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ROUGH ANTI-FUZZY SUBRINGS AND THEIR PROPERTIES

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Abstract. In this paper, we shall introduce the concept of rough anti-fuzzy subring and prove some theorems in this context. We have, if μ is an anti-fuzzy subring, then μ is a rough anti-fuzzy subring. Also we give some properties of homomorphism and anti-homomorphism on rough anti-fuzzy subring.

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1. Introduction

The fuzzy set introduced by L.A. Zadeh in 1965 and the rough set introduced by Z. Pawlak in 1982 are generalisations of the classical set theory. Both these set theories are new mathematical tool to deal the uncertain, vague and imprecise data. In Zadeh's fuzzy set theory, the degree of membership of elements of a set plays the key role, whereas in Pawlak's rough set theory, equivalence classes of a set are the building blocks for the upper and lower approximations of the set, in which a subset of universe is approximated by the pair of ordinary sets, called upper and lower approximations. Combining the theory of rough set with abstract algebra is one of the trends in the theory of rough set. Some authors studied the concept of rough algebraic structures. On the other hand, some authors substituted an algebraic structure for the universal set and studied the roughness in algebraic structure. Biswas and Nanda introduced the notion of rough subgroups. The concept of rough ideal in a semigroup was introduced by Kuroki. And then B. Davvaz studied relationship between rough sets and ring theory and considered ring as a universal set and introduced the notion of rough ideals of a rings in [4]. A further study of this work is done by Osman Kazanci and B Davaaz in [8]. Extensive researches has also been carried out to compare the theory of rough sets with other theories of uncertainty such as fuzzy sets and

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conditional events. There have been many papers studying the connections and differences of fuzzy set theory and rough set theory. Dubois and Prade were one of the first who combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets.

This paper deals with a relationship between rough sets, fuzzy sets and ring theory. In section 2, we review some basic definitions. Section 3 deals with some properties of rough anti-fuzzy subring. In section 4,we give some homomorphic properties of rough anti-fuzzy subring. Section 5 deals with anti-homomorphic properties of rough anti-fuzzy subring.

2. Preliminaries

Definition 2.1. Let θ be an equivalence relation on R, then θ is called a full congruence relation if
$(a, b) \in \theta$ implies$(a + x, b + x)$, (ax, bx), $(xa, xb) \in \theta$ for all $x \in R$.

A full congruence relation θ on R is called complete if $[ab]_\theta = [a]_\theta [b]_\theta$.

Definition 2.2. Let θ be a full congruence relation on R and A a subset of R. Then the sets

- $\theta_-(A) = \{ x \in R : [x]_\theta \subseteq A\}$ and
- $\theta^-(A) = \{x \in R : [x]_\theta \cap A \neq \phi\}$

are called, respectively , the θ - lower and θ - upper approximations of the set A. $\theta(A) = (\theta_-(A)\ ,\ \theta^-(A))$ is called a rough set with respect to θ if $\theta_-(A) \neq \theta^-(A)$

Definition 2.3 ([9])**.** Let X and Y be two non-empty sets, $f : X \to Y$, μ and σ be fuzzy subsets of X and Y respectively. Then
$f(\mu)$, the image of μ under f is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x);\ f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 \quad \text{otherwise} \end{cases}$$

$f^{-1}(\sigma)$, the pre-image of σ under f is a fuzzy subset of X defined by

$$f^{-1}(\sigma)(x) = \sigma(f(x)) \ \ \forall x \in X.$$

Definition 2.4 ([9])**.** For a function $f : R_1 \to R_2$, a fuzzy subset μ of a ring R_1 is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, x, $y \in R_1$.

We say that a fuzzy subset μ of a ring R_1 has the sup property if for any subset T of R_1 , there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t\in T} \mu(t)$.

Definition 2.5. A fuzzy subset μ of a ring R is called upper rough f-invariant if $\theta^-(\mu)$ is f-invariant and a lower rough f-invariant if $\theta_-(\mu)$ is a f-invariant.

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ a rough fuzzy set. If $\theta_-(\mu)$ and $\theta^-(\mu)$ are f-invariant, then $(\theta_-(\mu)\ ,\ \theta^-(\mu))$ is called rough f-invariant.

