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A QUADRATIC INTEGRAL EQUATION IN THE SPACE OF FUNCTIONS WITH TEMPERED MODULI OF CONTINUITY†

SHAN PENG, JINRONG WANG*, FULAI CHEN

ABSTRACT. In this paper, we investigate existence of solutions to a class of quadratic integral equation of Fredholm type in the space of functions with tempered moduli of continuity. Two numerical examples are given to illustrate our results.

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1. Introduction

Fractional integral and differential equations play increasingly important roles in the modeling of real world problems. Some problems in physics, mechanics and other fields can be described with the help of all kinds of fractional differential and integral equations. For more recent development on Riemann-Liouville, Caputo and Hadamard fractional calculus, the reader can refer to the monographs [1, 2, 3, 4, 5, 6].

Quadratic integral equations arise naturally in applications of real world problems. For example, problems in the theory of radiative transfer in the theory of neutron transport and in the kinetic theory of gases lead to the quadratic equation [7, 8, 9, 10]. There are many interesting existence results for all kinds of quadratic integral equations, one can refer to [11, 12, 13, 14, 15, 16, 17]. Our group extend to study the existence, local attractivity and stability of solutions

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to fractional version Urysohn type quadratic integral equations [18] and Erdélyi-Kober type quadratic integral equations [19] and Hadamard types quadratic integral equations [20] in the space of continuous functions.

Very recently, Banaś and Nalepa [21] study the space of real functions defined on a bounded metric space and having growths tempered by a modulus of continuity and derive the existence theorem for some quadratic integral equations of Fredholm type in the space of functions satisfying the Hölder condition. Further, Caballero et al. [22] study the solvability of a quadratic integral equation of Fredholm type in Hölder spaces.

The aim of the paper is to investigate the existence of solutions of the following integral equation of Fredholm type

$$x(t) = f(t, x(t)) + x(t)\int_a^b k(t,\tau)x(\tau)d\tau,\ t \in [a,b], \tag{1}$$

in $C_{\omega,g}[a,b]$ (see Section 2), where the functions f and k will be defined in the later.

By using a sufficient condition for the relative compactness in the space of functions with tempered moduli of continuity (see Theorem 2.5) and the classical Schauder fixed point theorem, we derive new existence result (see Theorem 3.5). Finally, two numerical examples are given to illustrate our results.

2. Preliminaries

Definition 2.1 (see Section 2 [21])**.** A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a modulus of continuity if $\omega(0) = 0$, $\omega(\epsilon) > 0$ for $\epsilon > 0$, and ω is nondecreasing on $\mathbb{R}$.

Let $C[a,b]$ be the space of continuous functions on $[a,b]$ equipped with $\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}$ for $x \in C[a,b]$. We denote $C_{\omega,g}[a,b]$ be the set of all real functions defined on $[a,b]$ such that their growths are tempered by the modulus of continuity ω with respect to a function g. That is, there exists a constant $H_x^{\omega,g} > 0$ such that

$$|x(t) - x(s)| \le H_x^{\omega,g}\omega(g(t) - g(s)) \tag{2}$$

for all $t, s \in [a,b]$ where $g : [a,b] \to \mathbb{R}$ is a monotonic function.

Without loss of generality, we suppose that the above g be a increasing function and $g(t) - g(s) \ge 0$ for $t \ge s$ in the this paper. Obviously, $C_{\omega,g}[a,b]$ is a linear subspace of $C[a,b]$.

For $x \in C_{\omega,g}[a,b]$, we denote $H_x^{\omega,g}$ be the least possible constant for which inequality (2) is satisfied. More precisely, we set

$$H_x^{\omega,g} = \sup\left\{\frac{|x(t) - x(s)|}{\omega(g(t) - g(s))} : t, s \in [a,b],\ t > s\right\}.$$

Next, the space $C_{\omega,g}[a,b]$ can be equipped with the norm

$$\|x\|_{\omega,g} = |x(a)| + \sup\left\{\frac{|x(t) - x(s)|}{\omega(g(t) - g(s))} : t, s \in [a,b],\ t > s\right\},$$

for $x \in C_{\omega,g}[a,b]$. Then $(C_{\omega,g}[a,b], \|\cdot\|_{\omega,g})$ is a Banach space.

Inspired by the properties of the space of Hölder functions in [21, see (41), (45)], we give the following sharp results.

Lemma 2.2. *For any $x \in C_{\omega,g}[a,b]$, the following inequality is satisfied*

$$\|x\|_\infty \le \max\{1, \omega(g(b)-g(a))\}\|x\|_{\omega,g}.$$

Proof. For any $x \in C_{\omega,g}[a,b]$ and $t \in [a,b]$ we obtain

$$\begin{aligned}
& |x(t)| \\
\le\ & |x(t)-x(a)| + |x(a)| \\
\le\ & \sup\{|x(t)-x(s)| : t,s \in [a,b]\} + |x(a)| \\
=\ & |x(a)| + \sup\left\{\frac{|x(t)-x(s)|}{\omega(g(t)-g(s))}\cdot \omega(g(t)-g(s)) : t,s\in[a,b],\ t>s\right\} \\
\le\ & |x(a)| + \omega(g(b)-g(a)) \sup\left\{\frac{|x(t)-x(s)|}{\omega(g(t)-g(s))} : t,s\in[a,b],\ t>s\right\} \\
\le\ & \max\{1, \omega(g(b)-g(a))\} \\
& \times\left\{|x(a)| + \sup\left\{\frac{|x(t)-x(s)|}{\omega(g(t)-g(s))} : t,s\in[a,b],\ t>s\right\}\right\} \\
\le\ & \max\{1, \omega(g(b)-g(a))\}\|x\|_{\omega,g}.
\end{aligned}$$

