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GLOBAL EXISTENCE OF SOLUTIONS FOR A SYSTEM OF SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECTS†

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Abstract. By employing a fixed point theorem in a weighted Banach space, we establish the existence of a solution for a system of impulsive singular fractional differential equations. Some examples are presented to illustrate the efficiency of the results obtained.

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1. Introduction

Fractional differential equation is a generalization of ordinary differential equation to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [14]. Fractional differential equations therefore find numerous applications in different branches of physics, chemistry and biological sciences such as visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles and neuron modelling [16]. The reader may refer to the books and monographs [15, 6, 8] for fractional calculus and developments on fractional differential and fractional integro-differential equations with applications.

On the other hand, the theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments.

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Processes with such characteristics arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the reader to [11].

On the other hand, in recently, there have been many papers [1, 2, 3, 4, 5, 7, 9, 12, 19, 17, 18] concerned with the existence of solutions for different initial value problems involving impulsive fractional differential equations. However, there has been few papers discussed the Global existence of solutions of initial value problems for impulsive fractional differential systems on half line.

Motivated by the above reason, we discuss the following initial value problem of singular fractional differential system on the half line

$$\begin{cases} {}^C D_{0^+}^{\alpha} u(t) = m(t) f(t, u(t), v(t)), \quad t \in (t_s, t_{s+1}],\ s \in N_0, \\ {}^C D_{0^+}^{\beta} v(t) = n(t) g(t, u(t), v(t)), \quad t \in (t_s, t_{s+1}],\ s \in N_0, \\ \lim_{t\to 0} u(t) = \int_0^{\infty} \phi(s) F(s, u(s), v(s)) ds, \\ \lim_{t\to 0} v(t) = \int_0^{\infty} \psi(s) G(s, u(s), v(s)) ds, \\ \lim_{t\to t_s^+} u(t) - u(t_s) = I(t_s, u(t_s), v(t_s)),\ s \in N, \\ \lim_{t\to t_s^+} v(t) - v(t_s) = J(t_s, u(t_s), v(t_s)),\ s \in N \end{cases} \tag{1.1}$$

where

- **(a)** $0 < \alpha, \beta \le 1$, ${}^C D_{0^+}^{\alpha}$ and ${}^C D_{0^+}^{\beta}$ are the Caputo fractional derivatives of orders α and β respectively,
- **(b)** $N = \{1, 2, \cdots\}$ and $N_0 = \{0, 1, 2, \cdots\}$, $0 = t_0 < t_1 < \cdots < t_s < \cdots$ with $\lim_{s\to\infty} t_s = \infty$,
- **(c)** $m, n : (0, \infty) \to R$ satisfy $m|_{(t_s, t_{s+1}]},\ n|_{(t_s, t_{s+1}]} \in C^0(t_s, t_{s+1}]$ $(s \in N_0)$, both m and n may be singular at $t = 0$, there exist constants $L_1, L_2 > 0$ and $k, l > -1$ such that $|m(t)| \le L_1 t^k$ and $|n(t)| \le L_2 t^l$, $t \in (0, \infty)$,
- **(d)** $\phi, \psi : (0, \infty) \to R$ satisfy $\phi, \psi \in L^1(0, \infty)$, and
- **(e)** f, g, F, G defined on $(0, \infty) \times R \times R$ are Caraéodory functions, I, J defined on $\{t_s : s \in N\} \times R^2$ Carathéodory sequences.

A pair of functions (x, y) with $x : (0, \infty) \to R$ and $y : (0, \infty) \to R$ is said to be a *solution* of (1.1) if

$$x|_{(t_s, t_{s+1}]},\ y|_{(t_s, t_{s+1}]} \in C^0(t_s, t_{s+1}],\ s \in N_0,\ \lim_{t\to t_s^+} x(s),\ \lim_{t\to t_s^+} y(t) \text{ exist},\ s \in N_0,$$
$${}^C D_{0^+}^{\alpha} x(t),\ {}^C D_{0^+}^{\beta} y(t) \text{ exist and } (x, y) \text{ satisfies all equations in (1.1).}$$

We construct a weighted Banach space and apply the Leray-Schauder nonlinear alternative to obtain the existence of at least one solution of (1.1). Our results are new and naturally complement the literature on fractional differential equations.

The paper is outlined as follows. Section 2 contains some preliminary results. The main results are presented in Section 3. Finally, in Section 4 we give two examples to illustrate the efficiency of the results obtained.

2. Preliminaries

For the convenience of the readers, we shall state the necessary definitions from fractional calculus theory.

For $h \in L^1(0,\infty)$, denote $\|h\|_1 = \int_0^\infty |h(s)|ds$. For $r > 0$, $p > 0, q > 0$, let the Gamma and beta functions $\Gamma(r)$ and $\mathbf{B}(p,q)$ be defined by

$$\Gamma(r) = \int_0^{+\infty} x^{r-1}e^{-x}dx, \qquad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Definition 2.1 ([6])**.** Let $a \in R$. The *Riemann-Liouville fractional integral of order* $\alpha > 0$ of a function $h : (a,\infty) \to R$ is given by

$$I_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

provided that the right-hand side exists.

Definition 2.2 ([6])**.** Let $a \in R$. The *Caputo fractional derivative of order* $\alpha > 0$ of a function $h : (a,\infty) \to R$ is given by

$${}^C D_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{h^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 \le \alpha < n$, provided that the right-hand side exists.

