

## TOPOLOGICAL CHARACTERIZATION OF CERTAIN CLASSES OF ALMOST DISTRIBUTIVE LATTICES

N. RAFI\* AND G.C. RAO

ABSTRACT. Proved that the two topologies (spectral topology and  $D$ -topology) coincide on  $SpecR$ ,  $MaxR$  and  $MinSpecR$  iff  $R$  is complemented, normal and stone ADL respectively.

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### 1. Introduction

After Boole's axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [7] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set  $PI(R)$  of all principal ideals of  $R$  forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. Swamy, G.C. Rao and G.N. Rao introduced the concept of Stone ADL and characterized it in terms of its ideals. In [6], Sambasiva Rao and G.C. Rao introduced  $\sigma$ -ideals in an ADL and proved their properties, in [5], G.C. Rao and Ravi kumar introduced the concept of normal ADL and studied its properties extensively. In [1], Al-Ezeh introduced  $SpecL$  can be endowed with two topologies, the spectral topology and  $D$ -topology and he proved that two topologies coincide on  $SpecL$ ,  $MaxL$  and  $MinL$  iff  $L$  is a Boolean, Stonian and Normal lattice respectively. In this paper, the concept of  $D$ -topology introduced on  $SpecR$ , where  $SpecR$  is the set of all prime ideals of

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\*Corresponding author.

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an ADL and studied the relationship between  $\sigma$ -ideals in an ADL and  $D$ -open subsets of  $SpecR$ . Characterized those ADLs for which topologies coincide on  $SpecR$ ,  $MaxR$  and  $MinspecR$ , where  $MaxR$  and  $MinspecR$  are the set of all maximal ideals and minimal prime ideals of an ADL respectively.

## 2. Preliminaries

**Definition 2.1** ([7]). An Almost Distributive Lattice with zero or simply ADL is an algebra  $(R, \vee, \wedge, 0)$  of type  $(2, 2, 0)$  satisfying

1.  $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2.  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3.  $(x \vee y) \wedge y = y$
4.  $(x \vee y) \wedge x = x$
5.  $x \vee (x \wedge y) = x$
6.  $0 \wedge x = 0$
7.  $x \vee 0 = x$ , for all  $x, y, z \in R$ .

Every non-empty set  $X$  can be regarded as an ADL as follows. Let  $x_0 \in X$ . Define the binary operations  $\vee, \wedge$  on  $X$  by

$$x \vee y \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \quad x \wedge y \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then  $(X, \vee, \wedge, x_0)$  is an ADL (where  $x_0$  is the zero) and is called a discrete ADL. If  $(R, \vee, \wedge, 0)$  is an ADL, for any  $a, b \in R$ , define  $a \leq b$  if and only if  $a = a \wedge b$  (or equivalently,  $a \vee b = b$ ), then  $\leq$  is a partial ordering on  $R$ .

**Theorem 2.2** ([7]). *If  $(R, \vee, \wedge, 0)$  is an ADL, for any  $a, b, c \in R$ , we have the following:*

- 1).  $a \vee b = a \Leftrightarrow a \wedge b = b$
- 2).  $a \vee b = b \Leftrightarrow a \wedge b = a$
- 3).  $\wedge$  is associative in  $R$
- 4).  $a \wedge b \wedge c = b \wedge a \wedge c$
- 5).  $(a \vee b) \wedge c = (b \vee a) \wedge c$
- 6).  $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
- 7).  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- 8).  $a \wedge (a \vee b) = a$ ,  $(a \wedge b) \vee b = b$  and  $a \vee (b \wedge a) = a$
- 9).  $a \leq a \vee b$  and  $a \wedge b \leq b$
- 10).  $a \wedge a = a$  and  $a \vee a = a$
- 11).  $0 \vee a = a$  and  $a \wedge 0 = 0$
- 12). If  $a \leq c$ ,  $b \leq c$  then  $a \wedge b = b \wedge a$  and  $a \vee b = b \vee a$
- 13).  $a \vee b = (a \vee b) \vee a$ .