□

Lemma 2.3. *Suppose that $\omega_2(g(t)-g(s)) \le G\omega_1(g(t)-g(s))$ for $t,s \in [a,b]$ where $G > 0$. Then we have*

$$C_{\omega_2,g}[a,b] \subset C_{\omega_1,g}[a,b] \subset C[a,b].$$

Moreover, for any $x \in C_{\omega_2,g}[a,b]$ the following inequality holds

$$\|x\|_{\omega_1,g} \le \max\{1, G\}\|x\|_{\omega_2,g}.$$

Proof. For any $x \in C_{\omega_2,g}[a,b]$, we obtain

$$|x(t)-x(s)| \le H_x^{\omega_2,g}\omega_2(g(t)-g(s)) \le GH_x^{\omega_2,g}\omega_1(g(t)-g(s)).$$

This shows that $x \in C_{\omega_1,g}[a,b]$ and hence we infer that inclusions hold. Further,

$$\begin{aligned}
\|x\|_{\omega_1,g} &= |x(a)| + \sup\left\{\frac{|x(t)-x(s)|}{\omega_1(g(t)-g(s))} : t,s\in[a,b],\ t>s\right\} \\
&\le |x(a)| + G\sup\left\{\frac{|x(t)-x(s)|}{\omega_2(g(t)-g(s))} : t,s\in[a,b],\ t>s\right\} \\
&\le \max\{1, G\}\|x\|_{\omega_2,g}.
\end{aligned}$$

□

Remark 2.1. In particular, if $\lim_{\epsilon\to 0}\frac{\omega_2(\epsilon)}{\omega_1(\epsilon)} = 0$ then the above imbedding relations also hold and for any $x \in C_{\omega_2,g}[a,b]$, we have $\|x\|_{\omega_1,g} \le \max\{1, M\}\|x\|_{\omega_2,g} = \|x\|_{\omega_2,g}$, where M is a arbitrarily small positive number.

Theorem 2.4 (see Theorem 5 [21]). *Assume that* ω_1, ω_2 *are moduli of continuity being continuous at zero and such that* $\lim_{\epsilon\to 0} \frac{\omega_2(\epsilon)}{\omega_1(\epsilon)} = 0$. *Further, assume that* (X, d) *is a compact metric space. Then, if* A *is a bounded subset of the space* $C_{\omega_2,g}(X)$ *then* A *is relatively compact in the space* $C_{\omega_1,g}(X)$.

Theorem 2.5. *Suppose that* $\lim_{\epsilon\to 0} \frac{\omega_2(\epsilon)}{\omega_1(\epsilon)} = 0$. *Denote* $B_r^{\omega_2,g} = \{x \in C_{\omega_2,g}[a,b] : \|x\|_{\omega_2,g} \le r\}$. *Then* $B_r^{\omega_2,g}$ *is compact in the space* $C_{\omega_1,g}[a,b]$.

Proof. By Theorem 2.4, since $B_r^{\omega_2,g}$ is a bounded subset in $C_{\omega_2,g}[a,b]$, it is a relatively compact subset of $C_{\omega_1,g}[a,b]$. Suppose that $(x_n) \subset B_r^{\omega_2,g}$ and

$$x_n \to x \text{ (according to } \|\cdot\|_{\omega_1,g})$$

with $x \in C_{\omega_1,g}[a,b]$. This means that for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$\|x_n - x\|_{\omega_1,g} \le \varepsilon,$$

for any $n \ge n_0$, or, equivalently

$$\begin{aligned} &|x_n(a) - x(a)| \\ &+ \sup\left\{\frac{|x_n(t) - x(t) - (x_n(s) - x(s))|}{\omega_1(g(t) - g(s))} : t, s \in [a,b],\ t > s\right\} \le \varepsilon, \end{aligned} \tag{3}$$

for any $n \ge n_0$.

This implies that $x_n(a) \to x(a)$.

Moreover, if in (3) we put $s = a$, then we get

$$\sup\left\{\frac{|x_n(t) - x(t) - (x_n(a) - x(a))|}{\omega_1(g(t) - g(s))} : t, s \in [a,b],\ t > s\right\} < \varepsilon,$$

for any $n \ge n_0$.

The last inequality implies that

$$|x_n(t) - x(t) - (x_n(a) - x(a))| < \varepsilon\omega_1(g(t) - g(s)) \le \varepsilon\omega_1(g(b) - g(a)), \tag{4}$$

for any $n \ge n_0$ and for any $t \in [a,b]$.

Therefore, for any $n \ge n_0$ and any $t \in [a,b]$ and taking into account (3) and (4), we have

$$\begin{aligned} |x_n(t) - x(t)| &\le |(x_n(t) - x(t)) - (x_n(a) - x(a))| + |x_n(a) - x(a)| \\ &< \varepsilon\omega_1(g(b) - g(a)) + \varepsilon. \end{aligned}$$

Consequently,

$$\|x_n - x\|_\infty \to 0. \tag{5}$$

Next, we will prove that $x \in B_r^{\omega_2,g}$.

