J. Appl. Math. & Informatics Vol. **33**(2015), No. 3 - 4, pp. 425 - 434
http://dx.doi.org/10.14317/jami.2015.425

TWO POINT FRACTIONAL BOUNDARY VALUE PROBLEM AT RESONANCE

A. GUEZANE-LAKOUD, S. KOUACHI* AND F. ELLAGGOUNE

Abstract. In this paper, a two-point fractional boundary value problem at resonance is considered. By using the coincidence degree theory some existence results of solutions are established.

AMS Mathematics Subject Classification : 34B15.
Key words and phrases : Fractional boundary value problem, Existence of solution, Mawhin Theorem, Topological degree.

1. Introduction

In this paper, we investigate the existence of solutions for the following two-point fractional boundary value problem (BVP) at resonance

$$\begin{gathered} {}^cD^\alpha_{0+}u(t) = f(t, u(t), u^{'}(t), u^{''}(t)), 0 < t < 1, \qquad (1.1) \\ u(0) = u^{'}(0) = 0, u^{''}(0) = 2u(1), \end{gathered}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $2 < \alpha < 3$ and ${}^cD^\alpha_{0+}$ denotes the Caputo's fractional derivative. The two-point boundary value problem (1.1) happens to be at resonance in the sense that its associated linear homogeneous boundary value problem

$$\begin{gathered} {}^cD^\alpha_{0+} = 0, 0 < t < 1, \\ u(0) = u^{'}(0) = 0, u^{''}(0) = 2u(1), \end{gathered}$$

has a nontrivial solution $u(t) = ct^2$, $c \in \mathbb{R}$. To solve this problem, we use the coincidence degree of Mawhin [12,13]. This method is based on an equivalent formulation in an abstract space and a theory of topological degree. This formulation generally leads to an abstract operator of the form $N + L$, where L is a Fredholm operator of index zero and N is generally a nonlinear operator having some compactness properties with respect to L. Fractional differential equations arise in different areas of sciences such as in rheology, fluid flows, viscoelasticity,

Received August 20, 2014. Revised November 21, 2014. Accepted November 27, 2014.
*Corresponding author.

chemical physics, and so on [1,6,8,11,14-19]. Recently, boundary value problems for fractional differential equations at nonresonance have been studied by many authors [1,3,4,10] by using fixed point theorems, lower and upper solution. Moreover, boundary value problems for differential equations at resonance have also been studied in some papers, see [2,7]. In [20], the authors studied, by using the coincidence degree theory, the following BVP of fractional equation at resonance

$$^cD^{\alpha}_{0+}x(t) = f(t, x(t), x^{'}(t)), 0 < t < 1,$$
$$u(0) = 0, u^{'}(0) = u^{'}(1),$$

where $^cD^{\alpha}_{0+}$ denotes the Caputo's fractional differential operator of order α, $1 < \alpha \leq 2$ and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

A similar boundary value problem at resonance involving Riemann-Liouville fractional derivative is considered in [7]. The author solved the problem

$$D^{\alpha}_{0+}x(t) = f(t, x(t), x^{'}(t)), 0 < t < 1, 1 < \alpha < 2$$
$$D^{\alpha-2}_{0+}u(0) = 0, \eta u(\xi) = u(1), 0 < \xi < 1, \eta\xi^{\alpha-1} = 1$$

by applying degree theory theorem for coincidences.

By the same method, in [5] the authors established the existence of solutions for the following third-order differential equation

$$x^{'''}(t) = f(t, x(t), x^{'}(t)), 0 < t < 1,$$
$$x(0) = x^{''}(0) = 0, x(1) = \frac{2}{\eta^2}\int_0^{\eta} x(t)dt, \eta \in (0, 1),$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Caratheodory function, and $\eta \in (0, 1)$.

The organization of this paper is as follows. We present in Section 2 some notation and some basic results involved in the reformulation of the problem. In section 3, we give the main theorem and some lemmas, then we will see that the proof of the main theorem is an immediate consequence of these lemmas and the coincidence degree of Mawhin. At the end of this section, we give an example to illustrate the main result.