3. Rough Anti-Fuzzy Subring

As it is well known in the fuzzy set theory established by Zadeh, a fuzzy subset μ of a set R is defined as a map from R to the unit interval $[0,1]$.

Definition 3.1 ([8])**.** Let θ be an equivalence relation on R and μ a fuzzy subset of R. Then we define the fuzzy sets $\theta_{-}(\mu)$, $\theta^{-}(\mu)$ as follows:

$$\theta_{-}(\mu)(x) = \bigwedge_{z\in[x]_\theta} \mu(z) \quad \text{and} \quad \theta^{-}(\mu)(x) = \bigvee_{z\in[x]_\theta} \mu(z).$$

The fuzzy sets $\theta_{-}(\mu)$ and $\theta^{-}(\mu)$ are called , respectively the θ- lower and θ-upper approximations of the fuzzy set μ. $\theta(\mu) = (\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ is called a rough fuzzy set with respect to θ if $\theta_{-}(\mu) \neq \theta^{-}(\mu)$.

Definition 3.2. A fuzzy subset μ of a ring R is called a fuzzy subring of R if

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$
(2) $\mu(xy) \geq \mu(x) \wedge \mu(y)$

for all $x, y \in R$.

Definition 3.3. A fuzzy subset μ of a ring R is called an anti-fuzzy subring of R if

(1) $\mu(x-y) \leq \mu(x) \vee \mu(y)$
(2) $\mu(xy) \leq \mu(x) \vee \mu(y)$

for all $x, y \in R$.

Definition 3.4. A fuzzy subset μ of a ring R is called an upper rough fuzzy subring of R if $\theta^{-}(\mu)$ is a fuzzy subring of R and a lower rough fuzzy subring of R if $\theta_{-}(\mu)$ is a fuzzy subring of R.

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ a rough fuzzy set. If $\theta_{-}(\mu)$ and $\theta^{-}(\mu)$ are fuzzy subrings of R, then $(\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ is called a rough fuzzy subring.

Definition 3.5. A fuzzy subset μ of a ring R is called an upper rough anti-fuzzy subring of R if $\theta^{-}(\mu)$ is an anti-fuzzy subring of R and a lower rough anti-fuzzy subring of R if $\theta_{-}(\mu)$ is an anti-fuzzy subring of R.

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ a rough fuzzy set. If $\theta_{-}(\mu)$ and $\theta^{-}(\mu)$ are anti-fuzzy subrings of R, then $(\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ is called a rough anti-fuzzy subring.

Theorem 3.6. *Let θ be a complete congruence relation on R. If μ is an anti-fuzzy subring of R, then $\theta^{-}(\mu)$ is an anti-fuzzy subring of R.*

Proof. For x,$y \in R$,

$$\begin{aligned} \theta^{-}(\mu)(x-y) &= \bigvee_{z\in[x-y]_\theta} \mu(z) \\ &= \bigvee_{z\in([x]_\theta-[y]_\theta)} \mu(z) \end{aligned}$$

$$\begin{aligned}
&= \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a-b)\\
&\le \bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a)\vee\mu(b)) \quad (\because \mu \text{ is an anti-fuzzy subring})\\
&= \bigvee_{a\in[x]_\theta} \mu(a) \vee \bigvee_{b\in[y]_\theta} \mu(b)\\
&= \theta^-(\mu)(x)\vee\theta^-(\mu)(y)
\end{aligned}$$

Hence $\theta^-(\mu)(x-y) \le \theta^-(\mu)(x) \vee \theta^-(\mu)(y)$. Also we have,

$$\begin{aligned}
\theta^-(\mu)(xy) &= \bigvee_{z\in[xy]_\theta} \mu(z)\\
&= \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(ab)\\
&\le \bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a)\vee\mu(b)) \quad (\because \mu \text{ is an anti-fuzzy subring })\\
&= \bigvee_{a\in[x]_\theta} \mu(a) \vee \bigvee_{b\in[y]_\theta} \mu(b)\\
&= \theta^-(\mu)(x)\vee\theta^-(\mu)(y)
\end{aligned}$$