In fact, since $(x_n) \subset B_r^{\omega_2,g} \subset C_{\omega_2,g}[a,b]$, we have that

$$\frac{|x_n(t) - x_n(s)|}{\omega_2(g(t) - g(s))} \le r,$$

for any $t, s \in [a,b]$ with $t > s$, and, accordingly,

$$|x_n(t) - x_n(s)| \le r\omega_2(g(t) - g(s)),$$

for any $t, s \in [a, b]$.

Letting in the above inequality with $n \to \infty$ and taking into account (5), we deduce that

$$|x(t) - x(s)| \leq r\omega_2(g(t) - g(s)),$$

for any $t, s \in [a, b]$.

Hence we get

$$\frac{|x(t) - x(s)|}{\omega_2(g(t) - g(s))} \leq r,$$

for any $t, s \in [a, b]$, and this means that $x \in B_r^{\omega_2, g}$. This proves that $B_r^{\omega_2, g}$ is a closed subset of $C_{\omega_1, g}[a, b]$. Thus, $x \in B_r^{\omega_2, g}$ is a compact subset of $C_{\omega_1, g}[a, b]$. This finishes the proof. □

3. Main results

In this section, we will study the solvability of the equation (1) in $C_{\omega, g}[a, b]$. We will use the following assumptions:

(H_1) $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a positive number k_1 such that

$$|f(t, x) - f(t, y)| \leq k_1|x - y|,$$

and set $k = |f(a, a)|$. Meanwhile, for any $t, s \in [a, b]$ and $t > s$, there exists a positive constant k_2 such that the inequality

$$\frac{|f(t, x(s)) - f(s, x(s))|}{\omega_2(g(t) - g(s))} \leq k_2|x(s)|.$$

(H_2) $k : [a, b] \times [a, b] \to \mathbb{R}$ is a continuous function satisfies the tempered by the modulus of continuity with respect to the first variable, that is, there exists a constant K_{ω_2} such that

$$|k(t, \tau) - k(s, \tau)| \leq K_{\omega_2}\omega_2(g(t) - g(s)),$$

for any $t, s, \tau \in [a, b]$.

(H_3) The following inequality is satisfied

$$\begin{aligned}&(2K + K_{\omega_2}(b - a))\max^2\{1, \omega_2(g(b) - g(a))\}r^2\\&+\Big[k_1 + (k_1 + k_2)\max\{1, \omega_2(g(b) - g(a))\} - 1\Big]r + k + |a|k_1 < 0, \qquad (6)\end{aligned}$$

where $K = \sup\left\{\int_a^b |k(t, \tau)|d\tau : t \in [a, b]\right\}$.

Consider the operator $\mathcal{F}$ defined on $C_{\omega_2, g}[a, b]$ by

$$(\mathcal{F}x)(t) = f(t, x(t)) + x(t)\int_a^b k(t, \tau)x(\tau)d\tau,\ t \in [a, b].$$

Lemma 3.1. *The operator $\mathcal{F}$ maps $C_{\omega_2, g}[a, b]$ into itself.*

Proof. In fact, we take $x \in C_{\omega_2,g}[a,b]$ and $t,s \in [a,b]$ with $t > s$. Then, by assumptions (H_1)-(H_3), we obtain

$$\begin{aligned}
&\frac{|(\mathcal{F}x)(t) - (\mathcal{F}x)(s)|}{\omega_2(g(t)-g(s))} \\
\leq\ & \frac{|f(t,x(t)) - f(s,x(s))|}{\omega_2(g(t)-g(s))} + \frac{|x(t)\int_a^b k(t,\tau)x(\tau)d\tau - x(s)\int_a^b k(t,\tau)x(\tau)d\tau|}{\omega_2(g(t)-g(s))} \\
&+ \frac{|x(s)\int_a^b k(t,\tau)x(\tau)d\tau - x(s)\int_a^b k(s,\tau)x(\tau)d\tau|}{\omega_2(g(t)-g(s))} \\
\leq\ & \frac{|f(t,x(t)) - f(t,x(s))| + |f(t,x(s)) - f(s,x(s))|}{\omega_2(g(t)-g(s))} \\
&+ \frac{|x(t)-x(s)|}{\omega_2(g(t)-g(s))}\int_a^b |k(t,\tau)||x(\tau)|d\tau + \frac{|x(s)|\int_a^b |k(t,\tau)-k(s,\tau)||x(\tau)|d\tau}{\omega_2(g(t)-g(s))} \\
\leq\ & \frac{k_1|x(t)-x(s)|}{\omega_2(g(t)-g(s))} + \frac{|f(t,x(s)) - f(s,x(s))|}{\omega_2(g(t)-g(s))} \\
&+ \frac{|x(t)-x(s)|}{\omega_2(g(t)-g(s))}\|x\|_\infty \int_a^b |k(t,\tau)|d\tau + \frac{\|x\|_\infty\|x\|_\infty\int_a^b |k(t,\tau)-k(s,\tau)|d\tau}{\omega_2(g(t)-g(s))} \\
\leq\ & k_1 H_x^{\omega_2,g} + k_2|x(s)| + K\|x\|_\infty \frac{|x(t)-x(s)|}{\omega_2(g(t)-g(s))} \\
&+ \frac{\|x\|_\infty^2 \int_a^b K_{\omega_2}\omega_2(g(t)-g(s))d\tau}{\omega_2(g(t)-g(s))} \\
\leq\ & k_1 H_x^{\omega_2,g} + k_2\|x\|_\infty + K\|x\|_\infty H_x^{\omega_2,g} + K_{\omega_2}(b-a)\|x\|_\infty^2.
\end{aligned}$$