2. Preliminaries

We begin by introducing the fundamental tools of fractional calculus and the coincidence degree theory which will be used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral is defined by

$$I^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g(s)}{(t-s)^{1-\alpha}}ds.$$

for $g \in C([a, b])$ and $\alpha > 0$.

Definition 2.2. Let $f \in C^n([a, b])$, the Caputo fractional derivative of order $\alpha \geq 0$ of f is defined by $^cD^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{g^n(s)}{(t-s)^{\alpha-n+1}}ds$, where $n = [\alpha] + 1$ ($[\alpha]$ is the integer part of α).

Lemma 2.3. *For $\alpha > 0$, $g \in C([0,1], \mathbb{R})$, the homogenous fractional differential equation ${}^cD^\alpha_{a^+}g(t) = 0$ has a solution $g(t) = c_0 + c_1t + c_2t^2 + ... + c_{n-1}t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, ..., n-1$, here n is the smallest integer greater than or equal to α.*

Definition 2.4. Let X and Y be real normed spaces. A linear mapping $L : domL \subset X \to Y$is called a Fredholm mapping if the following two conditions hold:

(i) $kerL$ has a finite dimension, and

(ii) ImL is closed and has a finite codimension.

If L is a Fredholm mapping, its index is the integer $IndL = dimkerL - codimImL$.

From here if L Fredholm mapping of index zero then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL$,$KerQ = ImL$,$X = KerL \oplus KerP$,$Y = ImL \oplus ImQ$ and that the mapping $L|_{domL \cap KerP}$: $domL \cap KerP \to ImL$ is one-to-one and onto. We denote its inverse by K_P. Moreover, since $dimImQ = codimImL$, there exists an isomorphism $J : ImQ \to KerL$.

If Ω is an open bounded subset of X, and $domL \cap \bar{\Omega} \neq \oslash$, the map $N : X \to Y$ will be called $L-compact$ on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact.

Lemma 2.5 ([13])**.** *Let $L : domL \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$, $L - compact$ on $\bar{\Omega}$. Assume that the following conditions are satisfied*

(1) $Lu \neq Nu$ for every $(u, \lambda) \in [(domL\ KerL)] \cap \partial\Omega \times (0, 1)$;

(2) $Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;

(3) $deg(JQN|_{KerL}, KerL \cap \Omega, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $ImL = KerQ$.

Then the equation $Lx = Nx$ has at least one solution in $domL \cap \bar{\Omega}$.

In this paper, we denote by X and Y the Banach spaces: $X = C^2([0, 1], \mathbb{R})$ equipped with the norm $\|u\|_X = max\{\|u\|_\infty,\ |u^{'}\|_\infty, \|u^{''}\|_\infty\}$ and $Y = C([0, 1], \mathbb{R})$ equipped with the norm $\|y\|_Y = \|y\|_\infty$, where $\|u\|_\infty = max_{t\in[0,1]}|u(t)|$.

Define the operator $L : domL \subset X \to Y$ by

$$Lu =^c D^\alpha_{0^+}u \tag{2.1}$$

where

$$domL = \{u \in X|D^\alpha_{0^+}u(t) \in Y, u(0) = u^{'} = (0) = 0, u^{''}(0) = 2u(1)\}.$$

Let $N : X \to Y$ be the operator

$$Nu(t) = f(t, u(t), u^{'}(t), u^{''}(t)), \forall t \in [0, 1].$$

Then BVP (1.1) is equivalent to the operator equation

$$Lu = Nu, u \in domL.$$

It is known [12] that the coincidence equation $Lu = Nu$ is equivalent to

$$u = (P + JQN)u + K_P(I - Q)Nu.$$

3. Main results

We can now state our result on the existence of a solution for the BVP (1.1).