Remark 2.1. Let $a, b \in R$ with $a < b$. From Theorem 2.14 in [6], we know that ${}^C D_{a^+}^{\alpha} I_{a^+}^{\alpha} h(t) = h(t)$ almost all $t \in (a,b)$ for every $h \in L^1(a,b)$. From Theorem 2.23 in [6], we have $I_{a^+}^{\alpha}\, {}^C D_{a^+}^{\alpha} h(t) = h(t) - \frac{\lim_{t\to a^+} I_{a^+}^{1-\alpha} h(t)}{\Gamma(\alpha)}$ if $I_{a^+}^{1-\alpha} h \in A^1(a,b)$ which is the set of all absolutely continuous functions see ([6], page 10), i.e. the functions h for which there exists (almost everywhere) a function $H \in L^1(a,b)$ such that $\int_a^t H(s)ds = h(t) - h(a)$.

Definition 2.3 ([10])**.** An odd homeomorphism Φ of the real line R onto itself is called a *sup-multiplicative-like function* if there exists a homeomorphism ω of $[0,+\infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \ge 0$,

$$\Phi(v_1 v_2) \ge \omega(v_1)\Phi(v_2). \tag{2.1}$$

ω is called the *supporting function* of Φ.

Remark 2.2. Note that any sup-multiplicative function is sup-multiplicative-like function. Also any function of the form $\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in R$ is sup-multiplicative-like, provided that $c_j \ge 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \ge 0$.

Remark 2.3. It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero.

Moreover, their inverses Φ^{-1} and ν respectively are increasing and such that for all $w_1, w_2 \geq 0$,

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2). \tag{2.2}$$

ν is called the *supporting function* of Φ^{-1}.

In this paper we always suppose that Φ is a sup-multiplicative-like function with its supporting function ω. The inverse function Φ^{-1} has its supporting function ν.

Definition 2.4. Let $\sigma > k + \alpha$ and $\delta > l + \beta$. We say $K : (0, +\infty) \times R^2 \to R$ is a *Carathéodory function* if it satisfies the following:

(i) $t \to K\left(t,\ (1+t^\sigma)x,\ (1+t^\delta)y\right)$ is continuous on $(t_s, t_{s+1}]$ $(s = 0, 1, 2, \cdots)$, and for any $(x, y) \in R^2$ there exist the limits

$$\lim_{t \to t_s^+} K\left(t,\ (1+t^\sigma)x,\ (1+t^\delta)y\right), \quad s = 0, 1, 2, \cdots;$$

(ii) $(x, y) \to K\left(t,\ (1+t^\sigma)x,\ (1+t^\delta)y\right)$ is continuous on R^2 for all $t \in (0, +\infty)$;

(iii) for each $r > 0$ there exists a constant $A_r > 0$ such that

$$\left|K\left(t,\ (1+t^\sigma)x,\ (1+t^\delta)y\right)\right| \leq A_r, \quad t \in (0, +\infty),\ |x|, |y| \leq r.$$

Definition 2.5. $\{G : \{t_s : s \in N\} \times R^2\}$ is called a *Carathéodory sequence* if

(i) $(x, y) \to G\left(t_s,\ (1+t_s^\sigma)x,\ (1+t_s^\delta)y\right)$ is continuous on R^2 for all $s \in N$;

(ii) for each $r > 0$ there exists $A_{r,s} \geq 0$ such that

$$\left|G\left(t_s,\ (1+t_s^\sigma)x,\ (1+t_s^\delta)y\right)\right| \leq A_{r,s}, s \in N,\ |x|, |y| \leq r \ \text{ and } \sum_{s=1}^{+\infty} A_{r,s} < +\infty.$$

To obtain the main results, we need the Leray-Schauder nonlinear alternative.

Lemma 2.6 (Leray-Schauder Nonlinear Alternative [13])**.** *Let X be a Banach space and $T : X \to X$ be a completely continuous operator. Suppose Ω is a nonempty open subset of X centered at zero. Then, either there exists $x \in \partial\Omega$ and $\lambda \in (0, 1)$ such that $x = \lambda Tx$, or there exists $x \in \overline{\Omega}$ such that $x = Tx$.*

3. Main Results

In this section we shall establish the existence of at least one solution of system (1.1). Throughout, we assume that the functions and parameters in (1.1) satisfy (a)–(e) (stated in Section 1) and the following:

(A) $f,\ g,\ F,\ G$ are Carathéodory functions;

(B) I, J are Carathéodory sequences.

Let

$$X = \left\{ x : \begin{array}{l} x|_{(t_s, t_{s+1}]} \in C^0(t_s, t_{s+1}]\ (s = 0, 1, 2, \cdots),\ \text{the following limits exsit} \\ \lim\limits_{t \to t_s^+} x(t)\ (s = 0, 1, 2, \cdots),\ \lim\limits_{t \to +\infty} \dfrac{x(t)}{1+t^\sigma} \end{array} \right\}.$$

It is easy to show that X is a real Banach space. Thus, $(X\times X, \|\cdot\|)$ is a Banach space with the norm defined by $||(x,y)|| = \max\{||x||_X,\ ||y||_X\},\ \ (x,y)\in X\times X$.