By Lemma 2.2, since $\|x\|_\infty \leq \max\{1, \omega_2(g(b)-g(a))\}\|x\|_{\omega_2,g}$ and, as $H_x^{\omega_2,g} \leq \|x\|_{\omega_2,g}$, we infer that

$$\begin{aligned}
&\frac{|(\mathcal{F}x)(t) - (\mathcal{F}x)(s)|}{\omega_2(g(t)-g(s))} \\
\leq\ & (k_1 + k_2\max\{1, \omega_2(g(b)-g(a))\})\|x\|_{\omega_2,g} \\
&+ (K + K_{\omega_2}(b-a))\max{}^2\{1, \omega_2(g(b)-g(a))\}\|x\|_{\omega_2,g}^2.
\end{aligned}$$

This proves that the operator $\mathcal{F}$ maps $C_{\omega_2,g}[a,b]$ into itself. □

Lemma 3.2. *Let* $B_{r_0}^{\omega_2,g} = \{x \in C_{\omega_2,g}[a,b] : \|x\|_{\omega_2,g} \leq r_0\}$ *where* $r_0 > 0$ *satisfying the inequality (6). Then* $\mathcal{F} : B_{r_0}^{\omega_2,g} \to B_{r_0}^{\omega_2,g}$.

Proof. For any $x \in B_{r_0}^{\omega_2,g}$, one has

$$\begin{aligned}
\|\mathcal{F}x\|_{\omega_2,g} \leq\ & |f(a,x(a))| + |x(a)|\int_a^b |k(a,\tau)||x(\tau)|d\tau \\
&+ (k_1 + k_2\max\{1, \omega_2(g(b)-g(a))\})\|x\|_{\omega_2,g} \\
&+ (K + K_{\omega_2}(b-a))\max{}^2\{1, \omega_2(g(b)-g(a))\}\|x\|_{\omega_2,g}^2
\end{aligned}$$

$$\begin{aligned}
&\leq |f(a,x(a))-f(a,a)|+|f(a,a)|+K\|x\|_\infty^2\\
&\quad+(k_1+k_2\max\{1,\omega_2(g(b)-g(a))\})\,\|x\|_{\omega_2,g}\\
&\quad+(K+K_{\omega_2}(b-a))\max^2\{1,\omega_2(g(b)-g(a))\}\|x\|_{\omega_2,g}^2\\
&\leq k_1|x(a)-a|+k+(k_1+k_2\max\{1,\omega_2(g(b)-g(a))\})\,\|x\|_{\omega_2,g}\\
&\quad+(2K+K_{\omega_2}(b-a))\max^2\{1,\omega_2(f(b)-f(a))\}\|x\|_{\omega_2,g}^2\\
&\leq |a|k_1+k_1\|x\|_\infty+k+(k_1+k_2\max\{1,\omega_2(g(b)-g(a))\})\,\|x\|_{\omega_2,g}\\
&\quad+(2K+K_{\omega_2}(b-a))\max^2\{1,\omega_2(g(b)-g(a))\}\|x\|_{\omega_2,g}^2\\
&= k+|a|k_1+\Big[k_1+(k_1+k_2)\max\{1,\omega_2(g(b)-g(a))\}\Big]\|x\|_{\omega_2,g}\\
&\quad+(2K+K_{\omega_2}(b-a))\max^2\{1,\omega_2(g(b)-g(a))\}\|x\|_{\omega_2,g}^2.
\end{aligned}$$

Consequently, from above it follows that $\digamma$ transforms the ball $B_{r_0}^{\omega_2,g}=\{x\in C_{\omega_2,g}[a,b]:\|x\|_{\omega_2,g}\leq r_0\}$ into itself, for any $r_0\in[r_1,r_2]$; i.e., $\digamma: B_{r_0}^{\omega_2,g}\to B_{r_0}^{\omega_2,g}$, where $r_1\leq r_0\leq r_2$. □

Lemma 3.3. *$B_{r_0}^{\omega_2,g}$ is a compact subset in $C_{\omega_1,g}[a,b]$.*

Proof. According to Theorem 2.5, we can know $B_{r_0}^{\omega_2,g}$ is a compact subset in $C_{\omega_1,g}[a,b]$. □

Lemma 3.4. *The operator $\digamma$ is continuous on $B_{r_0}^{\omega_2,g}$, where we consider the norm $\|\cdot\|_{\omega_1,g}$ in $B_{r_0}^{\omega_2,g}$.*

