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ON THE SPECIAL VALUES OF TORNHEIM'S MULTIPLE SERIES†

MIN-SOO KIM

Abstract. Recently, Jianxin Liu, Hao Pan and Yong Zhang in [On the integral of the product of the Appell polynomials, Integral Transforms Spec. Funct. 25 (2014), no. 9, 680–685] established an explicit formula for the integral of the product of several Appell polynomials. Their work generalizes all the known results by previous authors on the integral of the product of Bernoulli and Euler polynomials. In this note, by using a special case of their formula for Euler polynomials, we shall provide several reciprocity relations between the special values of Tornheim's multiple series.

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}, \tag{1}$$

and the Euler polynomials $E_k(x)$ are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} \tag{2}$$

(see Zhi-Wei Sun's lecture [17]).

Notice that the Bernoulli numbers $B_k = B_k(0)$ and the Euler numbers

$$E_k = 2^k E_k\left(\frac{1}{2}\right). \tag{3}$$

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In his classical book "Vorlesungen uber Differenzenrechnung", Nörlund presented the following formula for the integrals of two Bernoulli and Euler polynomials [13, p. 31 and 36]:

$$\begin{aligned}\int_0^1 B_k(z)B_m(z)dz &= (-1)^{k-1}\frac{k!m!}{(k+m+1)!}B_{k+m}, \quad k,m\in\mathbb{N},\\ \int_0^1 E_k(z)E_m(z)dz &= 2(-1)^{k+1}\frac{k!m!}{(k+m+1)!}E_{k+m+1}(0), \quad k,m\in\mathbb{N}_0.\end{aligned} \tag{4}$$

For the integral of two Bernoulli polynomials, Nielsen [12] and Mordell [11] provided two different proofs. In the appendix of a very recent book [1], Zagier also gave another interesting proof by using the Fourier expansion of Bernoulli polynomials (see [1, p. 250, Proposition A.8.]). In 1958, Mordell remarked: "The integrals containing the product of more than two Bernoulli polynomials do not appear to lead to simple results." (See [11, p. 375]). Later, Carlitz [4] presented a proof of formulas on the integrals of the products of three and four Bernoulli polynomials. Subsequently, Wilson [20] generalized Carlitz's result on the integral of the product of three Bernoulli polynomials by evaluating the integral

$$\int_0^1 \overline{B}_k(az)\overline{B}_l(bz)\overline{B}_m(cz)dz, \tag{5}$$

where $\overline{B}_k(x)$ is the periodic extension of $B_k(x)$ on $[0,1)$ and a,b,c are pairwise coprime integers. Carlitz's result becomes a special case when $a=b=c=1$. Similar integral evaluations have also been used by Espinosa and Moll [7] during their study on Tornheim's double sums.

We also see that it is a reasonable convention to set

$$C_{k_1,\ldots,k_r}(x)=0 \quad \text{when } \min\{k_1,\ldots,k_r\}<0.$$

In 2011, Agoh and Dilcher [3] generalized the result of Wilson and showed that

Proposition 1.1 (Agoh and Dilcher [3, Proposition 3]). *For $k,l,m\in\mathbb{N}_0$, we have*

$$\frac{1}{k!l!m!}\int_0^x B_k(z)B_l(z)B_m(z)dz=\sum_{a=0}^{k+l}(-1)^a\sum_{i=0}^{a}\binom{a}{i}\frac{C_{k-a+i,l-i,m+a+1}(x)}{(k-a+i)!(l-i)!(m+a+1)!}, \tag{6}$$

where $C_{k,l,m}(x)=B_k(x)B_l(x)B_m(x)-B_kB_lB_m$.

In 2012, Hu, Kim and Kim [8] generalized the above results to obtain the integral of the products of arbitrary many Bernoulli polynomials, in fact, they proved the explicit formula for

$$\int_0^x B_{k_1}(z)B_{k_2}(z)\cdots B_{k_r}(z)dz.$$

Recently, using this integral, Cihat Dagli and Can [6] established a connection between the reciprocity relations of sums of products of Bernoulli polynomials and of the Dedekind sums.