Hence $\theta^-(\mu)(xy) \le \theta^-(\mu)(x) \vee \theta^-(\mu)(y)$. Therefore, $\theta^-(\mu)$ is an anti-fuzzy subring of R. □

Theorem 3.7. *Let θ be a complete congruence relation on R. If μ is an anti-fuzzy subring of R, then $\theta_-(\mu)$ is an anti-fuzzy subring of R.*

Proof. For x,$y \in R$,

$$\begin{aligned}
\theta_-(\mu)(x-y) &= \bigwedge_{z\in[x-y]_\theta} \mu(z)\\
&= \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a-b)\\
&\le \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a)\vee\mu(b)) \quad (\because \mu \text{ is an anti-fuzzy subring })\\
&= \bigwedge_{a\in[x]_\theta} \mu(a) \vee \bigwedge_{b\in[y]_\theta} \mu(b)\\
&= \theta_-(\mu)(x)\vee\theta_-(\mu)(y)
\end{aligned}$$

Hence $\theta_-(\mu)(x-y) \le \theta_-(\mu)(x) \vee \theta_-(\mu)(y)$. Also we have,

$$\theta_-(\mu)(xy) = \bigwedge_{z\in[xy]_\theta} \mu(z)$$

$$\begin{aligned}
&= \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(ab) \\
&\leq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a) \vee \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy subring}) \\
&= \bigwedge_{a\in[x]_\theta} \mu(a) \vee \bigwedge_{b\in[y]_\theta} \mu(b) \\
&= \theta_-(\mu)(x) \vee \theta_-(\mu)(y)
\end{aligned}$$

Hence $\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \vee \theta_-(\mu)(y)$. Therefore, $\theta_-(\mu)$ is an anti-fuzzy subring of R. □

Corollary 3.8. *Let μ be an anti-fuzzy subring of R, then μ is a rough anti-fuzzy subring of R.*

Proof. This follows from Theorems 3.6 and 3.7. □

Remark 3.1. From here onwards θ, θ_1 and θ_2 denote full congruence relations on the rings R, R_1 and R_2 respectively.

Definition 3.9. Let μ be a fuzzy subset of R. Then the sets
$\mu_t = \{ x \in R \mid \mu(x) \leq t \}$, $\mu_t^s = \{ x \in R \mid \mu(x) < t \}$,
where $t \in [0, 1]$ are called respectively, t-lower level subset and t-strong lower level subset of μ.

Theorem 3.10. *Let μ be a fuzzy subset of R and $t \in [0, 1]$, then*

$$(1)\ (\theta^-(\mu))_t^s = \theta_-(\mu_t^s) \quad \textit{and} \quad (2)\ (\theta_-(\mu))_t = \theta^-(\mu_t).$$

Proof. (1) We have

$$\begin{aligned}
x \in (\theta^-(\mu))_t^s \quad &\Longleftrightarrow \theta^-(\mu)(x) < t \\
&\Longleftrightarrow \bigvee_{a\in[x]_\theta} \mu(a) < t \\
&\Longleftrightarrow \mu(a) < t \ \ \forall \text{ a} \in [x]_\theta \\
&\Longleftrightarrow [x]_\theta \subseteq \mu_t^s \\
&\Longleftrightarrow x \in \theta_-(\mu_t^s)
\end{aligned}$$

(2) Also we have

$$\begin{aligned}
x \in (\theta_-(\mu))_t \quad &\Longleftrightarrow \theta_-(\mu)(x) \leq t \\
&\Longleftrightarrow \bigwedge_{a\in[x]_\theta} \mu(a) \leq t \\
&\Longleftrightarrow \exists \ \ a \in [x]_\theta \text{ such that } \mu(a) \leq t \\
&\Longleftrightarrow [x]_\theta \cap \mu_t \neq \phi \\
&\Longleftrightarrow x \in \theta^-(\mu_t)
\end{aligned}$$

□

Theorem 3.11. *Let μ be a fuzzy subset of R. Then μ is an anti-fuzzy subring if and only if μ_t and μ_t^s are, if they are nonempty, subrings of R for every $t \in [0, 1]$.*

Proof. Suppose μ is an anti-fuzzy subring of R. Let $x, y \in \mu_t$. Then $\mu(x) \leq t$ and $\mu(y) \leq t$.
Since μ is an anti-fuzzy subring, we have $\mu(x-y) \leq \mu(x) \vee \mu(y) \leq t$. Therefore $x - y \in \mu_t$. Also since $\mu(xy) \leq \mu(x) \vee \mu(y)$, $\mu(xy) \leq t$. Therefore $xy \in \mu_t$. Hence μ_t is a subring of R. Similarly we can show that μ_t^s is also a subring of R.