It can be observed that an ADL  $R$  satisfies almost all the properties of a distributive lattice except the right distributivity of  $\vee$  over  $\wedge$ , commutativity of  $\vee$ , commutativity of  $\wedge$ . Any one of these properties make an ADL  $R$  a distributive lattice. That is

**Theorem 2.3** ([7]). *Let  $(R, \vee, \wedge, 0)$  be an ADL with 0. Then the following are equivalent:*

- 1).  $(R, \vee, \wedge, 0)$  is a distributive lattice
- 2).  $a \vee b = b \vee a$ , for all  $a, b \in R$
- 3).  $a \wedge b = b \wedge a$ , for all  $a, b \in R$
- 4).  $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ , for all  $a, b, c \in R$ .

As usual, an element  $m \in R$  is called maximal if it is a maximal element in the partially ordered set  $(R, \leq)$ . That is, for any  $a \in R$ ,  $m \leq a \Rightarrow m = a$ .

**Theorem 2.4** ([7]). *Let  $R$  be an ADL and  $m \in R$ . Then the following are equivalent:*

- 1).  $m$  is maximal with respect to  $\leq$
- 2).  $m \vee a = m$ , for all  $a \in R$
- 3).  $m \wedge a = a$ , for all  $a \in R$
- 4).  $a \vee m$  is maximal, for all  $a \in R$ .

As in distributive lattices [2, 3], a non-empty sub set  $I$  of an ADL  $R$  is called an ideal of  $R$  if  $a \vee b \in I$  and  $a \wedge x \in I$  for any  $a, b \in I$  and  $x \in R$ . Also, a non-empty subset  $F$  of  $R$  is said to be a filter of  $R$  if  $a \wedge b \in F$  and  $x \vee a \in F$  for  $a, b \in F$  and  $x \in R$ .

The set  $I(R)$  of all ideals of  $R$  is a bounded distributive lattice with least element  $\{0\}$  and greatest element  $R$  under set inclusion in which, for any  $I, J \in I(R)$ ,  $I \cap J$  is the infimum of  $I$  and  $J$  while the supremum is given by  $I \vee J := \{a \vee b \mid a \in I, b \in J\}$ . A proper ideal  $P$  of  $R$  is called a prime ideal if, for any  $x, y \in R$ ,  $x \wedge y \in P \Rightarrow x \in P$  or  $y \in P$ . A proper ideal  $M$  of  $R$  is said to be maximal if it is not properly contained in any proper ideal of  $R$ . It can be observed that every maximal ideal of  $R$  is a prime ideal. Every proper ideal of  $R$  is contained in a maximal ideal. For any subset  $S$  of  $R$  the smallest ideal containing  $S$  is given by  $(S) := \{(\bigvee_{i=1}^n s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$ . If  $S = \{s\}$ , we write  $(s)$  instead of  $(S)$ . Similarly, for any  $S \subseteq R$ ,  $[S] := \{x \vee (\bigwedge_{i=1}^n s_i) \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$ . If  $S = \{s\}$ , we write  $[s]$  instead of  $[S]$ .

**Theorem 2.5** ([7]). *For any  $x, y$  in  $R$  the following are equivalent:*

- 1).  $(x) \subseteq (y)$
- 2).  $y \wedge x = x$
- 3).  $y \vee x = y$
- 4).  $[y] \subseteq [x]$ .

For any  $x, y \in R$ , it can be verified that  $(x) \vee (y) = (x \vee y)$  and  $(x) \wedge (y) = (x \wedge y)$ . Hence the set  $PI(R)$  of all principal ideals of  $R$  is a sublattice of the distributive lattice  $I(R)$  of ideals of  $R$ .

### 3. Characterization of certain classes of ADLs

In this section, the concept of  $D$ -topology on  $\text{Spec}R$  introduced. We characterized Complemented ADL, normal ADL, Stone ADL in terms of  $\sigma$ -ideals topologically. we recall definition and important results.