Theorem 3.1. *Let* $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous. Assume that*

(H_1) *there exists nonnegative functions* $p, q, r, z \in C[0, 1]$ *with* $\Gamma(\alpha-1)-2q_1-2r_1 - 2z_1 > 0$ *such that*

$$|f(t, u, v, w)| \leq p(t) + q(t)|u| + r(t)|v| + z(t)|w|, \forall t \in [0, 1], (u, v, w) \in \mathbb{R}^3$$

where

$$p_1 = \|p\|_\infty, q_1 = \|q\|_\infty, r_1 = \|r\|_\infty, z_1 = \|z\|_\infty.$$

(H_2) *there exists a constant* $B > 0$ *such that for all* $c \in \mathbb{R}$ *with* $|2c| > B$ *either*

$$cf(t, u, v, w) > 0, \forall t \in [0, 1], (u, v, w) \in \mathbb{R}^3$$

or

$$cf(t, u, v, w) < 0, \forall t \in [0, 1], (u, v, w) \in \mathbb{R}^3$$

Then BVP(1.1) has at least one solution in X.

In order to prove Theorem 3.1, we need to prove some lemmas below.

Lemma 3.2. *Let* L *be defined by (2.1), then*

$$KerL = \{u \in X | u(t) = c_2t^2, c_2 \in \mathbb{R}, \forall t \in [0, 1]\} \tag{3.1}$$

$$ImL = \{y \in Y | \int_0^1 (1 - s)^{\alpha-1}y(s)ds = 0\} \tag{3.2}$$

Proof. By Lemma 2.3, $D^\alpha_{0+}u(t) = 0$ has solution

$$u(t) = c_0 + c_1t + c_2t^2 + ... + c_{n-1}t^{n-1}, c_0, c_1, c_2 \in \mathbb{R}$$

Combining with boundary condition of BVP (1.1), (3.1) holds. For $y \in ImL$, there exists $u \in domL$ such that $y = Lu \in Y$. We have

$$u(t) = \frac{1}{\Gamma(\alpha} \int_0^t (t - s)^{\alpha-1}y(s)ds + c_0 + c_1t + c_2t^2.$$

Differentiating two times and using the boundary conditions for BVP (1.1), we get

$$\int_0^1 (1 - s)^{\alpha-1}y(s)ds = 0$$

On the other hand, let $y \in Y$ satisfying $\int_0^1 (1 - s)^{\alpha-1}y(s)ds = 0$, then $u(t) = I^\alpha_{0+}y(t) \in domL$ and ${}^cD^a_{0+}u(t) = y(t)$, so $y \in ImL$. $\square$

Lemma 3.3. *If L be defined by (2.1), then L is a Fredholm operator of index zero and the linear continuous projector operators $P : X \longrightarrow X$ and Q: $Y \longrightarrow Y$ can be defined as*

$$Pu(t) = u(1)t^2, \forall t \in [0, 1],$$

$$Qy(t) = \alpha \int_0^1 (1 - s)^{\alpha-1} y(s)ds, \forall t \in [0, 1].$$

Furthermore, the operator $K_p :\ ImL \to domL \cap KerP$ can be written as

$$K_p y(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s)ds, \forall t \in [0, 1].$$

Proof. Obviously, $ImP = KerL$ and $P^2u = Pu$. It follows from $u = (u - Pu) + Pu$ that $X = KerP + KerL$. By simple calculation, we get that $KerL \cap KerP = 0$. Hence

$$X = KerL \oplus KerP$$

For $y \in Y$, it yields

$$Q^2 y = Q(Qy) = Qy\alpha \int_0^1 (1 - s)^{\alpha-1} ds = Qy$$

Let $y = (y - Qy) + Qy$, where $y - Qy \in KerQ = ImL$, $Qy \in ImQ$. It follows from $KerQ = ImL$ and $Q^2y = Qy$ that $ImQ \cap ImL = 0$. Then, we have

$$Y = ImL \oplus ImQ.$$

Thus

$$dimKerL = dimImQ = codimImL = 1.$$

This means that L is a Fredholm operator of index zero. From the definitions of P and K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in ImL$, we have

$$LK_P y =^c D^\alpha_{0+} I^\alpha_{0+} y = y \tag{3.3}$$

Moreover, for $u \in domL \cap KerP$, we get $u(0) = u^{'}(0) = u^{''}(0) = 0$. By Lemma 2.3, we obtain that

$$I^\alpha_{0+} Lu(t) = I^\alpha_{0+} D^\alpha_{0+} u(t) = u(t) + c_0 + c_1 t + c_2 t^2, c_0, c_1, c_2 \in \mathbb{R}.$$

