) = \lambda^2\mu^3 f(x, y)$ *for all* $x, y \in X$ *and all rational numbers* λ, μ.

Theorem 2.3. *Let $\phi_1, \phi_2 : X^3 \times Y \longrightarrow [0, \infty)$ be functions such that*

$$\phi_i(2x, 2y, 2z, w) \leq |2|^5 L\phi_i(x, y, z, w) \quad (i = 1, 2) \tag{7}$$

for all $x, y, z \in X$, $w \in Y$ and some L with $0 < L < 1$. Suppose that $f : X^2 \longrightarrow Y$ is a mapping such that $f(x, 0) = f(0, x) = 0$ for all $x \in X$,

$$\|f(x_1 + x_2, y) + f(x_1 - x_2, y) - 2f(x_1, y) - 2f(x_2, y), w\| \leq \phi_1(x_1, x_2, y, w), \tag{8}$$

and

$$\begin{aligned} &\|f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) - f(x, y_1 - 2y_2) - f(x, y_1 + y_2) \\ &\quad + f(x, y_1 - y_2) - 15f(x, y_1) - 6f(x, y_2), w\| \leq \phi_2(x, y_1, y_2, w) \end{aligned} \tag{9}$$

for all $w \in Y$ and all $x, y, x_1, x_2, y_1, y_2 \in X$. Then there exists a unique quintic mapping $T : X^2 \longrightarrow Y$ satisfying (3) *and*

$$\|f(x, y) - T(x, y), w\| \leq \frac{1}{1 - L}\Phi(x, y, w) \tag{10}$$

for all $w \in Y$ and all $x, y \in X$, where

$$\Phi(x, y, w) = max\{\ |2|^{-2}\phi_1(x, x, y, w),\ |2|^{-6}\phi_2(2x, y, 0, w)\}.$$

Proof. Putting $y_2 = 0$ and $y_1 = y$ in (9), we get

$$\|f(x, 2y) - 2^3 f(x, y), w\| \leq |2|^{-1}\phi_2(x, y, 0, w) \tag{11}$$

for all $w \in Y$ and all $x, y \in X$. Putting $x_1 = x_2 = x$ in (8), we get

$$\|f(2x, y) - 2^2 f(x, y), w\| \leq \phi_1(x, x, y, w) \tag{12}$$

for all $w \in Y$ and all $x, y \in X$. Thus by (11) and (12), we have

$$\begin{aligned} &\|f(2x, 2y) - 2^5 f(x, y), w\| \\ &= \|f(2x, 2y) - 2^3 f(2x, y) + 2^3[f(2x, y) - 2^2 f(x, y)], w\| \\ &\leq max\{\|f(2x, 2y) - 2^3 f(2x, y), w\|,\ |2|^3\|f(2x, y) - 2^2 f(x, y), w\|\} \\ &\leq max\{|2|^3\phi_1(x, x, y, w), |2|^{-1}\phi_2(2x, y, 0, w)\} \end{aligned} \tag{13}$$

for all $w \in Y$ and all $x, y \in X$. It follows from (13) that

$$\|2^{-5} f(2x, 2y) - f(x, y), w\|\ \leq \Phi(x, y, w) \tag{14}$$

for all $w \in Y$ and all $x, y \in X$.

Consider the set $S = \{h \mid h : X \times X \longrightarrow Y with\ h(x, 0) = h(0, x) = 0,\ \forall x \in X\}$ and the generalized metric d on S defined by

$$d(g, h) = inf\{\varepsilon \in [0, \infty) \mid \|g(x, y) - h(x, y), w\| \leq \varepsilon\ \Phi(x, y, w),\ \forall w \in Y,\ \forall x, y \in X\}.$$

Then (S, d) is a complete metric space [2]. Define a mapping $J : S \longrightarrow S$ by $Jg(x, y) = 2^{-5}g(2x, 2y)$ for all $x, y \in X$ and all $g \in S$. Let $g, h \in S$ and