**Definition 3.1** ([5]). For any non-empty subset  $S$  of  $R$ , write  $(S)^* = \{a \in R \mid a \wedge s = 0, \text{ for all } s \in S\}$ . Then  $(S)^*$  is an ideal of  $R$  and is called the annihilator of  $S$  in  $R$ . If  $S = \{s\}$ , then we write  $(s)^*$  for  $(\{s\})^*$ .

**Definition 3.2** ([6]). Let  $R$  be an ADL. For any  $I$  of  $R$ , define  $\sigma(I) = \{x \in R \mid (x)^* \vee I = R\}$ .

**Lemma 3.3** ([6]). For any ideal  $I$  of an ADL  $R$ ,  $\sigma(I)$  is an ideal of  $R$ .

**Lemma 3.4** ([6]). For any two ideals  $I, J$  of an ADL  $R$ , we have the following:

- (1).  $\sigma(I) \subseteq I$
- (2).  $I \subseteq J$  implies  $\sigma(I) \subseteq \sigma(J)$
- (3).  $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$
- (4).  $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$ .

**Definition 3.5** ([6]). Let  $R$  be an ADL. An ideal  $I$  of  $R$  is said to be an  $\sigma$ -ideal, if  $\sigma(I) = I$ .

**Example 3.6** ([6]). Let  $A = \{0, a\}$  and  $B = \{0, b_1, b_2\}$  be two discrete ADLs. Write  $R = A \times B = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$ . Then  $(R, \vee, \wedge, 0')$  is an ADL under point-wise operations, where the zero element of  $R$  is  $0' = (0, 0)$ . Consider the subset  $I = \{(0, 0), (a, 0)\}$  of  $R$ . Then clearly  $I$  is an ideal of  $R$ . Now  $\sigma(I) = \{x \in R \mid (x)^* \vee I = R\} = \{(0, 0), (a, 0)\} = I$ . Therefore  $I$  is an  $\sigma$ -ideal of  $R$ .

Let  $R$  be a non-trivial ADL. Let  $\text{Spec}R$  denote the set of all prime ideals of  $R$ . For any  $A \subseteq R$ , let  $D(A) = \{P \in \text{Spec}R \mid A \not\subseteq P\}$  and for any  $a \in R$ ,  $D(a) = \{P \in \text{Spec}R \mid a \notin P\}$ . Then we have the following;

**Lemma 3.7.** Let  $R$  be an ADL with maximal elements. Then for any  $a, b \in R$ , the following hold.

- (1).  $\bigcup_{x \in R} D(x) = \text{Spec}R$
- (2).  $D(a) \cap D(b) = D(a \wedge b)$
- (3).  $D(a) \cup D(b) = D(a \vee b)$ .

From the above lemma we can immediately say that the collection  $\{D(a) \mid a \in R\}$  forms a base for a topology on  $\text{Spec}R$ . The topology generated by this base is precisely  $\{D(A) \mid A \subseteq R\}$  and is called the hull-kernel topology on  $\text{Spec}R$ .

**Theorem 3.8.** Let  $R$  be an ADL with maximal elements. For any  $a \in R$ , the following hold.

- 1).  $D(a) = \emptyset$  if and only if  $a = 0$
- 2).  $D(a) = \text{Spec}R$  if and only if  $a$  is a maximal element.

Now we give the following definition.

**Definition 3.9.** Let  $R$  be an ADL. A subset  $X$  of  $\text{Spec}R$  is called  $S$ -stable if for any  $P, Q \in \text{Spec}R$ , whenever  $P \subseteq Q$  and  $P \in X, Q \in X$ .

**Lemma 3.10.** Let  $R$  be an ADL with maximal elements and  $I$ , an ideal of  $R$ . Then a spectrally open set  $D(I)$  is  $S$ -stable if and only if  $I$  is an  $\sigma$ -ideal of an ADL  $R$ .