This together with $u(0) = u^{'}(0) = u^{''}(0) = 0$ yields

$$K_P Lu = u \tag{3.4}$$

Combining (3.3) with (3.4), we get $K_P = (L|_{domL\cap KerP})^{-1}$. □

Lemma 3.4. *If $\Omega \subset X$ is an open bounded subset such that $domL \cap \bar{\Omega} \neq \oslash$, then N is $L -$ compact on $\bar{\Omega}$.*

Proof. By the continuity of f, we conclude that $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are bounded. In view of Arzelà-Ascoli theorem, we need only to prove that $K_P(I-Q)N(\bar{\Omega}) \subset X$ is equicontinuous. The continuity of f implies that exists a constant $A > 0$ such that $|(I-Q)Nu| \leq A$, $\forall u \in \bar{\Omega}$, $\forall t \in [0,1]$. Denote $K_{P,Q} = K_P(I-Q)N$ and let $0 \leq t_1 \leq t_2 \leq 1, u \in \bar{\Omega}$, then

$$\begin{aligned}
|(K_{P,Q}u)(t_2) - (K_{P,Q}u)(t_1)| &\leq \frac{1}{\Gamma(\alpha)} | \int_0^{t_2} (t_2 - s)^{\alpha-1}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{\alpha-1}(I-Q)Nu(s)ds| \\
&\leq \frac{A}{\Gamma(\alpha)} [\int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})ds \\
&\quad + \int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1} ds] \\
&= \frac{A}{\Gamma(\alpha+1)} (t_2^{\alpha} - t_1^{\alpha}) \left| (K_{P,Q}u)^{'}(t_2) - (K_{P,Q}u)^{'} t_1) \right| \\
&\leq \frac{A}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1})
\end{aligned}$$

and

$$\begin{aligned}
|(K_{P,Q}u)^{''}(t2) - (K_{P,Q}u)^{''}(t1)| &= \frac{1}{\Gamma(\alpha-2)} | \int_0^{t_2} (t_2 - s)^{\alpha-3}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{\alpha-3}(I-Q)Nu(s)ds| \\
&\leq \frac{A}{\Gamma(\alpha-2)} \Gamma(\alpha-2)) [\int_0^{t_1} ((t_1 - s)^{\alpha-3} \\
&\quad - (t_2 - s)^{\alpha-3})ds + \int_{t_1}^{t_2} ((t_2 - s)^{\alpha-3} ds] \\
&\leq \frac{A}{\Gamma(\alpha-1)} (t_1^{\alpha-2} - t_2^{\alpha-2} + 2(t_2 - t_1)^{\alpha-2})
\end{aligned}$$

Since $t^{\alpha}, t^{\alpha-1}$ and $t^{\alpha-2}$ are uniformly continuous on $[0,1]$, we get that $K_{P,Q}(\bar{\Omega}), (K_{P,Q})^{'}(\bar{\Omega})(K_{P,Q})^{''}(\bar{\Omega}) \in C[0,1]$ are equicontinuous. Hence $K_{P,Q} : \bar{\Omega} \to X$ is compact. □

Lemma 3.5. *Suppose* (H_1) *holds, then the set*

$$\Omega_1 = \{u \in domL \backslash KerL | Lu = \lambda Nu, \lambda \in (0,1)\}$$

is bounded.