In 2013, Liu, Pan and Zhang [9] extended Hu, Kim and Kim's result by establishing an explicit formula for the integral of the product of several Appell polynomials. If a polynomial sequence $\{A_n(x)\}, n \in \mathbb{N}_0$, satisfies that

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x),$$

then we say $\{A_n(x)\}$ is an Appell sequence. The Bernoulli polynomials, Euler polynomials, and the probablists' Hermite polynomials are both Appell polynomials. For Euler polynomials, their result is as follows.

Theorem 1.2 (Liu, Pan and Zhang [9, p. 682 (1.5)]).

$$\begin{aligned}
&\frac{1}{k_1!\cdots k_r!}\int_{1/2}^{x} E_{k_1}(z)E_{k_2}(z)\cdots E_{k_r}(z)dz \\
&\quad= \sum_{a=0}^{K}(-1)^a \sum_{\substack{i_1+\cdots+i_{r-1}=a \\ 0\le i_1,\dots,i_{r-1}\le a}} \binom{a}{i_1,\dots,i_{r-1}} \\
&\quad\times \left(\frac{E_{k_r+a+1}(x)}{(k_r+a+1)!}\prod_{j=1}^{r-1}\frac{E_{k_j-i_j}(x)}{(k_j-i_j)!} - \frac{E_{k_r+a+1}}{2^{K+1}(k_r+a+1)!}\prod_{j=1}^{r-1}\frac{E_{k_j-i_j}}{(k_j-i_j)!}\right),
\end{aligned} \tag{7}$$

where $K = k_1+\cdots+k_r$ and Euler numbers $E_k = 2^kE_k(1/2)$.

Proposition 1.3. *Let $k_1,\dots,k_r \in \mathbb{N}_0$, and let*

$$\begin{aligned}
I_{k_1,\dots,k_r}(x) &= \int_0^x E_{k_1}(z)\cdots E_{k_r}(z)dz, \\
C_{k_1,\dots,k_r}(x) &= E_{k_1}(x)\cdots E_{k_r}(x) - E_{k_1}(0)\cdots E_{k_r}(0), \\
\widetilde{I}_{k_1,\dots,k_r}(x) &= \frac{1}{k_1!\cdots k_r!}I_{k_1,\dots,k_r}(x), \\
\widetilde{C}_{k_1,\dots,k_r}(x) &= \frac{1}{k_1!\cdots k_r!}C_{k_1,\dots,k_r}(x).
\end{aligned}$$

Then we have

$$\begin{aligned}
\widetilde{I}_{k_1,\dots,k_r}(x) = &\sum_{a=0}^{k_1+\cdots+k_{r-1}}(-1)^a \sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,\dots,j_{r-1}} \\
&\times \widetilde{C}_{k_1-j_1,\dots,k_{r-1}-j_{r-1},k_r+a+1}(x).
\end{aligned}$$

Proof. This proposition is implied by the above Theorem, and it can also be proved following the same line as [8, Proposition 1.4]. □

In this paper, we shall apply the above result on the integral of the product of arbitrary many Euler polynomials to obtain several reciprocity relations between the special values of Tornheim's multiple series.

First, we recall the history and some background on Tornheim's series.

In 1950, Tornheim considered the double series $T(p,q,r)$ which was defined by

$$T(p,q,r)=\sum_{m,n=1}^{\infty}\frac{1}{m^p n^q (m+n)^r}, \tag{8}$$

where p,q,r are nonnegative integers with $p+r>1, q+r>1$ and $p+q+r>2$. In particular, he showed that $T(p,q,N-p-q)$ is a polynomial in $\{\zeta(j)\mid 2\leq j\leq N\}$ with rational coefficients, if N is an odd integer bigger than 3.