*Proof.* Assume that  $I$  is an  $\sigma$ -ideal of  $R$ . Let  $P, Q \in \text{Spec}R$  such that  $P \subseteq Q$  and  $P \in D(I)$ . Then  $I \not\subseteq P$ . We prove that  $Q \in D(I)$ . Since  $I \not\subseteq P$ , choose an element  $x \in I$  such that  $x \notin P$ . Since  $I$  is an  $\sigma$ -ideal of  $R$ , there exist  $x_1 \in I$  and  $y \in (x]^*$  such that  $x_1 \vee y$  is a maximal element in  $R$ . Then  $y \in P$  and  $P \subseteq Q$ . That implies  $x_1 \notin Q$  (since  $x_1 \vee y$  is maximal and  $y \in Q$ ). Therefore  $I \not\subseteq Q$  and hence  $Q \in D(I)$ . Thus  $D(I)$  is  $S$ -stable. Conversely, assume that  $D(I)$  is  $S$ -stable. We prove that  $I$  is an  $\sigma$ -ideal of  $R$  (i.e.  $\sigma(I) = I$ ). Suppose  $I$  is not an  $\sigma$ -ideal of  $R$ . Then there exists an element  $x \in I$  such that  $I \vee (x]^* \neq R$ . Since  $I \vee (x]^*$  is a proper ideal of  $R$ , there exists a maximal ideal  $M$  of  $R$  such that  $I \vee (x]^* \subseteq M$ . Then  $I \subseteq M$  and  $(x]^* \subseteq M$ . Now consider the set  $\mathfrak{F} := \{J \mid J \text{ is an ideal of } R, x \notin J \text{ and } J \subseteq M\}$ . Clearly  $(x]^* \in \mathfrak{F}$ . Then  $\mathfrak{F} \neq \emptyset$  and hence  $(\mathfrak{F}, \subseteq)$  is a poset in which every chain has an upper bound. By the Zorn's lemma,  $\mathfrak{F}$  has a maximal element  $P_1$  say. That is  $x \notin P_1$  and  $P_1 \subseteq M$ . Since  $x \in I$ , we have  $P_1 \in D(I)$ . Since  $D(I)$  is  $S$ -stable we get  $M \in D(I)$ . Then  $I \not\subseteq M$ , which is a contradiction. Therefore  $I$  is an  $\sigma$ -ideal of  $R$ .  $\square$

Now, we state the following result whose proof is straightforward.

**Theorem 3.11.** Let  $R$  be an ADL with maximal elements. Then the mapping  $I \rightarrow D(I)$  is bijection from the set of all  $\sigma$ -ideals of  $R$  to the set of all  $D$ -open subsets of  $\text{Spec}R$ .

Now we give the following definition.

**Definition 3.12.** Let  $R$  be an ADL with maximal elements. An element  $x$  of an ADL  $R$  is said to be complemented if there exists an element  $y \in R$  such that  $x \wedge y = 0$  and  $x \vee y$  maximal.

**Lemma 3.13.** Let  $R$  be an ADL with maximal elements. Then  $D(I)$  is spectrally clopen (both open and closed) if and only if  $D(I) = D(x)$ , for some complemented element  $x$  in  $R$ .

*Proof.* Assume that  $x$  is a complemented element in  $R$ . Then there exists an element  $x' \in R$  such that  $x \wedge x' = 0$  and  $x \vee x'$  is a maximal element. Now  $D(x) \cap D(x') = D(x \wedge x') = \emptyset$  and  $D(x) \cup D(x') = D(x \vee x') = \text{Spec}R$  (since  $x \wedge x' = 0$  and  $x \vee x'$  is maximal). Therefore  $D(x)$  is clopen. Conversely, assume that  $D(I)$  is clopen. Then  $\text{Spec}R \setminus D(I)$  is also an open set. Then there exists an  $\sigma$ -ideal  $J$  of  $R$  such that  $D(J) = \text{Spec}R \setminus D(I)$ . Now  $D(I) \cap D(J) = D(I \cap J) = \emptyset$ . That implies  $I \cap J = \{0\}$ . Also now,  $\text{Spec}R = D(I) \cup D(J) = D(I \vee J)$ . That implies  $I \vee J = R$ . Let  $m$  be any maximal element in  $R$ . So that  $a \vee b = m$ , for

some  $a \in I$ ,  $b \in J$  and  $a \wedge b = 0$  (since  $I \cap J = \{0\}$ ). We prove that  $I = (a]$ . Let  $x \in I$ . Now  $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) = x \wedge a$  (since  $x \wedge b = 0$ ). Then  $x \in (a]$ . Therefore  $(a] = I$  and hence  $D(I) = D((a])$ . Thus  $a$  is a complemented element in  $R$ .  $\square$