Proof. Take $u \in \Omega_1$, then $Nu \in ImL$. By (3.2), it yields

$$\int_0^1 (1-s)^{\alpha-1} f(s, u(s), u^{'}(s), u^{''}(s))ds = 0.$$

By the integral mean value theorem, we conclude that there exists a constant $h \in (0,1)$ such that $f(h, u(h), u^{'}(h), u^{''}(h)) = 0$. Then from (H_2), we have $|u^{''}(h)| \leq B$.
Since $u \in domL$, then $u(0) = u^{'}(0) = 0$. Therefore

$$|u(t)| = |u(0) + \int_0^t u^{'}(s)ds| \leq \|u^{'}\|_\infty.$$

and

$$|u^{'}(t)| = |u^{'}(0) + \int_0^t u^{''}(s)ds| \leq \|u^{''}\|_\infty.$$

That is

$$\|u\|_\infty \leq \|u^{'}\|_\infty \leq \|u^{''}\|_\infty \tag{3.5}$$

By $Lu = \lambda Nu$ and $u \in domL$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u^{'}(s), u^{''}(s))ds + \frac{1}{2}u^{''}(0)t^2,$$

and

$$u^{''}(t) = \frac{\lambda}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} f(s, u(s), u^{'}(s), u^{''}(s))ds + u^{''}(0).$$

Take $t = h$, we get

$$u^{''}(h) = \frac{\lambda}{\Gamma(\alpha-2)} \int_0^h (h-s)^{\alpha-3} f(s, u(s), u^{'}(s), u^{''}(s))ds + u^{''}(0)$$

Together with $|u^{''}(h)| \leq B$, (H_1) and (3.5), we have

$$\begin{aligned}
|u^{''}(0)| &\leq |u^{''}(h)| + \frac{1}{\Gamma(\alpha-2)} \int_0^h (h-s)^{\alpha-3} |f(s, u(s), u^{'}(s), u^{''}(s))|ds \\
&\leq B + \frac{1}{\Gamma(\alpha-2)} \int_0^h (h-s)^{\alpha-3} [p(s) + q(s)|u(s)| \\
&\qquad + r(s)|u^{'}(s)| + z(s)|u^{''}(s)|] \\
&\leq B + \frac{1}{\Gamma(\alpha-1)} [p_1 + q_1\|u\|_\infty + r_1\|u^{'}\|_\infty + z_1\|u^{''}\|_\infty] \\
&\leq B + \frac{1}{\Gamma(\alpha-1)} [p_1 + [q_1 + r_1 + z_1]\|u^{''}\|_\infty]
\end{aligned}$$

Then we have

$$\begin{aligned}
\|u^{''}\|_\infty &\leq \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} |f(s, u(s), u^{'}(s), u^{''}(s))|ds + |u^{''}(0)| \\
&\leq \frac{1}{\Gamma(\alpha-1)} [p_1 + [q_1 + r_1 + z_1]\|u^{''}\|_\infty + |u^{''}(0)|] \\
&\leq B + \frac{2}{\Gamma(\alpha-1)} [p_1 + [q_1 + r_1 + z_1]\|u^{''}\|_\infty]
\end{aligned}$$

Since $\Gamma(\alpha-1)-2q_1-2r_1-2z_1>0$, we obtain

$$\|u^{''}\|_\infty \le \frac{2p_1+B\Gamma(\alpha-1)}{\Gamma(\alpha-1)-2q_1-2r_1-2z_1}=M,$$

and

$$\|u\|_\infty \le \|u^{'}\|_\infty \le \|u^{''}\|_\infty \le M.$$

Therefore, $\|u\|_X \le M$, consequently Ω_1 is bounded. □

Lemma 3.6. *Suppose* (H_2) *holds, then the set*

$$\Omega_2=\{u|u\in KerL, Nu\in ImL\}$$

is bounded.

Proof. For $u\in\Omega_2$, we have $u(t)=ct^2$,$c\in\mathbb{R}$, and $Nu\in ImL$. Then we get

$$\int_0^1(1-s)^{\alpha-1}f(s,cs^2,2cs,2c)ds=0,$$

this together with (H_2) implies $|2c|\le B$. Thus, we have $\|u\|_X\le B$. Hence, Ω_2 is bounded. □

Lemma 3.7. *Suppose the first part of* (H_2) *holds, then the set*

$$\Omega_3=\{u|u\in KerL, \lambda Ju+(1-\lambda)QNu=0, \lambda\in[0,1]\}$$

is bounded. Where $J: KerL\to ImQ$*is an isomorphism defined by* $J(ct^2)=ct^2;\forall c\in\mathbb{R}, t\in[0,1]$.