Let $x\in X$ and $y\in X$. Then, there exists $r>0$ such that $||(x,y)|| = r < +\infty$ From (A), f is a Carathéodory function, thus there exists $A_r \geq 0$ such that

$$|f(t,x(t),y(t))| = \left|f\left(t,(1+t^{\sigma})\frac{x(t)}{1+t^{\sigma}},(1+t^{\delta})\frac{y(t)}{1+t^{\delta}}\right)\right| \leq A_r, t\in(0,+\infty). \tag{3.1}$$

Similarly, there exist positive constants A'_r and $A_{r,s}$ $(s=1,2,\cdots)$ such that

$$\begin{aligned}&|F(t,x(t),y(t))| \leq A'_r, |I(t_s,x(t_s),y(t_s))| \leq A_{r,s},\\ &(s=1,2,\cdots), t\in(0,\infty), \sum_{s=1}^{+\infty} A_{r,s} < \infty.\end{aligned} \tag{3.2}$$

Likewise, g, G and J_s are also Carathéodory functions, so there exist positive constants B_r, B'_r and $B_{r,s}$ $(s=1,2,\cdots)$ such that

$$\begin{aligned}&|g(t,x(t),y(t))| \leq B_r, \quad |G(t,x(t),y(t))| \leq B'_r,\\ &|J_s(t,x(t),y(t))| \leq B_{r,s}\ (s=1,2,\cdots), \quad t\in(0,+\infty), \quad \sum_{s=1}^{\infty} B_{r,s} < \infty.\end{aligned} \tag{3.3}$$

Lemma 3.1. *Suppose that $x,y\in X$. Then, $u\in X$ is a solution of*

$$\begin{cases} {}^C D^{\alpha}_{0^+}u(t) = m(t)f(t,x(t),y(t)), \quad t\in(t_s,t_{s+1}],\ s=1,2,\cdots,\\ \lim\limits_{t\to 0} u(t) = \displaystyle\int_0^{\infty} \phi(s)F(s,x(s),y(s))ds,\\ \Delta u(t_s) = I(t_s,x(t_s),y(t_s)),\ s=1,2,\cdots \end{cases} \tag{3.4}$$

if and only if $u\in X$ satisfies the integral equation

$$\begin{aligned}u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds + \int_0^{\infty}\phi(s)F(s,x(s),y(s))ds\\ &+\sum_{s=1}^{i} I(t_s,x(t_s),y(t_s)),\ t\in(t_i,t_{i+1}],\ i=0,1,2,\cdots.\end{aligned} \tag{3.5}$$

Proof. Let $u\in X$ be a solution of (3.4). Then, it follows from (3.4) that for $t\in(t_i,t_{i+1}]$ $(i=0,1,2,\cdots)$, there exist constants $c_i\in R$ such that

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)f(s,x(s),y(s))ds + c_i, t\in(t_i,t_{i+1}],\ i\in N_0. \tag{3.6}$$

From $\lim\limits_{t\to 0} u(t) = \int_0^{\infty}\phi(s)F(s,x(s),y(s))ds$, we get $c_0 = \int_0^{\infty}\phi(s)F(s,x(s),y(s))ds$. From $\Delta u(t_i) = I(t_i,x(t_i),y(t_i))$, we have $c_i - c_{i-1} = I(t_i,x(t_i),y(t_i))$, which leads to $c_i = c_0 + \sum\limits_{s=1}^{i} I(t_s,x(t_s),y(t_s))$. On substituting c_i into (3.6), we obtain for $t\in(t_i,t_{i+1}]$ $(i=0,1,2,\cdots)$,

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,m(s),y(s))ds$$

$$+ \int_0^\infty \phi(s)F(s,x(s),y(s))ds + \sum_{s=1}^{i} I(t_s,x(t_s),y(t_s)),$$

which is simply the same as (3.5). Moreover, since $x \in X$ and $y \in Y$, we have (3.1)–(3.3) which will lead to the expression of u in (3.5) is indeed in X.

On the other hand, if $x \in X,\ y \in Y$ and $u \in X$ satisfies (3.5), then we can prove that $u \in X$ is a solution of (3.4). The proof is complete. □

Lemma 3.2. *Suppose that $x, y \in X$. Then, $v \in Y$ is a solution of*

$$\begin{cases} D_{0^+}^{\beta} v(t) = n(t)g(t,x(t),y(t)), \quad t \in (0,\infty),\ t \neq t_s,\ s = 1,2,\cdots, \\ \lim_{t\to 0} v(t) = \int_0^\infty \psi(s)G(s,x(s),y(s))ds, \\ \Delta v(t_s) = J(t_s,x(t_s),y(t_s)),\ s = 1,2,\cdots \end{cases} \tag{3.7}$$

if and only if $v \in Y$ satisfies the integral equation

$$\begin{aligned} v(t) = &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} n(s)g(s,x(s),y(s))ds + \int_0^\infty \psi(s)G(s,x(s),y(s))ds \\ &+ \sum_{s=1}^{i} J(t_s,x(t_s),y(t_s)), t \in (t_i,t_{i+1}],\ i = 0,1,2,\cdots. \end{aligned} \tag{3.8}$$

Proof. The proof is similar to that of Lemma 3.1. □

Now, we define the operator T on $X \times Y$ by $T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t))$ where

$$\begin{aligned} T_1(x,y)(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds \\ &+ \int_0^\infty \phi(s)F(s,x(s),y(s))ds \\ &+ \sum_{s=1}^{i} I(t_s,x(t_s),y(t_s)), t \in (t_i,t_{i+1}],\ i \in N_0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} T_2(x,y)(t) = &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} n(s)g(s,x(s),y(s))ds \\ &+ \int_0^\infty \psi(s)G(s,x(s),y(s))ds \\ &+ \sum_{s=1}^{i} J(t_s,x(t_s),y(t_s)), t \in (t_i,t_{i+1}],\ i \in N_0. \end{aligned} \tag{3.10}$$

Remark 3.1. By Lemmas 3.1 and 3.2, $(x,y) \in X \times Y$ is a solution of system (1.1) if and only if $(x,y) \in X \times Y$ is a fixed point of the operator T.

Lemma 3.3. *The operator $T : X\times X \to X\times X$ is well defined and is completely continuous.*

Proof. The proof is long and will be divided into parts. First, we prove that T is well defined. Next, we show that T is continuous, and finally we prove that T is compact. Hence, T is completely continuous.