**Definition 3.14.** An ADL  $R$  is said to be complemented if every element of  $R$  is complemented.

**Theorem 3.15.** Let  $R$  be an ADL with maximal elements. Then the spectral topology and  $D$ -topology coincide on  $\text{Spec}R$  if and only if  $R$  is a complemented ADL.

*Proof.* Let  $\tau$  and  $\tau_D$  be spectral topology and  $D$ -topology on  $\text{Spec}R$ . Assume that  $\tau = \tau_D$ . Let  $x \in R$ . We know that  $D((x])$  is  $D$ -open. Then the ideal  $(x]$  generated by  $x$  is an  $\sigma$ -ideal of  $R$ . That implies  $(x]^* \vee (x] = R$ . Let  $m$  be any maximal element of  $R$ . Then there exists an element  $t \in (x]^*$  such that  $t \vee x$  is a maximal element. Therefore  $x$  is complemented. Hence  $R$  is a complemented ADL. Conversely, assume that  $R$  is a complemented ADL. We prove that  $\tau = \tau_D$ . It is enough to prove that every ideal of  $R$  is an  $\sigma$ -ideal. Consider an ideal  $I$  of  $R$ . Let  $x \in I$ . Then there exists an element  $x' \in R$  such that  $x \wedge x' = 0$  and  $x \vee x'$  is maximal. That implies  $(x]^* \vee I = R$ . Therefore  $x \in \sigma(I)$  and hence  $I \subseteq \sigma(I)$ . Thus  $I = \sigma(I)$ .  $\square$

The following two definitions taken from [8] and [9].

**Definition 3.16.** Let  $R$  be an ADL. Then a unary operation  $a \mapsto a^\perp$  on  $R$  is called a pseudo-complementation on  $R$  if, for any  $a, b \in R$ , it satisfies the following conditions:

- (P<sub>1</sub>).  $a \wedge b = 0 \implies a^\perp \wedge b = b$
- (P<sub>2</sub>).  $a \wedge a^\perp = 0$
- (P<sub>3</sub>).  $(a \vee b)^\perp = a^\perp \wedge b^\perp$ .

**Theorem 3.17.** Let  $R$  be an ADL with maximal elements and  $\perp$  a pseudo-complementation on  $L$ . Then, for any  $a, b \in R$ , we have the following:

- 1)  $0^\perp$  is a maximal element
- 2) If  $a$  is a maximal element then  $a^\perp = 0$
- 3)  $0^{\perp\perp} = 0$
- 4)  $0^\perp \wedge a = a$
- 5)  $a^{\perp\perp} \wedge a = a$
- 6)  $a^{\perp\perp\perp} = a^\perp$
- 7)  $a \leq b \implies b^\perp \leq a^\perp$
- 8)  $a^\perp \wedge b^\perp = b^\perp \wedge a^\perp$
- 9)  $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \wedge b^{\perp\perp}$
- 10)  $a^\perp \wedge b = (a \wedge b)^\perp \wedge b^\perp$ .

**Definition 3.18.** Let  $R$  be an ADL with a pseudo-complementation  $\perp$ . Then  $R$  is called a stone ADL if, for any  $x \in R$ ,  $x^\perp \vee x^{\perp\perp} = 0^\perp$ .

**Theorem 3.19** ([9]). *Let  $R$  be an ADL with maximal elements. Then  $R$  is stone ADL if and only if for any  $x \in R$ ,  $(x]^* \vee (x]^{**} = R$ .*

$MinspecR$  denote the set of all minimal prime ideals of an ADL  $R$ . For any  $x \in R$ , write  $D_M(x) = D(x) \cap MinspecR$ . Now we prove the following theorem.