$d(g,h) \le \varepsilon$ for some non-negative real number ε. Then by (7), we have

$$\begin{aligned}\|Jg(x,y) - Jh(x,y), w\| &= |2|^{-5}\|g(2x,2y) - h(2x,2y), w\| \\ &\le |2|^{-5}\varepsilon\Phi(2x,2y,w) \\ &= |2|^{-5}\varepsilon\ max\{|2|^{-2}\phi_1(2x,2x,2y,w), |2|^{-6}\phi_2(4x,2y,0,w)\} \\ &\le \varepsilon L\ \Phi(x,y,w),\end{aligned}$$

and so $d(Jg,Jh) \le \varepsilon L$. This mean that $d(Jg,Jh) \le Ld(g,h)$ for all $g,h \in S$ and so J is a strictly contractive mapping. By (14), we get $d(Jf,f) \le 1 < \infty$. By Theorem 1.6, there exists a mapping $T : X^2 \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, T) \to 0$ as $n \to \infty$, which implies the equality $T(x,y) = \lim_{n\to\infty} 2^{-5n} f(2^n x, 2^n y)$. Since $d(Jf,f) \le 1 < \infty$, by (4) in Theorem 1.6, we have (10). By (8) and (9), we get

$$\begin{aligned}&\|T(x_1+x_2,y) + T(x_1-x_2,y) - 2T(x_1,y) - 2T(x_2,y), w\| \\ &\le \lim_{n\to\infty} |2|^{-5n}\phi_1(2^n x_1, 2^n x_2, 2^n y, w) \\ &\le \lim_{n\to\infty} L^n \phi_1(x_1,x_2,y,w) = 0,\end{aligned}$$

and

$$\begin{aligned}&\|T(x,2y_1+y_2) + T(x,2y_1-y_2) - T(x,y_1-2y_2) - T(x,y_1+y_2) \\ &\quad + T(x,y_1-y_2) - 15T(x,y_1) - 6T(x,y_2), w\| \\ &\le \lim_{n\to\infty} |2|^{-5n}\phi_2(2^n x, 2^n y_1, 2^n y_2, w) \\ &\le \lim_{n\to\infty} L^n \phi_2(x,y_1,y_2,w) = 0\end{aligned}$$

for all $w \in Y$ and all $x,y,x_1,x_2,y_1,y_2 \in X$. Hence T satisfies (3).

To prove the uniquness of T, assume that $T_1 : X^2 \longrightarrow Y$ is another solution of (3) satisfying (10). Then T_1 is a fixed point of J and by (10),

$$d(Jf,T_1) = d(Jf,JT_1) \le \frac{L}{1-L} < \infty.$$

By (3) in Theorem 1.6, we have $T = T_1$. □

Theorem 2.4. *Let $\phi_1, \phi_2 : X^3 \times Y \longrightarrow [0,\infty)$ be functions such that*

$$\phi_i(x,y,z,w) \le |2|^{-5}L\phi_i(2x,2y,2z,w) \quad (i=1,2) \tag{15}$$

for all $x,y,z \in X$, $w \in Y$ and some L with $0 < L < 1$. Suppose that $f : X^2 \longrightarrow Y$ is a mapping satisfying $f(x,0) = f(0,x) = 0$ for all $x \in X$, (8) and (9). Then there exists a unique quintic mapping $T : X^2 \longrightarrow Y$ satisfying (3) and

$$\|f(x,y) - T(x,y), w\| \le \frac{L}{1-L}\Psi(x,y,w) \tag{16}$$

for all $w \in Y$ and all $x,y \in X$, where

$$\Psi(x,y,w) = max\{\ |2|^{-5}\phi_1(x,x,2y,w),\ |2|^{-4}\phi_2(x,y,0,w)\}.$$

Proof. Putting $y_2 = 0$ and $y_1 = \frac{y}{2}$ in (9), we get

$$\Big\|2^3 f\Big(x, \frac{y}{2}\Big) - f(x,y), w\Big\| \le |2|^{-1}\phi_2\Big(x, \frac{y}{2}, 0, w\Big) \tag{17}$$