Conversely, let μ_t be a subring of R. Let $x, y \in R$. Assume $\mu(x) \leq \mu(y)$ and $\mu(y) = t$. Then $x, y \in \mu_t$. Since μ_t is a subring, $x - y \in \mu_t$. Hence $\mu(x - y) \leq t = \mu(y) = \mu(x) \vee \mu(y)$. Thus $\mu(x - y) \leq \mu(x) \vee \mu(y)$. Again since $xy \in \mu_t$, $\mu(xy) \leq t = \mu(y) = \mu(x) \vee \mu(y)$. Hence $\mu(xy) \leq \mu(x) \vee \mu(y)$. Therefore μ is an anti-fuzzy subring of R. □

4. Homomorphism on Rough Anti-Fuzzy Subring

Definition 4.1. Let R and R' be any two rings. Then the function $f : R \to R'$ is said to be a homomorphism if for all $x, y \in R$

$$f(x + y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y)$$

Theorem 4.2 ([8])**.** *Let f be a homomorphism of a ring R_1 onto a ring R_2 and let A be a subset of R_1. Then*

(1) *$\theta_1 = \{(a, b)|(f(a), f(b)) \in \theta_2\}$ is a full congruence relation on R_1.*
(2) *$f(\theta_1^-(A)) = \theta_2^-(f(A))$*
(3) *$f(\theta_{1-}(A) \subseteq \theta_{2-}(f(A))$. If f is one to one, then $f(\theta_{1-}(A)) = \theta_{2-}(f(A))$*

Theorem 4.3 ([3])**.** *Let f be a homomorphism from ring R_1 onto a ring R_2 and let μ be a fuzzy subset of R_1. Then*

(1) *$f(\theta_1^-(\mu)) = \theta_2^-(f(\mu))$*
(2) *$f(\theta_{1-}(\mu)) \subseteq \theta_{2-}(f(\mu))$. If f is one to one, then $f(\theta_{1-}(\mu)) = \theta_{2-}(f(\mu))$*

Remark 4.1. Let f be a homomorphism from ring R_1 to a ring R_2 and μ be a fuzzy subset of R_1. Let $y \in (f(\mu))_t^s \iff f(\mu)(y) < t \iff sup_{f(x)=y}\, \mu(x) < t \iff \mu(x) < t\ \forall x$ such that $f(x) = y \iff x \in \mu_t^s$ for $f(x) = y \iff y \in f(\mu_t^s)$. Then $f(\mu_t^s) = (f(\mu))_t^s$

Remark 4.2. Let f be a homomorphism (anti-homomorphism) from ring R_1 onto a ring R_2, and let σ be a fuzzy subset of R_2. Then $f^{-1}(\sigma)$ is a fuzzy subset of R_1. Hence by theorem 4.3, we get $f(\theta_1^-(f^{-1}(\sigma)) = \theta_2^-(f(f^{-1}(\sigma)))$. Further if f is one to one and onto, $\theta_1^-(f^{-1}(\sigma)) = f^{-1}(\theta_2^-(\sigma))$.

Theorem 4.4. *Let f be an isomorphism from ring R_1 onto ring R_2 and let μ be a fuzzy subset of R_1. Then*

(1) *$\theta_1^-(\mu)$ is an anti-fuzzy subring of R_1 if and only if $\theta_2^-(f(\mu))$ is an anti-fuzzy subring of R_2.*

(2) $\theta_{1-}(\mu)$ *is an anti-fuzzy subring of* R_1 *if and only if* $\theta_{2-}(f(\mu))$ *is an anti-fuzzy subring of* R_2.