In 1958, Mordell [10] evaluated Tornheim's double series at $p=q=r=2k$, where k is a positive integer. In 1985, Subbarao and Sitaramachandrarao [16] extended Mordell's results by considering the alternating analogue of (8) which was defined by

$$R(p,q,r)=\sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^p n^q (m+n)^r} \tag{9}$$

and

$$S(p,q,r)=\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^p n^q (m+n)^r}. \tag{10}$$

In 2003, Tsumura [18] considered the following partial Tornheim's double series

$$\mathfrak{T}_{b_1,b_2}(p,q,r)=\sum_{m,n=0}^{\infty}\frac{1}{(2m+b_1)^p(2n+b_2)^q(2m+2n+b_1+b_2)^r}, \tag{11}$$

where $b_1,b_2\in\{1,2\}$. In particular, he wrote $\mathfrak{T}_{1,1}(p,q,r)$ as a rational linear combination of products of Riemann's zeta values at positive integers, when p and q are odd positive integers with $q\geq 3$ (see [18, Proposition 3.5]).

There exist the following two ways for the generalizations of above Tornheim's double series to the multiple cases:

$$\mathfrak{T}_r^{+}(s_1,\ldots,s_r;s)=\sum_{p_1,\ldots,p_r=0}^{\infty}\frac{1}{(2p_1+1)^{s_1}\cdots(2p_r+1)^{s_r}(2p_1+\cdots+2p_r+r)^{s}} \tag{12}$$

and

$$\mathfrak{T}_r^{-}(s_1,\ldots,s_r;s)=\sum_{p_1,\ldots,p_r=0}^{\infty}\frac{1}{(2p_1+1)^{s_1}\cdots(2p_r+1)^{s_r}(-2p_1+\cdots+2p_r+1)^{s}}. \tag{13}$$

We set $\mathfrak{T}^{+}(s)=\mathfrak{T}_1^{+}(s;0)$. Note that

$$\mathfrak{T}^{+}(s)=(1-2^{-s})\zeta(s), \tag{14}$$

where $\zeta(s)$ is the Riemann zeta function.

In 2004, Tsumura [19] obtained the following result on the special values of $\mathfrak{T}_r^{+}(s_1,\ldots,s_r;s)$ (for definition, see (12) above), so called Euler-Mordell-Tornhein zeta values.

Theorem 1.4 ([19, Theorem 1.1])**.** *The Euler-Mordell-Tornheim zeta value*

$$\mathfrak{T}_r^+(k_1,\ldots,k_r;k) \quad \textit{with } k\geq 2$$

can be expressed as a rational linear combination of products of Euler-Mordell-Tornheim zeta values of lower depth than r, when its depth r and its weight are of different parity.

In this paper, by using the formula on the integral of products of arbitrary many Euler polynomials, we obtain the following results on the relationships between the special values of the above Tornheim's multiple series (12) and (13).

The beta values $\beta(s)$ are defined by (see [2, p. 807, entry 23.2.21] and [19, (2.1)])

$$\beta(s)=\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)^s}. \tag{15}$$

According to Leibnitz's test for alternating series, this is a series which converges for all $\mathrm{Re}(s)>0$.

In this paper, we shall give an elementary proof of the following known result (see [18, Example 3.7] and [19, Proposition 2.1]).

Proposition 1.5. *For $m\in\mathbb{N}_0$, we have*

$$\beta(2m+1)=\frac{(-1)^m\pi^{2m+1}}{2^{2m+2}(2m)!}E_{2m},$$

where E_{2m} are the Euler numbers (see (3) above).

We shall also give an elementary proof of the following known result (see Shimura's book [15, (4.93)]).

Proposition 1.6. *For $m\in\mathbb{N}$, we have*

$$\mathfrak{T}^+(2m)=\frac{(-1)^m\pi^{2m}}{4(2m-1)!}E_{2m-1}(0).$$

For simplification of the notations, in what following, we shall denote by

$$\widetilde{E}_k(x)=\frac{1}{k!}E_k(x).$$

Theorem 1.7. (1) *For $l,m,n\in\mathbb{N}$, we have the following reciprocity relation:*

$$\begin{aligned}
&\mathfrak{T}_3^-(2l+1,2m+1,2n+1;1)+\mathfrak{T}_3^-(2m+1,2l+1,2n+1;1)\\
&+\mathfrak{T}_3^-(2n+1,2l+1,2m+1;1)-\mathfrak{T}_3^+(2l+1,2m+1,2n+1;1)\\
&=(-1)^{l+m+n}\frac{\pi^{2(l+m+n+2)}}{16}\sum_{a=0}^{2l+2m}(-1)^{a+1}\\
&\qquad\times\sum_{i=0}^{a}\binom{a}{i}\widetilde{E}_{2l-a+i}(0)\widetilde{E}_{2m-i}(0)\widetilde{E}_{2n+a+1}(0).
\end{aligned}$$