for all $w \in Y$ and all $x, y \in X$. Putting $x_1 = x_2 = \frac{x}{2}$ in (8), we get

$$\Big\|2^2 f\Big(\frac{x}{2}, y\Big) - f(x,y), w\Big\| \le \phi_1\Big(\frac{x}{2}, \frac{x}{2}, y, w\Big) \tag{18}$$

for all $w \in Y$ and all $x, y \in X$. Thus by (17) and (18), we have

$$\begin{aligned}
&\Big\|2^5 f\Big(\frac{x}{2}, \frac{y}{2}\Big) - f(x,y), w\Big\| \\
&= \Big\|2^5 f\Big(\frac{x}{2}, \frac{y}{2}\Big) - 2^2\ f\Big(\frac{x}{2}, y\Big) + 2^2\Big[f\Big(\frac{x}{2}, y\Big) - 2^{-2} f(x,y)\Big], w\Big\| \\
&\le max\Big\{\Big\|2^2\Big[2^3 f\Big(\frac{x}{2}, \frac{y}{2}\Big) -\ f\Big(\frac{x}{2}, y\Big)\Big], w\Big\|,\ \Big\|2^2 f\Big(\frac{x}{2}, y\Big) - f(x,y), w\Big\|\Big\} \\
&\le max\Big\{|2|\phi_2\Big(\frac{x}{2}, \frac{y}{2}, 0, w\Big), \phi_1\Big(\frac{x}{2}, \frac{x}{2}, y, w\Big)\Big\} \\
&\le L\ max\{|2|^{-5}\phi_1(x, x, 2y, w), |2|^{-4}\phi_2(x, y, 0, w)\}
\end{aligned}$$

for all $x, y \in X$ and all $w \in Y$. That is, we have

$$\|2^5 f\Big(\frac{x}{2}, \frac{y}{2}\Big) - f(x,y), w\|\ \le L\ \Psi(x, y, w) \tag{19}$$

for all $x, y \in X$ and all $w \in Y$.

Consider the set $S = \{h \mid h : X \times X \longrightarrow Y with\ h(x,0) = h(0,x) = 0,\ \forall x \in X\}$ and the generalized metric d on S defined by

$$d(g,h) = inf\{\varepsilon \in [0,\infty) \mid \|g(x,y) - h(x,y), w\| \le \varepsilon\Psi(x,y,w),\ \forall w \in Y, \forall x, y \in X\}.$$

Then (S, d) is a complete metric space([2]). Define a mapping $J : S \longrightarrow S$ by $Jg(x,y) = 2^5 g(\frac{x}{2}, \frac{y}{2})$ for all $x, y \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \le \varepsilon$ for some non-negative real number ε. Then by (15), we have

$$\begin{aligned}
\|Jg(x,y) - Jh(x,y), w\| &= |2|^5\|g\Big(\frac{x}{2}, \frac{y}{2}\Big) - h\Big(\frac{x}{2}, \frac{y}{2}\Big), w\| \\
&\le |2|^5\varepsilon\ \Phi\Big(\frac{x}{2}, \frac{y}{2}, w\Big) \\
&\le \varepsilon L\ \Psi(x, y, w),
\end{aligned}$$

and so $d(Jg, Jh) \le \varepsilon L$. This mean that $d(Jg, Jh) \le L\ d(g,h)$ for all $g, h \in S$ and so J is a strictly contractive mapping. By (19), we get $d(Jf, f) \le L < \infty$. By Theorem 1.6, there exists a mapping $T : X^2 \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, T) \to 0$ as $n \to \infty$, which imples the equality $T(x,y) = \lim_{n\to\infty}\ 2^{5n} f(\frac{x}{2^n}, \frac{y}{2^n})$. Since $d(Jf, f) \le L$, by (4) in Theorem 1.6, we have (16) and by (8) and (9), we get