Step 1. We shall prove that $T : X \times Y \to X \times Y$ is well defined. For $(x, y) \in X \times Y$, we have $\|(x, y)\| = r > 0$. Then, (3.1)–(3.4) hold. Hence,

$$\begin{aligned}
&\frac{t^{1-\alpha}}{1+t^\sigma}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds\right| \\
&\le \frac{t^{1-\alpha}}{1+t^\sigma}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |m(s)f(s,x(s),y(s))|ds \\
&\le A_rL_1\frac{t^{1-\alpha}}{1+t^\sigma}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \ \left(\text{let } w=\frac{s}{t}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{1+t^\sigma}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds\right| \\
&\le \frac{A_rL_1t^{\alpha+k}}{1+t^\sigma}\int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw
\end{aligned}\tag{3.11}$$

and

$$\frac{1}{1+t^\sigma}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds\right| \to 0 \text{ as } t\to+\infty. \tag{3.12}$$

We see that $T_1(x,y)(t)$ is defined on $(0,+\infty)$ and is continuous on $(t_s,t_{s+1}]$ $(s = 0,1,2,\cdots)$. Next, we have $\lim\limits_{t\to t_s^+} T_1(x,y)(t)(s\in N_0)$ exist.

Also, in view of (3.1)–(3.3) and (3.12), we find for $t\in(t_i,t_{i+1}]$

$$\begin{aligned}
&\frac{1}{1+t^\sigma}|T_1(x,y)(t)| \\
&\le \frac{1}{1+t^\sigma}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |m(s)f(s,x(s),y(s))|ds \\
&\quad + \frac{1}{1+t^\sigma}\left[\int_0^\infty |\phi(s)F(s,x(s),y(s))|ds + \sum_{s=1}^{i} |I_s(t_s,x(t_s),y(t_s))|\right] \\
&\le \frac{A_rL_1t^{\alpha+k}}{1+t^\sigma}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)} + \frac{\|\phi\|_1A'_r}{1+t^\sigma} + \frac{1}{1+t^\sigma}\sum_{s=1}^{\infty}A_{r,s} \ \to 0 \text{ as } t\to+\infty.
\end{aligned}\tag{3.13}$$

With this we have shown that $T_1(x,y)\in X$. Similarly we can show that $T_2(x,y)\in X$. Hence, $(T_1(x,y),T_2(x,y))\in X\times X$ and $T : X\times X\to X\times X$ is well defined.

Step 2. We shall prove that T is continuous. Let $(x_n,y_n)\in X\times X$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We shall show that $T(x_n,y_n)\to T(x_0,y_0)$ as $n\to\infty$, i.e., $T_1(x_n,y_n)\to T_1(x_0,y_0)$ and $T_2(x_n,y_n)\to T_2(x_0,y_0)$ as $n\to\infty$.

In fact, there exists $r > 0$ such that $\|(x_n, y_n)\| \le r$, $(n = 0, 1, 2, \cdots)$. Then, (3.1)–(3.4) hold for $(x, y) = (x_n, y_n)$. Also,

$$\sup_{t\in(0,+\infty)} \frac{|x_n(t) - x_0(t)|}{1 + t^{\sigma}} \to 0 \quad \text{and} \quad \sup_{t\in(0,+\infty)} \frac{|y_n(t) - y_0(t)|}{1 + t^{\delta}} \to 0$$

as $n \to +\infty$. Noting

$$T_1(x_n, y_n)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) f(s, x_n(s), y_n(s))ds + \int_0^{\infty} \phi(s) F(s, x_n(s), y_n(s))ds$$
$$+ \sum_{s=1}^{i} I_s(t_s, x_n(t_s), y_n(t_s)),\ t \in (t_i, t_{i+1}],\ i = 0, 1, 2, \cdots$$

from the Lebesgue dominated convergence theorem, we get

$$\sup_{t\in(0,+\infty)} \frac{|T_1(x_n, y_n)(t) - T_1(x_0, y_0)(t)|}{1 + t^{\sigma}} \to 0$$

as $n \to +\infty$. Similarly,

$$\sup_{t\in(0,+\infty)} \frac{|T_2(x_n, y_n)(t) - T_2(x_0, y_0)(t)|}{1 + t^{\delta}} \to 0$$

as $n \to +\infty$. Hence, T is continuous.

Step 3. We shall prove that T is compact, i.e., for each nonempty open bounded subset Ω of $X \times Y$, we shall prove that $T(\overline{\Omega})$ is relatively compact. For this, we shall show that $T(\overline{\Omega})$ is uniformly bounded, equicontinuous on each $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$, both $t \to \frac{(T_1(x,y))(t)}{1+t^{\sigma}}$ and $t \to \frac{(T_2(x,y))(t)}{1+t^{\mu}}$ are equiconvergent as $t \to +\infty$.

Let Ω be an open bounded subset of $X \times Y$. There exists $r > 0$ such that (3.1) holds for all $(x, y) \in \overline{\Omega}$. Hence, (3.2)–(3.4) also hold for all $(x, y) \in \overline{\Omega}$.