(2) *For $k, l, m, n \in \mathbb{N}$, we have the following reciprocity relation:*

$$\begin{aligned}
&\mathfrak{T}_3^{+}(2l, 2m, 2n; 2k) + \mathfrak{T}_3^{+}(2k, 2m, 2n; 2l) \\
&\qquad + \mathfrak{T}_3^{+}(2k, 2l, 2n; 2m) + \mathfrak{T}_3^{+}(2k, 2l, 2m; 2n) \\
&\qquad + \mathfrak{T}_3^{-}(2n, 2l, 2m; 2k) + \mathfrak{T}_3^{-}(2n, 2k, 2m; 2l) \\
&\qquad + \mathfrak{T}_3^{-}(2n, 2k, 2l; 2m) \\
&= \frac{(-1)^{k+l+m+n}\pi^{2(k+l+m+n)}}{16} \sum_{a=0}^{2k+2l+2m-3} (-1)^{a+1} \widetilde{E}_{2n+a}(0) \\
&\qquad \times \sum_{i_1+i_2+i_3=a} \binom{a}{i_1, i_2, i_3} \\
&\qquad \times \widetilde{E}_{2k-1-i_1}(0) \widetilde{E}_{2l-1-i_2}(0) \widetilde{E}_{2m-1-i_3}(0).
\end{aligned}$$

Remark 1.1. There exists a preceding research which has some similar ideas with the present work. That is, in some senses, Onodera [14] also connected the Mordell-Tornheim zeta function with the integral of the product of Bernoulli polynomials (see [14, p. 1468, Remark 2.2]). But his results are completely different with us.

2. Proof of Proposition 1.5, 1.6 and Theorem 1.7

In this section, we shall prove our main results.

1) Proof of Proposition 1.5 and 1.6:

The Euler polynomials are represented by the following Fourier series ([2, p. 805, entry 23.1.17 and 23.1.18] and [5, (14a) and (14b)])

$$E_{2m-1}(x) = a_m \sum_{p=0}^{\infty} \frac{\cos(2p+1)\pi x}{(2p+1)^{2m}}, \tag{16}$$

where $0 \le x \le 1$ for $m \in \mathbb{N}$, and

$$E_{2m}(x) = b_m \sum_{p=0}^{\infty} \frac{\sin(2p+1)\pi x}{(2p+1)^{2m+1}}, \tag{17}$$

where $0 \le x \le 1$ for $m \in \mathbb{N}$, $0 < x < 1$ for $m = 0$. Here

$$a_m = (-1)^m \frac{4(2m-1)!}{\pi^{2m}}, \quad b_m = (-1)^m \frac{4(2m)!}{\pi^{2m+1}}. \tag{18}$$

First, by (15) and taking $x = 1/2$ in (17), we have

$$\beta(2m+1) = \frac{(-1)^m \pi^{2m+1}}{2^{2m+2}(2m)!} E_{2m},$$

where E_{2m} are the Euler numbers (see (3) above), this is Proposition 1.5.

Next, by putting $x = 0$ in (16), we obtain

$$\mathfrak{T}^{+}(2m) = \frac{(-1)^m \pi^{2m}}{4(2m-1)!} E_{2m-1}(0),$$

this is Proposition 1.6.