Proof. To do this, we fix $x\in B_{r_0}^{\omega_2,g}$ and $\varepsilon>0$. Suppose that $y\in B_{r_0}^{\omega_2,g}$ and $\|x-y\|_{\omega_1,g}\leq\delta$, where δ is a positive number such that $\delta<\frac{\varepsilon}{2\rho}$ where $\rho=\max\{\rho_1,\rho_2\},\rho_1,\rho_2$ is defined below. Then, for any $t,s\in[a,b]$ with $t>s$, we have

$$\begin{aligned}
&\frac{|[(\digamma x)(t)-(\digamma y)(t)]-[(\digamma x)(s)-(\digamma y)(s)]|}{\omega_1(g(t)-g(s))}\\
\leq\;&\frac{\Big|k_1|x(t)-y(t)|-k_1|x(s)-y(s)|\Big|}{\omega_1(g(t)-g(s))}\\
&+\Bigg|\frac{[x(t)\int_a^b k(t,\tau)x(\tau)d\tau-y(t)\int_a^b k(t,\tau)x(\tau)d\tau]}{\omega_1(g(t)-g(s))}\\
&+\frac{[y(t)\int_a^b k(t,\tau)x(\tau)d\tau-y(t)\int_a^b k(t,\tau)y(\tau)d\tau]}{\omega_1(g(t)-g(s))}\\
&-\frac{[x(s)\int_a^b k(s,\tau)x(\tau)d\tau-y(s)\int_a^b k(s,\tau)x(\tau)d\tau]}{\omega_1(g(t)-g(s))}\\
&-\frac{[y(s)\int_a^b k(s,\tau)x(\tau)d\tau-y(s)\int_a^b k(s,\tau)y(\tau)d\tau]}{\omega_1(g(t)-g(s))}\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&= k_1 \frac{\Big| |x(t)-y(t)| - |x(s)-y(s)| \Big|}{\omega_1(g(t)-g(s))} \\
&\quad + \frac{1}{\omega_1(g(t)-g(s))} \Big| (x(t)-y(t)) \int_a^b k(t,\tau)x(\tau)d\tau \\
&\quad + y(t)\int_a^b k(t,\tau)(x(\tau)-y(\tau))d\tau \\
&\quad - (x(s)-y(s))\int_a^b k(s,\tau)x(\tau)d\tau - y(s)\int_a^b k(s,\tau)(x(\tau)-y(\tau))d\tau \Big| \\
&\leq k_1\|x-y\|_{\omega_1,g} + \frac{|(x(t)-y(t))-(x(s)-y(s))|}{\omega_1(g(t)-g(s))}\|x\|_\infty \int_a^b |k(t,\tau)|d\tau \\
&\quad + [|(x(s)-y(s))-(x(a)-y(a))| + |(x(a)-y(a))|]\|x\|_\infty \\
&\quad \times \int_a^b \frac{|k(t,\tau)-k(s,\tau)|}{\omega_1(g(t)-g(s))}d\tau \\
&\quad + \frac{\Big| y(t)\int_a^b k(t,\tau)(x(\tau)-y(\tau))d\tau - y(s)\int_a^b k(t,\tau)(x(\tau)-y(\tau))d\tau \Big|}{\omega_1(g(t)-g(s))} \\
&\quad + \frac{\Big| y(s)\int_a^b k(t,\tau)(x(\tau)-y(\tau))d\tau - y(s)\int_a^b k(s,\tau)(x(\tau)-y(\tau))d\tau \Big|}{\omega_1(g(t)-g(s))} \\
&\leq k_1\|x-y\|_{\omega_1,g} + K\|x-y\|_{\omega_1,g}\|x\|_\infty \\
&\quad + \sup\Big\{ |(x(t)-y(t))-(x(s)-y(s))| \Big\} \|x\|_\infty \int_a^b \frac{K_{\omega_2}\omega_2(g(t)-g(s))}{\omega_1(f(t)-f(s))}d\tau \\
&\quad + |(x(a)-y(a))|\|x\|_\infty \int_a^b \frac{K_{\omega_2}\omega_2(g(t)-g(s))}{\omega_1(g(t)-g(s))}d\tau \\
&\quad + \frac{|y(t)-y(s)|}{\omega_1(g(t)-g(s))}\int_a^b |k(t,\tau)||x(\tau)-y(\tau)|d\tau \\
&\quad + |y(s)|\int_a^b \frac{|k(t,\tau)-k(s,\tau)|}{\omega_1(g(t)-g(s))}|x(\tau)-y(\tau)|d\tau \\
&\leq k_1\|x-y\|_{\omega_1,g} + K\|x-y\|_{\omega_1,g}\|x\|_\infty \\
&\quad + M(b-a)\|x\|_\infty K_{\omega_2} \sup\left\{ \frac{|(x(t)-y(t))-(x(s)-y(s))|}{\omega_1(g(t)-g(s))}\omega_1(g(t)-g(s)) \right\} \\
&\quad + M(b-a)K_{\omega_2}\|x\|_\infty |(x(a)-y(a))| + KH_y^{\omega_1,g}\|x-y\|_\infty \\
&\quad + \|y\|_\infty\|x-y\|_\infty \int_a^b \frac{K_{\omega_2}\omega_2(g(t)-g(s))}{\omega_1(g(t)-g(s))}d\tau \\
&\leq k_1\|x-y\|_{\omega_1,g} + K\|x-y\|_{\omega_1,g}\|x\|_\infty \\
&\quad + M(b-a)\|x\|_\infty K_{\omega_2}\omega_1(g(b)-g(a))\|x-y\|_{\omega_1,g}
\end{aligned}
$$