Step 3a. We shall show that $T(\overline{\Omega})$ is uniformly bounded. Let $(x, y) \in \overline{\Omega}$. For $t \in (t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$, from (3.13) we have

$$\frac{1}{1 + t^{\sigma}} |T_1(x, y)(t)| \le A_r L_1 \frac{\mathbf{B}(\alpha, k+1)}{\Gamma(\alpha)} \sup_{t\in(0,+\infty)} \frac{t^{\alpha+k}}{1 + t^{\sigma}} + \|\phi\|_1 A'_r + \sum_{s=1}^{\infty} A_{r,s} < \infty.$$

Similarly, we can obtain for $t \in (t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$,

$$\frac{1}{1 + t^{\delta}} |T_2(x, y)(t)| \le B_r L_2 \frac{\mathbf{B}(\beta, l+1)}{\Gamma(\beta)} \sup_{t\in(0,+\infty)} \frac{t^{\beta+l}}{1 + t^{\mu}} + \|\psi\|_1 B'_r + \sum_{s=1}^{\infty} B_{r,s} < \infty.$$

Hence, it is easy to see that $T(\overline{\Omega})$ is uniformly bounded.

Step 3b. We shall prove that $T(\overline{\Omega})$ is equicontinuous on each $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$. We define

$$\overline{(T_1(x, y))}(t) = \begin{cases} (T_1(x, y))(t), t \in (t_i, t_{i+1}], \\ \lim\limits_{t \to t_i^+} (T_1(x, y))(t), t = t_i. \end{cases}$$

Then $\overline{(T_1(x,y))}$ is continuous on $[t_i, t_{i+1}]$. So $\{\overline{(T_1(x,y))} : (x,y) \in \overline{\Omega}\}$ is equicontinuous on $[t_i, t_{i+1}]$. Thus $\{(T_1(x,y)) : (x,y) \in \overline{\Omega}\}$ is equicontinuous on $[t_i, t_{i+1}]$.

Similarly, we can show that $\{(T_2(x,y)) : (x,y) \in \overline{\Omega}\}$ is equicontinuous on $[t_i, t_{i+1}]$.

So $T(\overline{\Omega})$ is equicontinuous on each $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$.

Step 3c. We shall show that $T(\overline{\Omega})$ is equiconvergent as $t \to +\infty$.

it comes from the following items:

$$\frac{1}{1+t^\sigma}|T_1(x,y)(t)| \le A_r L_1 \frac{\mathbf{B}(\alpha, k+1)}{\Gamma(\alpha)} \frac{t^{\alpha+k}}{1+t^\sigma} + \frac{\|\phi\|_1 A_r' + \sum_{s=1}^\infty A_{r,s}}{1+t^\sigma}$$
$$\to 0 \text{ uniformly as } t \to +\infty.$$

Similarly, we can obtain for $t \in (t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$,

$$\frac{1}{1+t^\delta}|T_2(x,y)(t)| \le B_r L_2 \frac{\mathbf{B}(\beta, l+1)}{\Gamma(\beta)} \frac{t^{\beta+l}}{1+t^\mu} + \frac{\|\psi\|_1 B_r' + \sum_{s=1}^\infty B_{r,s}}{1+t^m u}$$
$$\to 0 \text{ uniformly as } t \to +\infty.$$

We have established that $T(\overline{\Omega})$ is relatively compact. So T is completely continuous. This completes the proof. $\square$

We are now ready to present the main theorem.

Theorem 3.4. *Let (a)–(e) and (A)–(B) hold, $\Phi : R \to R$ be a sup-multiplicative-like function with supporting function ω, and its inverse function $\Phi^{-1} : R \to R$ with supporting function ν. Furthermore, suppose that*

(i) *there exist nonnegative numbers c_f, b_f, a_f, C_F, B_F, A_F, $C_{I,s}$, $B_{I,s}$ and $A_{I,s}$ such that $\sum\limits_{s=1}^{\infty} C_{I,s}$, $\sum\limits_{s=1}^{\infty} B_{I,s}$ and $\sum\limits_{s=1}^{\infty} A_{I,s}$ are convergent, and the following hold for all $(U,V) \in R^2$ and $t \in (0,\infty)$:*
$$\left|f\left(t,\ (1+t^\sigma)U,\ (1+t^\delta)V\right)\right| \le c_f + b_f|U| + a_f\Phi^{-1}(|V|),$$
$$\left|F\left(t,\ (1+t^\sigma)U,\ (1+t^\delta)V)\right)\right| \le C_F + B_F|U| + A_F\Phi^{-1}(|V|),$$
$$\left|I\left(t_s,\ (1+t_s^\sigma)U,\ (1+t_s^\delta)V\right)\right| \le C_{I,s} + B_{I,s}|U| + A_{I,s}\Phi^{-1}(|V|);$$

(ii) *there exist nonnegative numbers c_g, b_g, a_g, C_G, B_G, A_G, $C_{J,s}$, $B_{J,s}$ and $A_{J,s}$ such that $\sum\limits_{s=1}^{\infty} C_{J,s}$, $\sum\limits_{s=1}^{\infty} B_{J,s}$ and $\sum\limits_{s=1}^{\infty} A_{J,s}$ are convergent, and the following hold for all $(U,V) \in R^2$ and $t \in (0,\infty)$:*
$$\left|g\left(t,\ (1+t^\sigma)U,\ (1+t^\delta)V\right)\right| \le c_g + b_g\Phi(|U|) + a_g|V|,$$
$$\left|G\left(t,\ (1+t^\sigma)U,\ (1+t^\delta)V\right)\right| \le C_G + B_G\Phi(|U|) + A_G|V|,$$
$$\left|J\left(t_s,\ (1+t_s^\sigma)U,\ (1+t_s^\delta)V\right)\right| \le C_{J,s} + B_{J,s}\Phi(|U|) + A_{J,s}|V|.$$