Remark 2.1. By (4), (16) and (17), we have the following integral formulas for the Euler-Mordell-Tornheim zeta values

$$\mathfrak{T}^{+}(2m+2n) = \frac{2}{a_m a_n} I_{2m-1,2n-1}(1) = \frac{(-1)^{m+n}\pi^{2m+2n}}{4(2m+2n-1)!} E_{2m+2n-1}(0)$$

and

$$\mathfrak{T}^{+}(2m+2n+2) = \frac{2}{b_m b_n} I_{2m,2n}(1) = \frac{(-1)^{m+n+1}\pi^{2m+2n+2}}{4(2m+2n+1)!} E_{2m+2n+1}(0).$$

Remark 2.2. We setting $r = 1$ in (12), we obtain $\mathfrak{T}_1^{+}(k_1; k) = \mathfrak{T}^{+}(k_1 + k)$ for $k_1, k \in \mathbb{N}$. Putting $r = 2$ in (12), we have $\mathfrak{T}_2^{+}(k_1, k_2; k)$ $(k_1, k_2, k \in \mathbb{N})$, and this case has already been considered in [18].

2) Proof of Theorem 1.7 (1):

By (17), we obtain the expression

$$\frac{E_{2l}(x)E_{2m}(x)E_{2n}(x)}{b_l b_m b_n} = \sum_{p,q,r=0}^{\infty} \frac{\sin(2p+1)\pi x \sin(2q+1)\pi x \sin(2r+1)\pi x}{(2p+1)^{2l+1}(2q+1)^{2m+1}(2r+1)^{2n+1}}, \tag{19}$$

where $l, m, n \in \mathbb{N}$.

From

$$\begin{aligned} 4\sin A \sin B \sin C = {} & \sin(-A+B+C) + \sin(A-B+C) \\ & + \sin(A+B-C) - \sin(A+B+C), \end{aligned}$$

we have

$$\begin{aligned} & \int_0^1 \sin(2p+1)\pi x \sin(2q+1)\pi x \sin(2r+1)\pi x dx \\ & = \frac{1}{2}\left[\frac{1}{(-2p+2q+2r+1)\pi} + \frac{1}{(2p-2q+2r+1)\pi} \right. \\ & \qquad \left. + \frac{1}{(2p+2q-2r+1)\pi} - \frac{1}{(2p+2q+2r+3)\pi}\right]. \end{aligned} \tag{20}$$

By (13), (19) and (20), we have the following equality.

$$\begin{aligned} & \mathfrak{T}_3^{-}(2l+1, 2m+1, 2n+1; 1) + \mathfrak{T}_3^{-}(2m+1, 2l+1, 2n+1; 1) \\ & + \mathfrak{T}_3^{-}(2n+1, 2l+1, 2m+1; 1) - \mathfrak{T}_3^{+}(2l+1, 2m+1, 2n+1; 1) \\ & = \frac{2\pi}{b_l b_m b_n} I_{2l,2m,2n}(1). \end{aligned} \tag{21}$$

And from Proposition 1.3, we also have

$$\begin{aligned}
&C_{k_1-a+i,k_2-i,k_3+a+1}(1)\\
&=((-1)^{k_1+k_2+k_3+1}-1)E_{k_1-a+i}(0)E_{k_2-i}(0)E_{k_3+a+1}(0)\\
&=\begin{cases}-2E_{k_1-a+i}(0)E_{k_2-i}(0)E_{k_3+a+1}(0) & \text{if } k_1+k_2+k_3 \text{ even}\\ 0 & \text{if } k_1+k_2+k_3 \text{ odd.}\end{cases}
\end{aligned} \tag{22}$$

By Proposition 1.3 and (22), the integral $\widetilde{I}_{2l,2m,2n}(1)$ can be expressed by

$$\begin{aligned}
\widetilde{I}_{2l,2m,2n}(1) &= \int_0^1 \widetilde{E}_{2l}(z)\widetilde{E}_{2m}(z)\widetilde{E}_{2n}(z)dz\\
&=\sum_{a=0}^{2l+2m}(-1)^a\sum_{i=0}^{a}\binom{a}{i}\frac{C_{2l-a+i,2m-i,2n+a+1}(1)}{(2l-a+i)!(2m-i)!(2n+a+1)!}\\
&=2\sum_{a=0}^{2l+2m}(-1)^{a+1}\sum_{i=0}^{a}\binom{a}{i}\frac{E_{2l-a+i}(0)E_{2m-i}(0)E_{2n+a+1}(0)}{(2l-a+i)!(2m-i)!(2n+a+1)!}\\
&=2\sum_{a=0}^{2l+2m}(-1)^{a+1}\sum_{i=0}^{a}\binom{a}{i}\widetilde{E}_{2l-a+i}(0)\widetilde{E}_{2m-i}(0)\widetilde{E}_{2n+a+1}(0),
\end{aligned} \tag{23}$$