$$
\begin{aligned}
&+M(b-a)K_{\omega_2}\|x\|_\infty\|x-y\|_\infty + K\|y\|_{\omega_1,g}\|x-y\|_\infty \\
&+M(b-a)K_{\omega_2}\|y\|_\infty\|x-y\|_\infty \\
\leq\ & k_1\|x-y\|_{\omega_1,g} + K\max\{1,\omega_2(g(b)-g(a))\}\|x-y\|_{\omega_1,g}\|x\|_{\omega_2,g} \\
&+M(b-a)K_{\omega_2}\omega_1(g(b)-g(a))\max\{1,\omega_2(g(b)-g(a))\}\|x-y\|_{\omega_1,g}\|x\|_{\omega_2,g} \\
&+M(b-a)K_{\omega_2}\max\{1,\omega_2(g(b)-g(a))\} \\
&\times\max\{1,\omega_1(g(b)-g(a))\}\|x-y\|_{\omega_1,g}\|x\|_{\omega_2,g} \\
&+K\max\{1,\omega_1(g(b)-g(a))\}\|x-y\|_{\omega_1,g}\|y\|_{\omega_1,g} \\
&+M(b-a)K_{\omega_2}\max\{1,\omega_2(g(b)-g(a))\} \\
&\times\max\{1,\omega_1(g(b)-g(a))\}\|x-y\|_{\omega_1,g}\|y\|_{\omega_2,g}.
\end{aligned}
$$

Define

$$
\begin{aligned}
\rho_1 = \ & k_1 + K\max\{1,\omega_2(g(b)-g(a))\}r_0 \\
&+M(b-a)K_{\omega_2}\omega_1(g(b)-g(a))\max\{1,\omega_2(g(b)-g(a))\}r_0 \\
&+2M(b-a)K_{\omega_2}\max\{1,\omega_2(g(b)-g(a))\}\max\{1,\omega_1(g(b)-g(a))\}r_0 \\
&+K\max\{1,\omega_1(g(b)-g(a))\}r_0.
\end{aligned}
$$

Since $\|y\|_{\omega_1,g} \leq \|y\|_{\omega_2,g}$ (see Remark 2.1) and $x, y \in B_{r_0}^{\omega_2,g}$, from the above inequality we infer that

$$
\frac{|[(\mathcal{F}x)(t)-(\mathcal{F}y)(t)]-[(\mathcal{F}x)(s)-(\mathcal{F}y)(s)]|}{\omega_1(g(t)-g(s))} \leq \rho_1\delta < \frac{\varepsilon}{2}. \tag{7}
$$

On the other hand,

$$
\begin{aligned}
&|(\mathcal{F}x)(a)-(\mathcal{F}y)(a)| \\
\leq\ & |f(a,x(a))-f(a,y(a))| + \left|x(a)\int_a^b k(a,\tau)x(\tau)d\tau - x(a)\int_a^b k(a,\tau)y(\tau)d\tau\right| \\
&+\left|x(a)\int_a^b k(a,\tau)y(\tau)d\tau - y(a)\int_a^b k(a,\tau)y(\tau)d\tau\right| \\
\leq\ & k_1|x(a)-y(a)| + \left|x(a)\int_a^b k(a,\tau)(x(\tau)-y(\tau))d\tau\right| \\
&+\left|(x(a)-y(a))\int_a^b k(a,\tau)y(\tau)d\tau\right| \\
\leq\ & k_1\|x-y\|_\infty + K\|x\|_\infty\|x-y\|_\infty + K\|y\|_\infty\|x-y\|_\infty \\
\leq\ & k_1\max\{1,\omega_1(g(b)-g(a))\}\|x-y\|_{\omega_1,g} \\
&+K\max\{1,\omega_2(g(b)-g(a))\}\max\{1,\omega_1(g(b)-g(a))\}\|x\|_{\omega_2,g}\|x-y\|_{\omega_1,g} \\
&+K\max\{1,\omega_2(g(b)-g(a))\}\max\{1,\omega_1(g(b)-g(a))\}\|y\|_{\omega_2,g}\|x-y\|_{\omega_1,g} \\
\leq\ & \rho_2\delta,
\end{aligned}
$$

where

$$
\rho_2 = k_1\max\{1,\omega_1(g(b)-g(a))\}
$$

$$+2K\max\{1,\omega_2(g(b)-g(a))\}\max\{1,\omega_1(g(b)-g(a))\}r_0.$$

which yields that

$$|(\digamma x)(a)-(\digamma y)(a)|\le\rho_2\delta<\frac{\varepsilon}{2}. \tag{8}$$

By (7) and (8), we have

$$\begin{aligned}
&\|\digamma x-\digamma y\|_{\omega_1,g}\\
=\;&|(\digamma x)(a)-(\digamma y)(a)|\\
&+\sup\left\{\frac{|[(\digamma x)(t)-(\digamma y)(t)]-[(\digamma x)(s)-(\digamma y)(s)]|}{\omega_1(g(t)-g(s))}:t,s\in[a,b],\ t>s\right\}\\
<\;&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

This proves the operator $\digamma$ is continuous at the point $x\in B_{r_0}^{\omega_2,g}$ for the norm $\|\cdot\|_{\omega_1,g}$. □

Theorem 3.5. *Under assumptions (H_1)-(H_3), the equation (1) has at least one solution in the space $C_{\omega_1,g}[a,b]$.*

Proof. According to Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, the operator $\digamma$ is continuous at the point $x\in B_{r_0}^{\omega_2,g}$ for the norm $\|\cdot\|_{\omega_1,g}$. Since $B_{r_0}^{\omega_2,g}$ is compact in $C_{\omega_2,g}[a,b]$, applying the classical Schauder fixed point theorem we obtain the desired result. □

4. Examples

Now we make two examples illustrating the main results in the above section.