$$\begin{aligned}&\|T(x_1+x_2,y)+T(x_1-x_2,y)-2T(x_1,y)-2T(x_2,y),w\|\\ \leq&\lim_{n\to\infty}|2|^{5n}\phi_1\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{y}{2^n},w\Big)\\ \leq&\lim_{n\to\infty}L^n\phi_1(x_1,x_2,y,w)=0,\end{aligned}$$

and

$$\begin{aligned}&\|T(x,2y_1+y_2)+T(x,2y_1-y_2)-T(x,y_1-2y_2)-T(x,y_1+y_2)\\ &+T(x,y_1-y_2)-15T(x,y_1)-6T(x,y_2),w\|\\ \leq&\lim_{n\to\infty}|2|^{5n}\phi_2\Big(\frac{x}{2^n},\frac{y_1}{2^n},\frac{y_2}{2^n},w\Big)\\ \leq&\lim_{n\to\infty}L^n\phi_2(x,y_1,y_2,w)=0\end{aligned}$$

for all $w\in Y$ and all $x,y,x_1,x_2,y_1,y_2\in X$. Hence T satisfies (3).

To prove the uniquness of T, assume that $T_1:X^2\longrightarrow Y$ is another solution of (3) satisfying (16). Then T_1 is a fixed point of J and by (16),

$$d(Jf,T_1)=d(Jf,JT_1)\leq\frac{L^2}{1-L}<\infty.$$

By (3) in Theorem 1.6, we have $T=T_1$. □

As example of $\phi_1(x_1,x_2,y,w)$ and $\phi_2(x,y_1,y_2,w)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi_1(x_1,x_2,y,w)=\theta\ (\|x_1\|^p+\|x_2\|^p+\|y\|^p)\|w\|$ and $\phi_2(x,y_1,y_2,w)=|2|^4\ \theta\ (\|x\|^p+\|y_1\|^p+\|y_2\|^p)\|w\|$ for all $x,y,x_1,x_2,y_1,y_2\in X$, all $w\in Y$ and some positive real number θ. Then we have the following corollary.

Corollary 2.5. *Let θ,p be positive real numbers with $p\neq 5$. Suppose that $f:X^2\longrightarrow Y$ is a mapping satisfying $f(x,0)=f(0,x)=0$,*

$$\begin{aligned}&\|f(x_1+x_2,y)+f(x_1-x_2,y)-2f(x_1,y)-2f(x_2,y),w\|\\ \leq&\ \theta(\|x_1\|^p+\|x_2\|^p+\|y\|^p)\|w\|,\end{aligned}$$

and

$$\begin{aligned}&\|f(x,2y_1+y_2)+f(x,2y_1-y_2)-f(x,y_1-2y_2)-f(x,y_1+y_2)+f(x,y_1-y_2)\\ &-15f(x,y_1)-6f(x,y_2),w\|\leq|2|^4\theta(\|x\|^p+\|y_1\|^p+\|y_2\|^p)\|w\|\end{aligned}$$

for all $w\in Y$ and all $x,y,x_1,x_2,y_1,y_2\in X$. Then there exists a unique quintic mapping $T:X^2\longrightarrow Y$ satisfying

$$\begin{aligned}&\|f(x,y)-T(x,y),w\|\\ \leq&\begin{cases}\frac{|2|^3\theta}{|2|^5-|2|^p}\Big[max\{2,\ |2|^p\}\|x\|^p+\|y\|^p\Big]\|w\|, & p>5\\ \frac{\theta}{|2|^p-|2|^5}\ max\{2\|x\|^p+|2|^p\|y\|^p,\ |2|^5(\|x\|^p+\|y\|^p)\}\|w\|, & p<5\end{cases}\end{aligned}\tag{20}$$

for all $w\in Y$ and $x,y\in X$.