Proof. For $u\in\Omega_3$, we have $u(t)=ct^2$,$c\in\mathbb{R}$, and

$$\lambda ct^2+(1-\lambda)\alpha\int_0^1(1-s)^{\alpha-1}f(s,cs^2,2cs,2c)ds=0. \tag{3.6}$$

If $\lambda=0$, then $\alpha\int_0^1(1-s)^{\alpha-1}f(s,cs^2,2cs,2c)ds=0$, thus $|2c|\le B$ in view of the first part of (H_2). If $\lambda\in(0,1]$, we can also obtain $|2c|\le B$. Otherwise, if $|2c|>B$, in view of the first part of (H_2), one has

$$\lambda ct^2+(1-\lambda)\alpha\int_0^1(1-s)^{\alpha-1}f(s,cs^2,2cs,2c)ds>0,$$

which contradicts (3.6). Therefore, Ω_3 is bounded. □

Lemma 3.8. *Suppose the second part of* (H_2) *hold, the set*

$$\Omega_3^{'}=\{u|u\in KerL, -\lambda Ju+(1-\lambda)QNu=0, \lambda\in[0,1]\}$$

is bounded.

Proof. Using similar argument as in the proof of Lemma 3.7, we prove that $\Omega_3^{'}$ is bounded. □

Now we are able to give the proof of Theorem 3.1, which is an immediate consequence of Lemmas 3.2-3.8 and Lemma 2.5.

Proof. Set $\Omega = \{u \in X | \; \|u\|_X < max\{M, B\} + 1\}$. It follows from Lemma 3.2 and Lemma 3.3 that L is a Fredholm operator of index zero and N is $L-compact$ on $\bar{\Omega}$. By Lemma 3.4 and Lemma 3.5, we get that the following two conditions are satisfied

(1) $Lu \neq Nu$ for every $(u, \lambda) \in [(domL \; KerL)] \cap \partial\Omega \times (0, 1)$;

(2) $Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$;

Define $H(u, \lambda) = \lambda Idu + (1 - \lambda)JQNu$. According to Lemma 3.6 (or Lemma 3.7), we see that $H(u, \lambda) \neq 0$ for $u \in KerL \cap \partial\Omega$. By the degree property of invariance under a homotopy, it yields

$$\begin{aligned} deg(JQN|_{KerL}, KerL \cap \Omega, 0) &= deg(H(., 0), KerL \cap \Omega, 0) \\ &= deg(H(., 1), KerL \cap \Omega, 0) \\ &= deg(Id, KerL \cap \Omega, 0) \neq 0. \end{aligned}$$

So that, the condition (3) of Lemma 2.1 is satisfied. By Lemma 2.1, we can get that $Lu = Nu$ has at least one solution in $domL \cap \bar{\Omega}$. Therefore by Lemma 2.5, the BVP (1.1) has at least one solution. The proof is complete. $\square$

Example 3.9. Let us consider the following fractional boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}} u(t) = \frac{t}{5}(|u^{'}(t)| - 15) + \frac{t^2}{5} e^{-|u(t)| - |u^{''}(t)|}, \; 0 < t < 1, \\ u(0) = u^{'}(0) = 0, \; u^{''}(0) = 2u(1) \end{cases} \tag{3.7}$$

We have $f(t, u, v, w) = \frac{t}{5}(|v| - 15) + \frac{t^2}{5} e^{-|u| - |w|}$, then

$|f(t, u, v, w)| \leq p(t) + q(t)|u| + r(t)|v| + z(t)|w(t)|, \forall t \in [0, 1], (u, v, w) \in \mathbb{R}^3$, where $p(t) = 3t, q(t) = 0, r(t) = \frac{t}{5}$ and $z(t) = \frac{t}{7}$. We get that $q_1 = 0, r_1 = \frac{1}{5}, z_1 = \frac{1}{7}$ and

$$\Gamma(\frac{5}{2} - 1) - 2q_1 - 2r_1 - 2z_1 = \Gamma(\frac{3}{2}) - \frac{2}{5} - \frac{2}{7} = 0.20051 > 0.$$

If we choose $B = 15$ then for $|2c| > B$ it yields

$$cf(t, u, v, w) > 0, \forall t \in [0, 1], u \in R$$

Then, the conditions of Theorem 3.1 are satisfied, so BVP(3.7) has at least one solution.