Example 4.1. Let us consider the quadratic integral equation

$$x(t)=\frac{1}{100}\sqrt{\ln t}\arctan x(t)+x(t)\int_1^e\sqrt{\ln t+\ln\tau}\frac{x(\tau)}{\tau}d\tau,\ t\in[1,e]. \tag{9}$$

Set $f(t,x(t))=\frac{1}{100}\sqrt{\ln t}\arctan x(t)$ and $k(t,\tau)=\frac{\sqrt{\ln t+\ln\tau}}{\tau}$ for $t,\tau\in[1,e]$. It is easy to see that

$$|k(t,\tau)-k(s,\tau)|\le|\ln t-\ln s|^{\frac{1}{2}},$$

which implies $K_{\omega_2}=1$ and

$$g(t)=\ln t,\ \omega_2(g(t)-g(s))=|\ln t-\ln s|^{\frac{1}{2}},\ \omega_2(g(e)-g(1))=1.$$

So we can choose

$$\omega_1(g(t)-g(s))=|\ln t-\ln s|^{\alpha},\ 0<\alpha<\frac{1}{2}.$$

Moreover, $K=\sup\{\int_1^e|\frac{\sqrt{\ln t+\ln\tau}}{\tau}|d\tau:t\in[1,e]\}=\frac{2}{3}(2\sqrt{2}-1)$.

On the other hand,

$$|f(t,x(t))-f(t,y(t))|=\frac{1}{100}\sqrt{\ln t}|\arctan x(t)-\arctan y(t)|\le\frac{1}{100}|x(t)-y(t)|,$$

so we can get $k_1 = \frac{1}{100}, k = |f(1,1)| = 0$ and

$$\frac{|\frac{1}{100}\sqrt{\ln t}\arctan x(s) - \frac{1}{100}\sqrt{\ln s}\arctan x(s)|}{|\ln t - \ln s|^{\frac{1}{2}}} \leq \frac{1}{100}|\arctan x(s)| \leq \frac{1}{100}|x(s)|,$$

so $k_2 = \frac{1}{100}$.

In what follows, the condition (H_3) reduce to the inequality

$$\left(\frac{8\sqrt{2}-7}{3} + e\right) r^2 - \frac{97}{100} r + \frac{1}{100} < 0.$$

Obviously, there exist a positive number satisfying these conditions. For example, one can choose $r = 0.1$.

Finally, applying Theorem 3.5, we conclude that the quadratic integral equation has at least one solution in the space $C_{|\cdot|^\alpha, \ln\cdot}[1, e]$ and displayed in Fig.1.

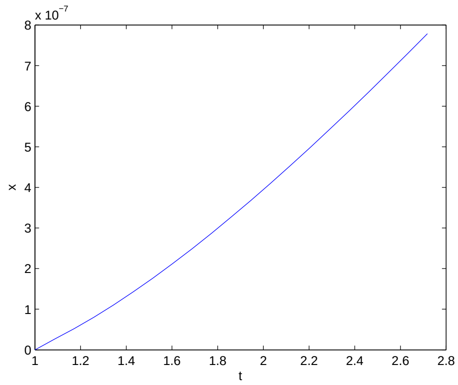

Fig.1 The solution of the equation (9).

Example 4.2. Consider another quadratic integral equation

$$x(t) = \frac{1}{10}\sqrt[5]{t+1}\sin x(t) + x(t)\int_0^1 \sqrt[5]{3t^2+\tau}x(\tau)d\tau,\ t \in [0,1]. \tag{10}$$

Set $f(t, x(t)) = \frac{1}{10}\sqrt[5]{t+1}\sin x(t)$ and $k(t,\tau) = \sqrt[5]{3t^2+\tau},\ t \in [0,1]$. Obviously,

$$|k(t,\tau) - k(s,\tau)| \leq |3t^2 - 3s^2|^{\frac{1}{5}} \leq \sqrt[5]{6}|t-s|^{\frac{1}{5}},$$

which gives $K_{\omega_2} = \sqrt[5]{6}$, $g(t) = t,\ \omega_2(g(t)-g(s)) = |t-s|^{\frac{1}{5}},\ \omega_2(g(1)-g(0)) = 1$. Then we choose

$$\omega_1(g(t)-g(s)) = |t-s|^\alpha,\ 0 < \alpha < \frac{1}{5}.$$

Moreover, $K = \sup\{\int_0^1 |\sqrt[5]{3t^2+\tau}|d\tau\ :\ t \in [0,1]\} = \sup\{\frac{5}{6}(3t^2+\tau)^{\frac{6}{5}}|_0^1\ :\ t \in [0,1]\} = \frac{5}{6}(4\sqrt[5]{4} - 3\sqrt[5]{3})$.