Proof. Let $\phi_1(x_1, x_2, y, w) = \theta\ (\|x_1\|^p + \|x_2\|^p + \|y\|^p)\|w\|$ and $\phi_2(x, y_1, y_2, w) = |2|^4\ \theta\ (\|x\|^p + \|y_1\|^p + \|y_2\|^p)\|w\|$. Note that

$$\begin{aligned}\phi_i(2x, 2y, 2z, w) &= |2|^p \phi_i(x, y, z, w),\\ &= |2|^5\ |2|^{p-5}\phi_i(x, y, z, w) \quad (i = 1, 2).\end{aligned}$$

So if $p > 5$, by Theorem 2.3, we have (20). Note that

$$\begin{aligned}\phi_i\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, w\Big) &= |2|^{-p}\phi_i(x, y, z, w),\\ &= |2|^{-5}\ |2|^{5-p}\phi_i(x, y, z, w) \quad (i = 1, 2).\end{aligned}$$

So if $p < 5$, by Theorem 2.4, we have (20). □

As another example of $\phi_1(x, y, z, w)$ and $\phi_2(x, y, z, w)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi_1(x, y, z, w) = \phi_2(x, y, z, w) = \theta\ \|x\|^p\|y\|^q\|z\|^r\|w\|$ for all $x, y, z \in X$, all $w \in Y$ and some positive real number p, q, r, θ. Then we have the following corollary:

Corollary 2.6. *Let p, q, r and θ be positive real numbers with $p + q + r \neq 5$. Suppose that $f : X^2 \longrightarrow Y$ is a mapping satisfying $f(x, 0) = 0$,*

$$\begin{aligned}&\|f(x_1 + x_2, y) + f(x_1 - x_2, y) - 2f(x_1, y) - 2f(x_2, y), w\|\\ &\leq \theta\ \|x_1\|^p\|x_2\|^q\|y\|^r\|w\|,\end{aligned}$$

and

$$\begin{aligned}&\|f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) - f(x, y_1 - 2y_2) - f(x, y_1 + y_2) + f(x, y_1 - y_2)\\ &- 15f(x, y_1) - 6f(x, y_2), w\| \leq \theta\ \|x\|^p\|y_1\|^q\|y_2\|^r\|w\|\end{aligned}$$

for all $w \in Y$ and all $x, y, x_1, x_2, y_1, y_2 \in X$. Then there exists a unique quintic mapping $T : X^2 \longrightarrow Y$ satisfying

$$\|f(x, y) - T(x, y), w\| \leq \begin{cases} \frac{|2|^3\theta}{|2|^5 - |2|^{p+q+r}}\ \|x\|^{p+q}\|y\|^r\|w\|, & p + q + r > 5 \\ \frac{|2|^r\theta}{|2|^{p+q+r} - |2|^5}\ \|x\|^{p+q}\|y\|^r\|w\|, & p + q + r < 5 \end{cases} \tag{21}$$

for all $w \in Y$ and all $x, y \in X$.

Proof. Let $\phi_1(x, y, z, w) = \phi_2(x, y, z, w) = \theta\ \|x\|^p\|y\|^q\|z\|^r\|w\|$. Then we have

$$\begin{aligned}\phi_i(2x, 2y, 2z, w) &= |2|^{p+q+r}\phi_i(x, y, z, w)\\ &= |2|^5\ |2|^{p+q+r-5}\ \phi_i(x, y, z, w) \quad (i = 1, 2).\end{aligned}$$

Hence if $p + q + r > 5$, by Theorem 2.3, we have (21). Note that

$$\begin{aligned}\phi_i(x, y, z, w) &= |2|^{-(p+q+r)}\phi_i(2x, 2y, 2z, w),\\ &= |2|^{-5}\ \ |2|^{5-p-q-r}\ \phi_i(2x, 2y, 2z, w) \quad (i = 1, 2).\end{aligned}$$

Thus if $p + q + r < 5$, by Theorem 2.4, we have (21). □

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