Then, the system (1.1) has at least one solution in $X \times X$ if

$$\Sigma_3 < 1, \qquad \Theta_2 < 1, \qquad \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right) < 1 \tag{3.14}$$

or

$$\Sigma_3 < 1, \qquad \Theta_2 < 1, \qquad \frac{\Sigma_2}{1-\Sigma_3}\frac{1}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)} < 1, \tag{3.15}$$

where

$$\begin{aligned}\Theta_2 &= L_1\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}b_f + \|\phi\|_1 B_F + \sum_{s=1}^{\infty} B_{I,s},\\ \Theta_3 &= L_1\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}a_f + \|\phi\|_1 A_F + \sum_{s=1}^{\infty} A_{I,s},\\ \Sigma_2 &= L_2\frac{\mathbf{B}(\beta,l+1)}{\Gamma(\beta)}b_g + \|\psi\|_1 B_G + \sum_{s=1}^{\infty} B_{J,s},\\ \Sigma_3 &= L_2\frac{\mathbf{B}(\beta,l+1)}{\Gamma(\beta)}a_g + \|\psi\|_1 A_G + \sum_{s=1}^{\infty} A_{J,s}.\end{aligned} \tag{3.16}$$

Proof. We shall apply Lemma 2.6. From Lemma 3.3 we note that T is completely continuous. Let us consider the operator equation

$$(x,y) = \lambda T(x,y) \tag{3.17}$$

where $\lambda \in (0,1)$. We shall show that any solution (x,y) of (3.17) satisfies

$$\|(x,y)\| \le M \tag{3.18}$$

where M is a constant independent of λ. Now, in the context of Lemma 2.1, let

$$\Omega = \{(x,y) \in X \times X : \|(x,y)\| < M+1\}.$$

In view of (3.18), it is not possible to have $(x,y) \in \partial\Omega$ satisfying $(x,y) = \lambda T(x,y)$, hence we conclude by Lemma 2.6 that there exists $(x,y) \in \overline{\Omega}$ such that $(x,y) = T(x,y)$, i.e., the system (1.1) has a solution in $X \times X$. This completes the proof.

We shall now proceed to prove (3.18). Let (x,y) be a solution of the operator equation (3.17). It follows that $x = \lambda T_1(x,y)$ and $y = \lambda T_2(x,y)$, i.e.,

$$\begin{aligned}x(t) &= \lambda T_1(x,y)(t)\\ &= \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) f(s,x(s),y(s))ds\\ &\quad + \lambda\left[\int_0^{\infty}\phi(s)F(s,x(s),y(s))ds + \sum_{s=1}^{i} I(t_s,x(t_s),y(t_s))\right],\\ &\quad t \in (t_i,t_{i+1}],\ i = 0,1,2,\cdots\end{aligned} \tag{3.19}$$

and

$$\begin{aligned} y(t) &= \lambda T_2(x,y)(t) \\ &= \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} n(s) g(s, x(s), y(s)) ds \\ &\quad + \lambda \left[\int_0^\infty \psi(s) G(s, x(s), y(s)) ds + \sum_{s=1}^{i} J(t_s, x(t_s), y(t_s)) \right], \\ &\quad t \in (t_i, t_{i+1}],\ i = 0, 1, 2, \cdots. \end{aligned} \tag{3.20}$$

It is easy to see from condition (i) that

$$\begin{aligned} |f(t, x(t), y(t))| &= \left| f\left(t,\ (1+t^\sigma)\frac{x(t)}{1+t^\sigma}, (1+t^\delta)\frac{y(t)}{1+t^\delta}\right)\right| \\ &\le c_f + b_f \frac{|x(t)|}{1+t^\sigma} + a_f \Phi^{-1}\left(\frac{|y(t)|}{1+t^\delta}\right) \\ &\le c_f + b_f ||x|| + a_f \Phi^{-1}(||y||). \end{aligned} \tag{3.21}$$

Similarly, we get

$$\begin{aligned} |F(t, x(t), y(t))| &\le C_F + B_F ||x|| + A_F \Phi^{-1}(||y||), \\ |I(t_s, x(t_s), y(t_s))| &\le C_{I,s} + B_{I,s}\ ||x|| + A_{I,s} \Phi^{-1}(||y||). \end{aligned} \tag{3.22}$$

From (3.19), using (3.21) and (3.22), we find for $t \in (t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$,

$$\begin{aligned} \frac{|x(t)|}{1+t^\sigma} &\le \frac{1}{1+t^\sigma} |T_1(x,y)(t)| \\ &\le \frac{1}{1+t^\sigma} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |m(s) f(s, x(s), y(s))| ds \\ &\quad + \frac{1}{1+t^\sigma} \left[\int_0^\infty |\phi(s) F(s, x(s), y(s))| ds + \sum_{s=1}^{i} |I(t_s, x(t_s), y(t_s))| \right] \\ &\le \frac{1}{1+t^\sigma} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L_1 s^k \left[c_f + b_f ||x|| + a_f \Phi^{-1}(||y||)\right] ds \\ &\quad + \frac{1}{1+t^\sigma} \Bigg\{ \int_0^\infty |\phi(s)| \left[C_F + B_F ||x|| + A_F \Phi^{-1}(||y||)\right] ds \\ &\quad + \sum_{s=1}^{i} \left[C_{I,s} + B_{I,s} ||x|| + A_{I,s} \Phi^{-1}(||y||)\right] \Bigg\} \\ &\le L_1 \frac{1}{1+t^\sigma} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \left[c_f + b_f \|x\| + a_f \Phi^{-1}(\|y\|)\right] \\ &\quad + \|\phi\|_1 \left[C_F + B_F \|x\| + A_F \Phi^{-1}(\|y\|)\right] \\ &\quad + \sum_{s=1}^{\infty} \left[C_{I,s} + B_{I,s} \|x\| + A_{I,s} \Phi^{-1}(\|y\|)\right] \end{aligned}$$