On the other hand,

$$|f(t,x(t))-f(t,y(t))|=\frac{1}{10}\sqrt[5]{t+1}|\sin x(t)-\sin y(t)|\leq\frac{\sqrt[5]{2}}{10}|x(t)-y(t)|,$$

we can get $k_1=\frac{\sqrt[5]{2}}{10}, k=|f(0,0)|=0$ and

$$\frac{|\frac{1}{10}\sqrt[5]{t+1}\sin x(s)-\frac{1}{10}\sqrt[5]{s+1}\sin x(s)|}{|t-s|^{\frac{1}{5}}}\leq\frac{1}{10}|\sin x(s)|\leq\frac{1}{10}|x(s)|,$$

so derive $k_2=\frac{1}{10}$.

In what follows, the condition (H_3) reduce to the inequality

$$(200\sqrt[5]{4}-150\sqrt[5]{3}+30\sqrt[5]{6})r^2+(6\sqrt[5]{2}-27)r<0.$$

The condition reduce to $r<0.1676$. Obviously, there exist a positive number satisfying these conditions. For example, one can choose $r=0.16$.

Finally, applying Theorem 3.5, we conclude that the quadratic integral equation has at least one solution in the space $C_{|\cdot|^\alpha,\cdot}[0,1]$ and displayed in Fig.2.

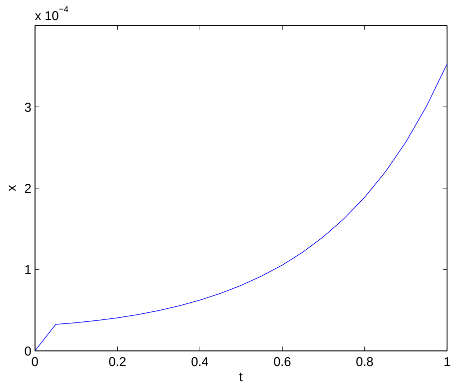

Fig.2 The solution of the equation (10).

References

1. D. Baleanu, J.A.T. Machado and A.C.-J. Luo, *Fractional Dynamics and Control*, Springer, 2012.
2. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of frac- tional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B. V., Amsterdam, 2006.
3. V. Lakshmikantham, S. Leela and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.
4. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
5. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
6. V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles*, Fields and Media, Springer, HEP, 2010.

7. K.M. Case, P.F. Zweifel, *Linear Transport Theory*, Addison Wesley, Reading, M. A, 1967.
8. S. Chandrasekhar, Radiative transfer, Dover Publications, New York, 1960.
9. S. Hu, M. Khavani, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal., **34**(1989), 261-266.
10. C.T. Kelly, *Approximation of solutions of some quadratic integral equations in transport theory*, J. Int. Eq., **4**(1982), 221-237.
11. J. Banaś, M. Lecko and W.G. El-Sayed, *Existence theorems of some quadratic integral equation*, J. Math. Anal. Appl., **222**(1998), 276-285.
12. J. Banaś, J. Caballero, J. Rocha and K. Sadarangani, *Monotonic solutions of a class of quadratic integral equations of Volterra type*, Comput. Math. Appl., **49**(2005), 943-952.
13. J. Caballero, J. Rocha and K. Sadarangani, *On monotonic solutions of an integral equation of Volterra type*, J. Comput. Appl. Math., **174**(2005), 119-133.
14. M.A. Darwish, *On solvability of some quadratic functional-integral equation in Banach algebras*, Commun. Appl. Anal., **11**(2007), 441-450.
15. M.A, Darwish and S.K. Ntouyas, *On a quadratic fractional Hammerstein-Volterra integral equations with linear modification of the argument*, Nonlinear Anal., **74**(2011), 3510-3517.
16. M.A. Darwish, *On quadratic integral equation of fractional orders*, J. Math. Anal. Appl., **311**(2005), 112-119.
17. R.P. Agarwal, J. Banaś, K. Banaś and D. O'Regan, *Solvability of a quadratic Hammerstein integral equation in the class of functions having limits at infinity*, J. Int. Eq. Appl., **23**(2011), 157-181.
18. J. Wang, X. Dong and Y. Zhou, *Existence, attractiveness and stability of solutions for quadratic Urysohn fractional integral equations*, Commun. Nonlinear Sci. Numer. Simul., **17**(2012), 545-554.
19. J. Wang, X. Dong and Y. Zhou, *Analysis of nonlinear integral equations with Erdélyi-Kober fractional operator*, Commun. Nonlinear Sci. Numer. Simul., **17**(2012), 3129-3139.
20. J. Wang, C. Zhu and Y. Zhou, *Study on a quadratic Hadamard types fractional integral equation on an unbounded interval*, Topological Methods in Nonlinear Analysis, **42**(2013), 257-275.
21. J. Banaś and R. Nalepa, *On the space of functions with growths tempered by a modulus of continuity and its applications*, Journal of Function Spaces and Applications, 2013(2013), Article ID 820437, 13 pages.
22. J. Caballero, M.A. Darwish and K. Sadarangani, *Solvability of a quadratic integral equation of Fredholm type in Hölder spaces*, Electronic Journal of Differential Equations, 2014(2014), No.31, 1-10.

Shan Peng

Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China.
e-mail: spengmath@126.com

JinRong Wang

Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China.
Department of Mathematics, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China.
e-mail: sci.jrwang@gzu.edu.cn

Fulai Chen

Department of Mathematics, Xiangnan University, Chenzhou, Hunan 423000, P.R. China.
e-mail: cflmath@163.com