**Theorem 3.20.** *Let  $R$  be an ADL with maximal elements. Then  $R$  is stone if and only if the spectral topology and the  $D$ -topology coincides on  $MinspecR$ .*

*Proof.* Assume that  $R$  is a stone ADL. Let  $x \in R$ . We prove that  $D_M(x) = D(x) \cap MinspecR$  is  $S$ -stable. Since  $x \in R$  and  $R$  is stone  $(x]^* = (y]$ , for some complemented element  $y$  in  $R$ . Since  $y \in (x]^*$ , there exists  $z \in (x]^{**}$  such that  $y \vee z$  is maximal and also  $y \wedge z = 0$ . Let  $P \in D_M(x)$ . Then  $x \notin P$ . So that  $y \in P$  (since  $x \wedge y = 0$ ) and hence  $z \notin P$  (since  $y \vee z$  is maximal). That implies  $P \in D_M(z)$ . Therefore  $D_M(x) \subseteq D_M(z)$ . Let  $P \in D_M(z)$ . Then  $z \notin P$ . That implies  $y \in P$ . Since  $P$  is a minimal prime ideal of  $R$ , we get  $P \in D_M(x)$ . Therefore  $D_M(z) \subseteq D_M(x)$  and hence  $D_M(x) = D_M(z)$ . Thus  $D_M(x)$  is  $D$ -open (since  $D_M(z)$  is  $D$ -open). Finally, we prove that  $D(z)$  is  $S$ -stable. Let  $P, Q \in SpecR$  such that  $P \in D(z)$  and  $P \subseteq Q$ . Then  $z \notin P$  and hence  $y \in P$ . That implies  $y \in Q$ . So that  $z \notin Q$ . Therefore  $Q \in D(z)$ . Hence  $D(z)$  is  $S$ -stable and  $D(z)$  is open in  $specR$ .  $D_M(z)$  is open in  $MinspecR$ . Thus  $\tau = \tau_D$  on  $MinspecR$ . Conversely, assume that  $\tau = \tau_D$  on  $MinspecR$ . Let  $x \in R$ . We prove that  $R$  is a Stone ADL. It is enough to show that  $(x]^* \vee (x]^{**} = R$ . Since  $\tau = \tau_D$  on  $MinspecR$ ,  $D_M(x)$  is  $D$ -open. By the lemma 3.10, there exists an  $\sigma$ -ideal  $I$  of  $R$  such that  $D_M(x) = D_M(I)$ . That implies  $(x]^* = I^*$ . We prove that  $(x]^* \vee I = R$  and  $(x]^* \cap I = \{0\}$ . Suppose  $(x]^* \vee I \neq R$ . Then there exists a maximal ideal  $M$  of  $R$  such that  $(x]^* \vee I \subseteq M$ . So that  $((x]^* \vee I) \cap (R \setminus M) = \emptyset$ . Then there exists  $Q \in MinspecR$  such that  $(x]^* \vee I \subseteq M$ ,  $Q \subseteq M$ . So that  $(x]^* \subseteq M$ ,  $I \subseteq M$  and  $Q \subseteq M$ . That implies  $x \notin Q$  and hence  $I \not\subseteq Q$  which is a contradiction. Therefore  $(x]^* \vee I = R$ . Let  $a \in (x]^* \cap I$ . Then  $a \in (x]^*$  and  $a \in I$ . so that  $a \wedge a = 0$  and hence  $a = 0$ , (since  $(x]^* = I^*$ ). Therefore  $(x]^* \cap I = \{0\}$ .  $\square$

We observe that, if  $R$  has maximal elements, then the Zorn's lemma helps us in proving the existence of maximal ideals. However, if  $R$  has no maximal elements, then there is no such guarantee. For, consider the following.

**Example 3.21.** If  $R$  is a chain with least element which is not bounded above, then  $R$  has no maximal ideals. For, if  $I$  is any proper ideal of  $R$  and  $a \in R \setminus I$ , then  $I \subsetneq (a] \subsetneq R$  and hence  $I$  is not maximal.