$$
\begin{aligned}
&= L_1 \frac{t^{\alpha+k}}{1+t^\sigma} \frac{\mathbf{B}(\alpha, k+1)}{\Gamma(\alpha)} \left[c_f + b_f\|x\| + a_f \Phi^{-1}(\|y\|)\right] \\
&\quad + \|\phi\|_1 \left[C_F + B_F\|x\| + A_F \Phi^{-1}(\|y\|)\right] \\
&\quad + \sum_{s=1}^{\infty} \left[C_{I,s} + B_{I,s}\|x\| + A_{I,s} \Phi^{-1}(\|y\|)\right] \\
&\le L_1 \frac{\mathbf{B}(\alpha, k+1)}{\Gamma(\alpha)} \left[c_f + b_f\|x\| + a_f \Phi^{-1}(\|y\|)\right] \\
&\quad + \|\phi\|_1 \left[C_F + B_F\|x\| + A_F \Phi^{-1}(\|y\|)\right] \\
&\quad + \sum_{s=1}^{\infty} \left[C_{I,s} + B_{I,s}\|x\| + A_{I,s} \Phi^{-1}(\|y\|)\right] \\
&= \Theta_1 + \Theta_2\|x\| + \Theta_3 \Phi^{-1}(\|y\|)
\end{aligned}
$$

where

$$\Theta_1 = L_1 \frac{\mathbf{B}(\alpha, k+1)}{\Gamma(\alpha)} c_f + \|\phi\|_1 C_F + \sum_{s=1}^{\infty} C_{I,s}.$$

It follows that

$$\|x\| = \sup_{t\in(0,\infty)} \frac{1}{1+t^\sigma}|x(t)| \le \Theta_1 + \Theta_2\|x\| + \Theta_3\Phi^{-1}(\|y\|),$$

or equivalently

$$\|x\| \le \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}(\|y\|). \tag{3.23}$$

Similarly, from (3.20) we can show that

$$\|y\| \le \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi(\|x\|) \tag{3.24}$$

where

$$\Sigma_1 = L_2 \frac{\mathbf{B}(\beta, l+1)}{\Gamma(\beta)} c_g + \|\psi\|_1 C_G + \sum_{s=1}^{\infty} t_s^{1-\beta} C_{J,s}.$$

Case 1. Suppose (3.14) holds. If $\|x\| \le \Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right)$, by (3.24) we have $||y|| \le \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\frac{\Sigma_1}{\Sigma_2}$. Then (3.18) holds with $M_1 = \max\left\{\Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right), \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\frac{\Sigma_1}{\Sigma_2}\right\}$. If

$$\|x\| \ge \Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right). \tag{3.25}$$

Then, using (3.24) in (3.23) as well as (3.25) and (2.2), we get

$$\begin{aligned} \|x\| &\le \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}\left(\frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi(\|x\|)\right) \\ &\le \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}\left(\frac{2\Sigma_2}{1-\Sigma_3}\Phi(\|x\|)\right) \\ &\le \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right)\Phi^{-1}\left(\Phi(\|x\|)\right) \\ &= \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right)\|x\|. \end{aligned} \tag{3.26}$$

From (3.14) we have $\frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right) < 1$, therefore it follows from (3.26) that

$$\|x\| \le \frac{\Theta_1}{1-\Theta_2}\left[1 - \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right)\right]^{-1} \equiv W. \tag{3.27}$$

From the above discussion, we have either $\|x\| \le W$. Substituting (3.27 into (3.24) yields

$$\|y\| \le \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi\left(W\right) \equiv M_2. \tag{3.28}$$

Combining (3.27) and (3.28), we have proved that (3.18) holds with $M = \max\{W, M_1, M_2\}$.

<u>Case 2</u>. Suppose (3.16) holds. If $||y|| \le \Phi\left(\frac{\Theta_1}{\Theta_3}\right)$, then (3.23) implies that $||x|| \le \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\frac{\Theta_1}{\Theta_3}$. Thus (3.18) holds with $M_3 = \max\left\{\frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\frac{\Theta_1}{\Theta_3}, \Phi\left(\frac{\Theta_1}{\Theta_3}\right)\right\}$. If

$$\|y\| \ge \Phi\left(\frac{\Theta_1}{\Theta_3}\right). \tag{3.29}$$

Then, using (3.23) in (3.24) and together with (3.29) and (2.1), we find

$$\begin{aligned} \|y\| &\le \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi\left(\frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}(\|y\|)\right) \\ &\le \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi\left(\frac{2\Theta_3}{1-\Theta_2}\Phi^{-1}(\|y\|)\right) \\ &\le \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\frac{\Phi\left(\Phi^{-1}(\|y\|)\right)}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)} \\ &= \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\frac{1}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)}\|y\|. \end{aligned} \tag{3.30}$$

Since $\frac{\Sigma_2}{1-\Sigma_3}\left[w\left(\frac{1-\Theta_2}{2\Theta_3}\right)\right]^{-1}<1$, it is clear from (3.30) that

$$\|y\| \leq \frac{\Sigma_1}{1-\Sigma_3}\left[1-\frac{\Sigma_2}{1-\Sigma_3}\frac{1}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)}\right]^{-1} \equiv W', \tag{3.31}$$

which on substituting into (3.23) gives

$$\|x\| \leq \frac{\Theta_1}{1-\Theta_2}+\frac{\Theta_3}{1-\Theta_2}\Phi^{-1}\left(W'\right) \equiv M_4. \tag{3.32}$$

Coupling (3.31) and (3.32), we have shown that (3.18) holds with $M=\max\{W', M_3, M_4\}$. The proof is complete. □

4. An example

In this section, we present an example to illustrate the main theorem.