Proof. (1) By theorem 3.11, $\theta_1^-(\mu)$ is an anti-fuzzy subring of R_1 if and only if $(\theta_1^-(\mu))_t^s$ is, if it is non-empty, a subring of R_1 for every $t \in [0,1]$. Again by theorem 3.10, we have $(\theta_1^-(\mu))_t^s = \theta_{1-}(\mu_t^s)$. We know that $\theta_{1-}(\mu_t^s)$ is a subring of R_1 if and only if $f(\theta_{1-}(\mu_t^s))$ is a subring of R_2. Now by theorem 4.2, $f(\theta_{1-}(\mu_t^s)) = \theta_{2-}f(\mu_t^s)$. By remark 4.1, $f(\mu_t^s) = (f(\mu))_t^s$. From this and theorem 3.10, we have, $\theta_{2-}f(\mu_t^s) = \theta_{2-}(f(\mu)_t^s) = (\theta_2^- f(\mu))_t^s$. By theorem 3.11, we obtain $(\theta_2^- f(\mu))_t^s$ is a subring of R_2 for every $t \in [0,1]$ if and only if $\theta_2^-(f(\mu))$is an anti-fuzzy subring of R_2.
The proof of (2) is similar to the proof of (1). □

Theorem 4.5. *Let* f *be a homomorphism from a ring* R_1 *onto a ring* R_2 *and let* σ *be an upper rough anti-fuzzy subring of* R_2. *Then* $f^{-1}(\sigma)$ *is an upper rough anti-fuzzy subring of* R_1.

Proof. Let σ be an upper rough anti-fuzzy subring of R_2. Then $\theta_2^-(\sigma)$ is an anti-fuzzy subring of R_2. For $x, y \in R_1$,

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(x-y) &= \theta_2^-(\sigma)f(x-y) \\ &= \theta_2^-(\sigma)(f(x)-f(y)) \quad (\because \text{ f is a homomorphism}) \\ &\leq \theta_2^-(\sigma)f(x) \vee \theta_2^-(\sigma)f(y) \\ &\qquad (\because \theta_2^-(\sigma) \text{ is an anti-fuzzy subring}) \\ &= f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y) \end{aligned}$$

Therefore, $f^{-1}(\theta_2^-(\sigma))(x-y) \leq f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y)$. Also

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(xy) &= \theta_2^-(\sigma)f(xy) \\ &= \theta_2^-(\sigma)(f(x)f(y)) \quad (\because \text{ f is a homomorphism}) \\ &\leq \theta_2^-(\sigma)f(x) \vee \theta_2^-(\sigma)f(y) \\ &\qquad (\because \theta_2^-(\sigma) \text{ is an anti-fuzzy subring}) \\ &= f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y) \end{aligned}$$

Therefore $f^{-1}(\theta_2^-(\sigma))(xy) \leq f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y)$. Thus $f^{-1}(\theta_2^-(\sigma))$ is an anti-fuzzy subring of R_1. By Remark 4.2, $f^{-1}(\theta_2^-(\sigma)) = \theta_1^-(f^{-1}(\sigma))$ is an anti-fuzzy subring of R_1. Therefore, $f^{-1}(\sigma)$ is an upper rough anti-fuzzy subring of R_1. Hence the theorem is proved. □

Theorem 4.6. *Let* f *be an isomorphism from a ring* R_1 *onto a ring* R_2 *and let* σ *be a lower rough anti-fuzzy subring of* R_2. *Then* $f^{-1}(\sigma)$ *is a lower rough anti-fuzzy subring of* R_1.

Proof. The proof is similar to that of theorem 4.5. □

Corollary 4.7. *Isomorphic pre-image of a rough anti-fuzzy subring is a rough anti-fuzzy subring.*

Proof. This follows from Theorems 4.5 and 4.6. □

Theorem 4.8. *Let f be a homomorphism from a ring R_1 onto a ring R_2 and let μ be an upper rough f-invariant anti-fuzzy subring of R_1 with sup property. Then $f(\mu)$ is an upper rough anti-fuzzy subring of R_2.*