For this reason, we consider in this section only ADLs with maximal elements. Let  $MaxR$  denote the space of all maximal ideals of  $R$ . Then  $MaxR$  is a subspace of  $SpecR$ . This subspace topology on  $MaxR$  is given by  $\{D^M(X) | X \subseteq R\}$ , where  $D^M(X) = \{M \in MaxR | X \not\subseteq M\} = D(X) \cap MaxR$ . For this subspace topology on  $MaxR$ ,  $\{D^M(x) | x \in R\}$  forms a base where for any  $x \in R$ ,  $D^M(x) = D^M(\{x\})$ . Here after words we treat  $MaxR$  is a topological space with this hull-kernel topology. We recall the definition of normal ADL in [5].

**Definition 3.22.** An ADL  $R$  is said to be normal if every prime ideal of  $R$  contained in unique maximal ideal of  $R$ .

**Theorem 3.23.** Let  $R$  be an ADL with maximal elements. Then the spectral topology and the  $D$ -topology coincides on  $MaxR$  if and only  $R$  is a normal ADL.

*Proof.* Assume that  $R$  be a normal ADL. Let  $x \in R$ . Consider  $D^M(x)$  and let  $V = \{P \in SpecR \mid M_P \in D^M(x)\} = \{P \in SpecR \mid x \notin M_P\}$ . We prove that  $D^M(x)$  is a  $D$ -open set in  $MaxR$ . Let  $P_1, P_2 \in SpecR$  such that  $P_1 \subseteq P_2$  and  $P_1 \in V$ . Since  $P_1$  and  $P_2$  are proper ideals in  $R$ , there exist maximal ideals  $M_{P_1}$  and  $M_{P_2}$  in  $R$  such that  $P_1 \subseteq M_{P_1}$  and  $P_2 \subseteq M_{P_2}$ . Since  $P_1 \subseteq P_2$ , we have  $P_1 \subseteq M_{P_2}$ . By normality of  $R$ , we get  $M_{P_1} = M_{P_2}$ . That implies  $M_{P_2} \in D^M(x)$ . So that  $P_2 \in V$ . Therefore  $D^M(x)$  is a  $D$ -open set in  $MaxR$  (since  $V \cap MaxR = D^M(x)$ ). Hence  $\tau = \tau_D$  on  $MaxR$ . Conversely, assume that  $\tau = \tau_D$  on  $MaxR$ . Let  $P$  be a prime ideal of  $R$ . Suppose  $M_1, M_2$  are two disjoint maximal ideals of  $R$  such that  $P \subseteq M_1$  and  $P \subseteq M_2$ . Since  $M_1 \neq M_2$ , choose  $a \in M_1$  such that  $a \notin M_2$ . Then  $M_1 \notin D^M(a)$  and  $M_2 \in D^M(a)$ . Then  $D^M(a)$  is  $D$ -open in  $MaxR$ . Then there exists an  $\sigma$ -ideal  $I$  of  $R$  such that  $D^M(a) = D^M(I)$ . Since  $M_2 \in D^M(a)$ , we get  $M_2 \in D^M(I)$ . That implies  $P \in D^M(I)$ . Since  $D(I)$  is  $S$ -stable and  $P \subseteq M_1$ , we get  $M_1 \in D(I)$ . That implies  $I \not\subseteq M_1$ . So that  $M_1 \in D^M(a)$ , which is a contradiction.  $M_1 = M_2$ . Hence  $R$  is normal.  $\square$

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**Noorbhasha Rafi** received M.Sc. from Andhra University and Ph.D from Andhra University. He is currently working as Assistant professor at Bapatla Engineering College since 2012. His research interests include Almost Distributive lattices, G-algebras.

Department of Mathematics, Bapatla Engineering College, Bapatla, Andhra Pradesh, India - 522 101.

e-mail: rafimaths@gmail.com

**G.C. Rao** received M.Sc. from Andhra University, and Ph.D. from Andhra University. He is currently a professor at Andhra University His research interests are Almost Distributive Lattices, C-algebras and Sheaf Theory.

Department of Mathematics, Andhra University, Visakhapatnam, Andhra Pradesh, India - 530003.

e-mail: gcraomaths@yahoo.co.in