Example 4.1. Consider the following initial value problem of impulsive fractional differential system:

$$\begin{cases} {}^C D_{0^+}^{\frac{2}{5}} u(t)=t^{-\frac{1}{2}}\left[c_0+b_0\dfrac{u(t)}{1+t^{2/3}}+a_0\dfrac{(v(t))^3}{(1+t^{2/3})^3}\right],\ t\in(s,s+1],\ s\in N_0, \\ {}^C D_{0^+}^{\frac{3}{5}} v(t)=t^{-\frac{1}{2}}\left[c_1+b_1\dfrac{(u(t))^{\frac{1}{3}}}{(1+t^{2/3})^{1/3}}+a_1\dfrac{v(t)}{1+t^{2/3}}\right],\ t\in(s,s+1],\ s\in N_0, \\ \lim\limits_{t\to 0} t^{\frac{3}{5}}u(t)=B_0\displaystyle\int_0^\infty e^{-s}\frac{u(s)}{1+s^{2/3}}ds, \\ \lim\limits_{t\to 0} t^{\frac{2}{5}}v(t)=B_1\displaystyle\int_0^\infty e^{-s}\frac{(u(s))^{\frac{1}{3}}}{(1+s^{2/3})^{1/3}}ds, \\ \Delta u(s)=2^{-s}, \Delta v(s)=3^{-s},\ s=1,2,\cdots, \end{cases} \tag{4.1}$$

where $c_0,\ b_0,\ a_0,\ c_1,\ b_1,\ a_1,\ B_0$ and B_1 are constants.

Corresponding to system (1.1) we have

- **(a)** $\alpha=\frac{2}{5},\ \beta=\frac{3}{5}$,
- **(b)** $0=t_0<t_1=1<\cdots<t_s=s<\cdots$ with $\lim\limits_{s\to\infty} s=\infty$,
- **(c)** $m(t)=t^{-\frac{1}{2}}=n(t)$ are singular at $t=0$, $|m(t)|=|n(t)|\leq L_1t^k=L_2t^l$ with $L_1=L_2=1$ and $k=l=-\frac{1}{2}$,
- **(d)** $\phi(t)=e^{-t}=\psi(t)$ satisfy $\phi,\psi\in L^1(0,\infty)$, and
- **(e)** $f,\ g,\ F,\ G,\ I_s$ and J_s are defined by

$$f(t,x,y)=c_0+b_0\frac{x}{1+t^{2/3}}+a_0\frac{y^3}{(1+t^{2/3})^3},$$

$$g(t,x,y)=c_1+b_1\frac{x^{1/3}}{(1+t^{2/3})^{1/3}}+a_1\frac{y}{1+t^{2/3}},$$

$$F(t,x,y)=B_0\frac{x}{1+t^{2/3}},\quad G(t,x,y)=B_1\frac{x^{1/3}}{(1+t^{2/3})^{1/3}},$$

$$I(s,x,y) = 2^{-s}, \quad J(s,x,y) = 3^{-s}, \quad s = 1, 2, \cdots.$$

Choose $\sigma = \delta = \frac{2}{3}$. Then, $\sigma > k+1$ and $\delta > l+1$. It is easy to show that

(A) $f,\ g,\ F,\ G$ are Carathéodory functions,

(B) $I(s,x,y), J(s,x,y)$ are Caraéodory sequences.

Furthermore, in the context of Theorem 3.1, we have $\Phi^{-1}(x) = x^3$ with supporting function $w(x) = x^{\frac{1}{3}}$, and $\Phi(x) = x^{\frac{1}{3}}$ with supporting function $\nu(x) = x^3$. It is easy to see that conditions (i) and (ii) in Theorem 3.1 are satisfied with

$$\begin{aligned}
&c_f = |c_0|,\ \ b_f = |b_0|,\ \ a_f = |a_0|,\ \ C_F = 0,\ \ B_F = |B_0|,\ \ A_F = 0,\\
&C_{I,s} = 2^{-s},\ \ B_{I,s} = A_{I,s} = 0,\\
&c_g = |c_1|,\ \ b_g = |b_1|,\ \ a_g = |a_1|,\ \ C_G = 0,\ \ B_G = |B_1|,\ A_G = 0,\\
&C_{J,s} = 3^{-s},\ \ B_{J,s} = A_{J,s} = 0.
\end{aligned}$$

By direct computation, we get

$$\begin{aligned}
\Theta_2 &= \frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| + |B_0|, \quad \Theta_3 = \frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|a_0|,\\
\Sigma_2 &= \frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|b_1| + |B_1|, \quad \Sigma_3 = \frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1|.
\end{aligned}$$

Applying Theorem 3.1, we see that system (4.1) has at least one solution if (3.14) or (3.15) holds, i.e., if

$$\begin{gathered}
\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1| < 1, \quad \frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| + |B_0| < 1, \quad \text{and}\\
\frac{\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|a_0|}{1 - \frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| - |B_0|}\left[\frac{2\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|b_1| + 2|B_1|}{1 - \frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1|}\right]^3 < 1 \quad \text{or}\\
\frac{\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|b_1| + |B_1|}{1 - \frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1|}\left[\frac{1 - \frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| - |B_0|}{2\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|a_0|}\right]^{-\frac{1}{3}} < 1.
\end{gathered} \tag{4.2}$$

Remark 4.1. It is easy to see from (4.2) that system (4.1) has at least one solution for sufficiently small $|a_0|,\ |b_0|,\ |a_1|,\ |b_1|,\ |B_0|$ and $|B_1|$.

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