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EXISTENCE OF SOLUTION FOR IMPULSIVE FRACTIONAL DYNAMIC EQUATIONS WITH DELAY ON TIME SCALES†

ZHI-JUAN GAO, XU-YANG FU, QIAO-LUAN LI*

Abstract. This paper is mainly concerned with the existence of solution for nonlinear impulsive fractional dynamic equations on a special time scale. We introduce the new concept and propositions of fractional q-integral, q-derivative, and α-Lipschitz in the paper. By using a new fixed point theorem, we obtain some new existence results of solutions via some generalized singular Gronwall inequalities on time scales. Further, an interesting example is presented to illustrate the theory.

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1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988, in order to unify continuous and discrete analysis. A time scale is an arbitrary nonempty closed subset of the real numbers. In recent years, there has been much research activity concerning some different equations on time scales. We refer the reader to the paper [3].

In the last few decades, fractional differential equations have gained considerable importance and attention due to their applications in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex mediums. See the monographs of Kilbas, Miller and Ross [10], Podlubny and the papers of Daftardar-Gejji and Jafari [6], Diethelm [5], Lakshmikantham.

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The concept of fractional q-calculus is not new. Recently and after the appearance of time scale calculus(see for example [4]), some authors started to pay attention and apply the techniques of time scale to discrete fractional calculus [1,2] benefitting from the results announced before in [7].

In paper [11], JinRong Wang discussed the impulsive fractional differential equations with order $q \in (1, 2)$ as follows:

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), \;\; t \in J' := J \backslash \{t_1, \ldots, t_m\}, \;\; J := [0, T], \\ \Delta u(t_k) = y_k, \;\; \Delta u'(t_k) = \overline{y}_k, \qquad k = 1, 2, 3, ..., m, \\ u(0) = u_0, \;\; u'(0) = \overline{u}_0, \quad y_k, \; \overline{y}_k \in \mathbb{R}. \end{cases} \tag{1.1}$$

A unique solution u of (1.1) is given by

$$u(t) = \begin{cases} u_0 + \overline{u}_0 t + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, u(s)) ds, \qquad \text{for} \quad t \in [0, t_1), \\ \quad \vdots \\ u_0 + \overline{u}_0 t + \displaystyle\sum_{i=1}^k y_i + \sum_{i=1}^k \overline{y}_i (t - t_i) \\ \quad + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, u(s)) ds, \qquad \text{for} \quad t \in (t_k, t_{k+1}]. \end{cases} \tag{1.2}$$

Motivated by the above result, we reconsider the existence of solution for impulsive fractional dynamic equations with delay on time scales

$$\begin{cases} \nabla_q^\nu u(t) = f(t, u(\alpha(t)), u(\beta(t))), \;\; t \in \mathbb{T}_a' := \mathbb{T}_a \backslash \{t_1, \ldots, t_m\}, \\ \Delta\Big(u(t_k) - \overline{G}_1(t_k)\Big) = I_k(u(t_k)), \\ \Delta\Big(\nabla_q u(t_k) - \overline{G}_2(t_k)\Big) = J_k(u(t_k)), \qquad k = 1, 2, 3, ..., m, \\ u(0) = u_0, \;\; \nabla_q u(0) = \overline{u}_0, \end{cases} \tag{1.3}$$

where $a \in \mathbb{R}^+$, $\nu \in (1, 2)$, $q \in (0, 1)$, $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $\mathbb{T}_a = \{t : t = aq^n, \, n \in N_0\} \cup \{0\}$, $N_0 = \{0, 1, 2, \cdots\}$, $I_k, J_k : \mathbb{R} \to \mathbb{R}$, t_k satisfies $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$, and $\alpha(t), \beta(t) \leq t$, u_0, $\overline{u}_0$ are fixed real numbers. For $t \in \mathbb{T}_a$, we define the forward jump operator $\sigma : \mathbb{T}_a \to \mathbb{T}_a$ by $\sigma(t) := \inf\{s \in \mathbb{T}_a \, : \, s > t\}$. For any function υ, we define

$$\begin{aligned} &\Delta \upsilon(t_k) = \upsilon(\sigma(t_k)) - \upsilon(t_k), \\ &\overline{G}_1(t) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-1)}} f(s, u(\alpha(s)), u(\beta(s))) \nabla s, \\ &\overline{G}_2(t) = \frac{1 - q^{\nu-1}}{(1-q)\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-2)}} f(s, u(\alpha(s)), u(\beta(s))) \nabla s. \end{aligned} \tag{1.4}$$

The rest of this paper is organized as follows. In Section 2, we give some notations, recall some concepts and preparation results. In Section 3, we give four main results. At last, we give an example to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminaries. Throughout this paper, let $C(\mathbb{T}_a, \mathbb{R})$ be the Banach space of all continuous functions from $\mathbb{T}_a$ to $\mathbb{R}$ with the norm $\|u\|_C := \sup\{|u(t)| : t \in \mathbb{T}_a\}$ for $u \in C(\mathbb{T}_a, \mathbb{R})$. We also introduce the Banach space $PC(\mathbb{T}_a, \mathbb{R}) = \{u : \mathbb{T}_a \to \mathbb{R} : u \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, 2, \ldots, m$ with the norm $\|u\|_{PC} := \sup\{|u(t)| : t \in \mathbb{T}_a\}\}$.