Proof. Let μ be an upper rough anti-fuzzy subring of R_1. Then $\theta_1^-(\mu)$ is an anti-fuzzy subring of R_1. Let $x_0 \in f^{-1}[f(x)]$ and $y_0 \in f^{-1}[f(y)]$ be such that

$$\theta_1^-(\mu)(x_0) = \sup_{t\in f^{-1}[f(x)]} \theta_1^-(\mu)(t), \quad \theta_1^-(\mu)(y_0) = \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t)$$

For $f(x), f(y) \in R_2$,

$$\begin{aligned}
f(\theta_1^-(\mu))(f(x)-f(y)) &= f(\theta_1^-(\mu))(f(x-y)) \qquad (\because \text{ f is a homomorphism})\\
&= \sup_{t\in f^{-1}[f(x-y)]} \theta_1^-(\mu)(t)\\
&= \theta_1^-(\mu)(x_0-y_0)\\
&\leq \theta_1^-(\mu)(x_0) \;\vee\; \theta_1^-(\mu)(y_0)\\
&\qquad\qquad (\because \theta_1^-(\mu) \text{ is an anti-fuzzy subring})\\
&= \sup_{t\in f^{-1}[f(x)]} \theta_1^-(\mu)(t) \;\vee\; \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t)\\
&= f(\theta_1^-(\mu))f(x) \;\vee\; f(\theta_1^-(\mu))f(y)
\end{aligned}$$

Therefore, $f(\theta_1^-(\mu))(f(x)-f(y)) \leq f(\theta_1^-(\mu))f(x) \;\vee\; f(\theta_1^-(\mu))f(y)$. Also

$$\begin{aligned}
f(\theta_1^-(\mu))(f(x)f(y)) &= f(\theta_1^-(\mu))(f(xy)) \qquad (\because \text{ f is a homomorphism})\\
&= \sup_{t\in f^{-1}[f(xy)]} \theta_1^-(\mu)(t)\\
&= \theta_1^-(\mu)(x_0y_0)\\
&\leq \theta_1^-(\mu)(x_0) \;\vee\; \theta_1^-(\mu)(y_0)\\
&\qquad\qquad (\because \theta_1^-(\mu) \text{ is an anti-fuzzy subring})\\
&= \sup_{t\in f^{-1}[f(x)]} \theta_1^-(\mu)(t) \;\vee\; \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t)\\
&= f(\theta_1^-(\mu))f(x) \;\vee\; f(\theta_1^-(\mu))f(y)
\end{aligned}$$

Hence, $f(\theta_1^-(\mu))(f(x)f(y)) \leq f(\theta_1^-(\mu))f(x) \;\vee\; f(\theta_1^-(\mu))f(y)$.
Therefore, $f(\theta_1^-(\mu))$ is an anti-fuzzy subring of R_2. By theorem 4.3, $f(\theta_1^-(\mu))= \theta_2^-(f(\mu))$ is an anti-fuzzy subring of R_2. Hence $f(\mu)$ is an upper rough anti-fuzzy subring of R_2. This proves the theorem. □

Theorem 4.9. *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be a lower rough f-invariant anti-fuzzy subring of R_1 with sup property. Then $f(\mu)$ is a lower rough anti-fuzzy subring of R_2.*

Proof. Let μ be a lower rough anti-fuzzy subring of R_1. Then $\theta_{1-}(\mu)$ is an anti-fuzzy subring of R_1. For $f(x), f(y) \in R_2$,

$$\begin{aligned}
f(\theta_{1-}(\mu))(f(x)-f(y)) &= f(\theta_{1-}(\mu))(f(x-y)) \quad (\because \text{ f is a homomorphism}) \\
&= \sup_{t\in f^{-1}[f(x-y)]} \theta_{1-}(\mu)(t) \\
&= \theta_{1-}(\mu)(x-y) \\
&\leq \theta_{1-}(\mu)(x) \ \vee \ \theta_{1-}(\mu)(y) \\
&\qquad (\because \theta_{1-}(\mu) \text{ is an anti-fuzzy subring}) \\
&= \sup_{t\in f^{-1}[f(x)]} \theta_{1-}(\mu)(t) \ \vee \sup_{t\in f^{-1}[f(y)]} \theta_{1-}(\mu)(t) \\
&= f(\theta_{1-}(\mu))f(x) \ \vee \ f(\theta_{1-}(\mu))f(y)
\end{aligned}$$