References

1. Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495-505.
2. W. Feng and J.R.L. Webb. Pang, *Solvability of three point boundary value problems at resonance*, Nonlinear Anal. **30** (1997), 3227-3238.
3. A. Guezane-Lakoud and R. Khaldi, *Positive solution to a higher order fractional boundary value problem with fractional integral condition*, Romanian Journal of Mathematics and Computater Sciences **2** (2012), 28-40.
4. A. Guezane-Lakoud and R. Khaldi, *Solvability of two-point fractional boundary value problem*, The Journal of Nonlinear Science and Applications **5** (2012), 64-73.

5. A. Guezane-Lakoud, A. Frioui and R. Khaldi, *Third Order Boundary Value Prolem with Integral Condition at Resonance*, Theory and Applications of Mathematics & Computer Sciences **3(1)** (2013), 56-64.
6. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).
7. N. Kosmatov, *A boundary value problem of fractional order at resonance*, Electron. J. Differ. Equ. **135** (2010), 1-10.
8. V. Lakshmikantham and A.S. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equations*, Appl. Math. Lett. **21** (2008), 828-834.
9. V. Lakshmikantham, S. Leela and JV. Devi, *Theory of fractional Dynamic Systems*, Cambridge Academic, Cambridge (2009).
10. G.B. Loghmani and S. Javanmardi, *Numerical Methods for Sequential Fractional Differential Equations for Caputo Operator*, Bull. Malay. Math. Sci. Soc. **2** Accepted.
11. F. Mainardi, *Fractional diffusive waves in viscoelastic solids*, Wegner, JL, Norwood, FR (eds) Nonlinear Waves in Solids ASME/AMR, Fairfield (1995), 93-97.
12. J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, NSFCBMS Regional Conference Series in Mathematics. Am. Math. Soc, Providence. (1979).
13. J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations*, Lecture Notes in Mathematics. **1537** (Springer, Berlin (1991)), 74-142.
14. R. Metzler and J. Klafter, *Boundary value for fractional diffusion equations*, Physica A **278** (2000), 107-125.
15. K.S. Miller and B. Ross, *An introduction to Fractional Calculus and Differential Equations*, Wiley, New York (1993).
16. Podlubny, *Fractional Differential Equation*, Academic Press, Sain Diego. (1999).
17. S.G. Samko, A.A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, Theory and Applications, Gordon & Breach, Switzerland (1993)
18. H. Scher and Montroll, *Anomalous transit-time dispersion in amorphous solids*, Phys. Rev. B. **12** (1975), 2455-2477.
19. X.Z. Zhang, C.X. Zhu and Z.Q. Wu, *The Cauchy problem for a class of fractional impulsive differential equations with delay*, Electron. J. Qual. Theory Differ. Equ. **37** (2012), 1-13.
20. H. Zhigang, L. Wenbin and C. Taiyong *Two-point Boundary Value Problems for Fractional Differential Equations at resonance*, Bull. Malays. Math. Sci. Soc. **(2).36.(3)** (2013), 747-755.

A. Guezane-Lakoud Her research interests are on differential equations and their applications.

Department of Mathematics, Faculty of Sciences, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria.
e-mail: a_guezane@yahoo.fr

S. Kouachi Her research interests are on differential equations and their applications.

Department of Mathematics, Faculty of Sciences, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria.
e-mail: sa.kouachi@gmail.com

F. Ellaggoune His research interests in difference equations and its applications.

Department of Mathematics, University 8 mai 1945 - Guelma P.O. Box 401, Guelma 24000, Algeria.
e-mail: fellaggoune@gmail.com