Let us recall the following known definitions. For more details see [2,13].

Definition 2.1. For a function $f : \mathbb{T} \to \mathbb{R}$, the nabla q-derivative of f is

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t},$$

for all $t \in \mathbb{T}\backslash\{0\},\ q \in (0, 1)$.
The q-factorial function is defined in the following way.

Definition 2.2. If n is a positive integer, $t, s \in \mathbb{T}\backslash\{0\},\ q \in (0, 1)$, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)(t-q^2 s)\cdots(t-q^{n-1}s).$$

If ν is not a positive integer, then

$$(t-s)^{\underline{(\nu)}} = t^{\nu} \prod_{n=0}^{\infty} \frac{1 - \frac{s}{t}q^n}{1 - \frac{s}{t}q^{\nu+n}}.$$

We state several properties of the q-factorial function.

Proposition 2.3. *(i)* $(t-s)^{\underline{(\beta+\gamma)}} = (t-s)^{\underline{(\beta)}}(t-q^{\beta}s)^{\underline{(\gamma)}}$,
(ii) $(at-as)^{\underline{(\beta)}} = a^{\beta}(t-s)^{\underline{(\beta)}}$,
(iii) The nabla q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(\nu)}} = \frac{1-q^{\nu}}{1-q}(t-s)^{\underline{(\nu-1)}},$$

(iv) The nabla q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(\nu)}} = -\frac{1-q^{\nu}}{1-q}(t-qs)^{\underline{(\nu-1)}},$$

where $\beta, \gamma \in \mathbb{R}$.

Definition 2.4. The q-Gamma function is defined by

$$\Gamma_q(\alpha) = \frac{1}{1-q}\int_0^1 (\frac{t}{1-q})^{\alpha-1} e_q(qt)\nabla t,$$

where $\alpha \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$, $q \in (0,1)$, $\quad e_q(t) = \prod_{n=0}^{\infty}(1-q^n t)$, $e_q(0) = 1$.
The q-Beta function is defined by

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)^{\underline{(s-1)}}\nabla x.$$

Proposition 2.5. *(i)* $\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q}\Gamma_q(\alpha)$, $\quad \Gamma_q(1) = 1$, *where* $\alpha \in \mathbb{R}^+$.

$$(ii) \quad B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

Definition 2.6. The fractional q-integral is defined by

$$\nabla_q^{-\nu}(f(t)) = \frac{1}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}} f(s)\nabla s,$$

where $q \in (0,1)$.

Definition 2.7. If X is a Banach space and $\mathcal{B}$ is a family of all its bounded sets, the function $\alpha \;:\mathcal{B} \to \mathbb{R}^+$ defined by $\alpha(B) = \inf\{d > 0 \;:\; B$ admits a finite cover by sets of diameter $\leq d\}$, $B \in \mathcal{B}$, is called the Kuratowski measure of noncompactness.

Consider $\Omega \subset X$ and $F \;:\Omega \to X$ is a continuous bounded map. We say that F is α-Lipschitz, if there exists $\kappa \geq 0$ such that $\alpha(F(B)) \leq \kappa\alpha(B)$ for all $B \subset \Omega$ bounded. If, in addition, $\kappa < 1$, then we say that F is strict α-contraction.

We say that F is α-condensing if $\alpha(F(B)) < \alpha(B)$ for all $B \subset \Omega$ bounded with $\alpha(B) > 0$. In other words, $\alpha(F(B)) \geq \alpha(B)$ implies $\alpha(B) = 0$.

The class of all strict α-contractions $F \;:\; \Omega \to X$ is denoted by $C^1_\alpha(\Omega)$ and the class of all α-condensing maps $F \;:\; \Omega \to X$ is denoted by $C^2_\alpha(\Omega)$. We remark that $C^1_\alpha(\Omega) \subset C^2_\alpha(\Omega)$ and every $F \in C^2_\alpha(\Omega)$ is α-Lipschitz with constant $\kappa = 1$.