Therefore, $f(\theta_{1-}(\mu))(f(x)-f(y)) \leq f(\theta_{1-}(\mu))f(x) \ \vee \ f(\theta_{1-}(\mu))f(y)$. Also

$$\begin{aligned}
f(\theta_{1-}(\mu))(f(x)f(y)) &= f(\theta_{1-}(\mu))(f(xy)) \quad (\because \text{ f is a homomorphism}) \\
&= \sup_{t\in f^{-1}[f(xy)]} \theta_{1-}(\mu)(t) \\
&= \theta_{1-}(\mu)(xy) \\
&\leq \theta_{1-}(\mu)(x) \ \vee \ \theta_{1-}(\mu)(y) \\
&\qquad (\because \theta_{1-}(\mu) \text{ is an anti-fuzzy subring}) \\
&= \sup_{t\in f^{-1}[f(x)]} \theta_{1-}(\mu)(t) \ \vee \sup_{t\in f^{-1}[f(y)]} \theta_{1-}(\mu)(t) \\
&= f(\theta_{1-}(\mu))f(x) \ \vee \ f(\theta_{1-}(\mu))f(y)
\end{aligned}$$

Hence, $f(\theta_{1-}(\mu))(f(x)f(y)) \leq f(\theta_{1-}(\mu))f(x) \ \vee \ f(\theta_{1-}(\mu))f(y)$.
Therefore, $f(\theta_{1-}(\mu))$ is an anti-fuzzy subring of R_2. By theorem 4.3, $f(\theta_{1-}(\mu))=\theta_{2-}(f(\mu))$ is an anti-fuzzy subring of R_2. Hence $f(\mu)$ is a lower rough anti-fuzzy subring of R_2. This proves the theorem. □

Corollary 4.10. *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be a rough f-invariant anti-fuzzy subring of R_1 with sup property. Then $f(\mu)$ is a rough anti-fuzzy subring of R_2.*

Proof. This follows from theorems 4.8 and 4.9. □

5. Anti-homomorphism on Rough Anti-Fuzzy Subring

Definition 5.1. Let R and R' be any two rings. Then the function $f : R \to R'$ is said to be an anti homomorphism if for all $x, y \in R$

$$f(x+y) = f(x) + f(y) \quad \text{and} \quad f(xy) = f(y)f(x).$$

Theorem 5.2 ([3])**.** *Let f be an anti-homomorphism from a ring R_1 onto a ring R_2 and let μ be a fuzzy subset of R_1 . Then*

(1) $f(\theta_1^-(\mu)) = \theta_2^-(f(\mu))$
(2) $f(\theta_{1-}(\mu)) \subseteq \theta_{2-}(f(\mu))$. *If f is one to one $f(\theta_{1-}(\mu)) = \theta_{2-}(f(\mu))$.*

The following theorems in anti-homomorphisms can be proved in similar way as the corresponding theorems in homomorphism.

Theorem 5.3. *Anti homomorphic image of an upper rough f-invariant anti-fuzzy subring with sup property is an upper rough anti-fuzzy subring.*

Theorem 5.4. *Anti isomorphic image of a lower rough f-invariant anti-fuzzy subring with sup property is a lower rough anti-fuzzy subring.*

Corollary 5.5. *Anti isomorphic image of a rough f-invariant anti-fuzzy subring with sup property is a rough anti-fuzzy subring.*

Theorem 5.6. *Anti homomorphic pre-image of an upper rough anti-fuzzy subring is an upper rough anti-fuzzy subring.*

Theorem 5.7. *Anti isomorphic pre-image of a lower rough anti-fuzzy subring is a lower rough anti-fuzzy subring.*

Theorem 5.8. *Anti isomorphic pre-image of a rough anti-fuzzy subring is a rough anti-fuzzy subring.*

6. Conclusion

In this paper, we have shown that the theory of rough sets can be extended to rings. We discussed the concept of rough anti-fuzzy subring. Also, we discussed homomorphic and anti-homomorphic properties of rough anti-fuzzy subrings. In a similar fashion the theory of rough sets can be extended to other topics in ring theory.

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