since $2l+2m+2n \equiv 0 \pmod 2$. This is equivalent to

$$\begin{aligned}
I_{2l,2m,2n}(1) = 2(2l)!(2m)!(2n)! &\sum_{a=0}^{2l+2m}(-1)^{a+1}\\
\times\sum_{i=0}^{a}\binom{a}{i}&\widetilde{E}_{2l-a+i}(0)\widetilde{E}_{2m-i}(0)\widetilde{E}_{2n+a+1}(0).
\end{aligned} \tag{24}$$

Finally by comparing (21) with (24), we obtain the following identity

$$\begin{aligned}
&\mathfrak{T}_3^-(2l+1,2m+1,2n+1;1)+\mathfrak{T}_3^-(2m+1,2l+1,2n+1;1)\\
&+\mathfrak{T}_3^-(2n+1,2l+1,2m+1;1)-\mathfrak{T}_3^+(2l+1,2m+1,2n+1;1)\\
&=(-1)^{l+m+n}\frac{\pi^{2(l+m+n+2)}}{16}\sum_{a=0}^{2l+2m}(-1)^{a+1}\\
&\quad\times\sum_{i=0}^{a}\binom{a}{i}\widetilde{E}_{2l-a+i}(0)\widetilde{E}_{2m-i}(0)\widetilde{E}_{2n+a+1}(0),
\end{aligned}$$

which is Theorem 1.7 (1).

3) Proof of Theorem 1.7 (2):

Letting $x=1$ in Proposition 1.3, we have the following equality:

$$\begin{aligned}\widetilde{I}_{2k-1,2l-1,2m-1,2n-1}(1) = &\sum_{a=0}^{2k+2l+2m-3}(-1)^a \sum_{i_1+i_2+i_3=a}\binom{a}{i_1,i_2,i_3}\\ &\times((-1)^{2k+2l+2m+2n-3}-1)\\ &\times \widetilde{E}_{2k-1-i_1}(0)\widetilde{E}_{2l-1-i_2}(0)\widetilde{E}_{2m-1-i_3}(0)\widetilde{E}_{2n+a}(0),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\widetilde{I}_{2k-1,2l-1,2m-1,2n-1}(1) = 2 &\sum_{a=0}^{2k+2l+2m-3}(-1)^{a+1}\widetilde{E}_{2n+a}(0)\\ &\times \sum_{i_1+i_2+i_3=a}\binom{a}{i_1,i_2,i_3}\\ &\times \widetilde{E}_{2k-1-i_1}(0)\widetilde{E}_{2l-1-i_2}(0)\widetilde{E}_{2m-1-i_3}(0),\end{aligned} \tag{25}$$

where $k,l,m,n\in\mathbb{N}$.

From (16), we have

$$\begin{aligned}&\frac{E_{2k-1}(x)E_{2l-1}(x)E_{2m-1}(x)E_{2n-1}(x)}{a_ka_la_ma_n}\\ &= \sum_{p,q,r,s=0}^{\infty}\frac{\cos(2p+1)\pi x\cos(2q+1)\pi x\cos(2r+1)\pi x\cos(2s+1)\pi x}{(2p+1)^{2k}(2q+1)^{2l}(2r+1)^{2m}(2s+1)^{2n}}.\end{aligned} \tag{26}$$

The series on the right hand side converges uniformly for $0\le x\le 1$, thus can be integrated term wise. Also notice that by integration of the terms from 0 and 1, the series vanishes except $p=q+r+s+1$ or $q=p+r+s+1$ or $r=p+q+s+1$ or $s=p+q+r+1$ or $p=q+r-s$ or $q=p+r-s$ or $r=p+q-s$.

This is because

$$\begin{aligned}8\cos A\cos B\cos C\cos D = &\cos(A+B+C+D)+\cos(A+B+C-D)\\ &+\cos(-A+B+C+D)+\cos(-A+B+C-D)\\ &+\cos(A-B+C+D)+\cos(A-B+C-D)\\ &+\cos(A+B-C+D)+\cos(A+B-C-D).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{a_ka_la_ma_n}{8}\{\mathfrak{T}_3^+(2l,2m,2n;2k)+\mathfrak{T}_3^+(2k,2m,2n;2l)\\ &\qquad+\mathfrak{T}_3^+(2k,2l,2n;2m)+\mathfrak{T}_3^+(2k,2l,2m;2n)\\ &\qquad+\mathfrak{T}_3^-(2n,2l,2m;2k)+\mathfrak{T}_3^-(2n,2k,2m;2l)\\ &\qquad+\mathfrak{T}_3^-(2n,2k,2l;2m)\}\\ &= I_{2k-1,2l-1,2m-1,2n-1}(1).\end{aligned} \tag{27}$$

From (25) and (27) we have

$$
\begin{aligned}
\mathfrak{T}_3^{+}&(2l,2m,2n;2k)+\mathfrak{T}_3^{+}(2k,2m,2n;2l)\\
&\quad +\mathfrak{T}_3^{+}(2k,2l,2n;2m)+\mathfrak{T}_3^{+}(2k,2l,2m;2n)\\
&\quad +\mathfrak{T}_3^{-}(2n,2l,2m;2k)+\mathfrak{T}_3^{-}(2n,2k,2m;2l)\\
&\quad +\mathfrak{T}_3^{-}(2n,2k,2l;2m)\\
&=\frac{8(2k-1)!(2l-1)!(2m-1)!(2n-1)!}{a_k a_l a_m a_n}\\
&\quad \times 2\sum_{a=0}^{2k+2l+2m-3}(-1)^{a+1}\widetilde{E}_{2n+a}(0)\\
&\quad \times \sum_{i_1+i_2+i_3=a}\binom{a}{i_1,i_2,i_3}\\
&\quad \times \widetilde{E}_{2k-1-i_1}(0)\widetilde{E}_{2l-1-i_2}(0)\widetilde{E}_{2m-1-i_3}(0).
\end{aligned}
\tag{28}
$$

Then we have the following reciprocity relation:

$$
\begin{aligned}
\mathfrak{T}_3^{+}&(2l,2m,2n;2k)+\mathfrak{T}_3^{+}(2k,2m,2n;2l)\\
&\quad +\mathfrak{T}_3^{+}(2k,2l,2n;2m)+\mathfrak{T}_3^{+}(2k,2l,2m;2n)\\
&\quad +\mathfrak{T}_3^{-}(2n,2l,2m;2k)+\mathfrak{T}_3^{-}(2n,2k,2m;2l)\\
&\quad +\mathfrak{T}_3^{-}(2n,2k,2l;2m)\\
&=\frac{(-1)^{k+l+m+n}\pi^{2(k+l+m+n)}}{16}\sum_{a=0}^{2k+2l+2m-3}(-1)^{a+1}\widetilde{E}_{2n+a}(0)\\
&\quad \times \sum_{i_1+i_2+i_3=a}\binom{a}{i_1,i_2,i_3}\\
&\quad \times \widetilde{E}_{2k-1-i_1}(0)\widetilde{E}_{2l-1-i_2}(0)\widetilde{E}_{2m-1-i_3}(0),
\end{aligned}
$$

which is Theorem 1.7 (2).

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Min-Soo Kim received Ph.D. degree from Kyungnam University. His research interests focus on the p-adic numbers, p-adic analysis and zeta-functions.

Center for General Education, Kyungnam University, 7(Woryeong-dong) kyungnamdaehak-ro, Masanhappo-gu, Changwon-si, Gyeongsangnam-do 631-701, Korea
e-mail: mskim@kyungnam.ac.kr