Proposition 2.8. *If F, $G \;:\Omega \to X$ are α-Lipschitz maps with constants κ, respectively κ', then $F+G \;:\Omega \to X$ are α-Lipschitz with constant $\kappa+\kappa'$.*

Proposition 2.9. *If $F \;:\Omega \to X$ is compact, then F is α-Lipschitz with constant $\kappa = 0$.*

Proposition 2.10. *If $F \;:\Omega \to X$ is Lipschitz with constant κ, then F is α-Lipschitz with the same constant κ.*

Theorem 2.11 (PC-type Ascoli-Arzela theorem, Theorem 2.1 of [12])**.** *Let X be a Banach space and $\mathcal{W} \subset PC(J,X)$. If the following conditions are satisfied:*
(i) $\mathcal{W}$ is uniformly bounded subset of $PC(J,X)$;
(ii) $\mathcal{W}$ is equicontinuous in (t_k, t_{k+1}), $k = 0,1,\cdots,m$, where $t_0 = 0, t_{m+1} = T$;
(iii) $\mathcal{W}(t) = \{u(t)|u \in \mathcal{W}, t \in J\backslash\{t_1,\cdots,t_m\}\}$, $\mathcal{W}(t_k^+) = \{u(t_k^+)|u \in \mathcal{W}\}$ and $\mathcal{W}(t_k^-) = \{u(t_k^-)|u \in \mathcal{W}\}$ is a relatively compact subsets of X,
then $\mathcal{W}$ is a relatively compact subset of $PC(J,X)$.

Theorem 2.12 (Theorem 2, [8]). *Let $F : X \to X$ be α-condensing and*
$$S = \{x \in X \ : \text{exists } \lambda \in [0,1] \text{ such that } x = \lambda F x\}.$$
If S is a bounded set in X, so there exists $r > 0$ such that $S \subset B_r(0)$, then F has at least one fixed point and the set of the fixed points of F lies in $B_r(0)$.

For measurable functions $m : \mathbb{T}_a \to \mathbb{R}$, define the norm

$$\|m\|_{L^p(\mathbb{T}_a)} = \begin{cases} \left(\displaystyle\int_{\mathbb{T}_a} |m(t)|^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \displaystyle\inf_{\mu(\overline{\mathbb{T}}_a)=0} \{\sup |m(t)|\}, & p = \infty, \ t \in \mathbb{T}_a - \overline{\mathbb{T}}_a, \end{cases}$$

where $\mu(\overline{\mathbb{T}}_a)$ is the Lebesgue measure on $\overline{\mathbb{T}}_a$. Let $L^p(\mathbb{T}_a, \mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m : \mathbb{T}_a \to \mathbb{R}$ with $\|m\|_{L^p(\mathbb{T}_a)} < \infty$.

Theorem 2.13 (Hölder's inequality). *Assume that $1 \le p, \ q \le \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. For any $l \in L^p(\mathbb{T}_a, \mathbb{R})$ and $m \in L^q(\mathbb{T}_a, \mathbb{R})$, $lm \in L^1(\mathbb{T}_a, \mathbb{R})$ with $\|lm\|_{L^1(\mathbb{T}_a)} \le \|l\|_{L^p(\mathbb{T}_a)} \|m\|_{L^q(\mathbb{T}_a)}$.*

Lemma 2.14. *Suppose $u(t), b(t), g(t), f(t)$ are nonnegative on $\mathbb{T}_a$, $b(t)$ is nondecreasing and locally integrable on $\mathbb{T}_a$, $g(t), f(t)$ are nondecreasing and continuous on $\mathbb{T}_a$, $a \in \mathbb{R}^+$, and $h(t) := g(t) + f(t) \le M_0 < \frac{1}{a^\nu (1-q)\Gamma_q(\nu)}$ for any $\nu > 1$. If*

$$u(t) \le b(t) + g(t) \int_0^t (t - qs)^{\underline{(\nu-1)}} u(\alpha(s)) \nabla s + f(t) \int_0^t (t - qs)^{\underline{(\nu-1)}} u(\beta(s)) \nabla s$$

for any $\alpha, \beta : \mathbb{T}_a \to \mathbb{T}_a$ with $\alpha(t) \le t, \beta(t) \le t$, then

$$u(t) \le b(t) + \int_0^t \big[\sum_{n=1}^{\infty} \frac{(h(t)\Gamma_q(\nu))^n}{\Gamma_q(n\nu)} (t - qs)^{\underline{(n\nu-1)}} b(s) \big] \nabla s \quad \forall \ t \in \mathbb{T}_a.$$

Proof. Let

$$z(t) := b(t) + g(t) \int_0^t (t - qs)^{\underline{(\nu-1)}} u(\alpha(s)) \nabla s + f(t) \int_0^t (t - qs)^{\underline{(\nu-1)}} u(\beta(s)) \nabla s.$$

Since $\alpha(t) \le t, \ \beta(t) \le t$, we have

$$u(\alpha(t)) \le z(\alpha(t)) \le z(t) \quad \text{and} \quad u(\beta(t)) \le z(\beta(t)) \le z(t),$$

then

$$z(t) \le b(t) + h(t) \int_0^t (t - qs)^{\underline{(\nu-1)}} z(s) \nabla s,$$

where $h(t) := g(t) + f(t)$, $h(t)$ is a nondecreasing continuous function.
Let $D\phi(t) := h(t) \int_0^t (t - qs)^{\underline{(\nu-1)}} \phi(s) \nabla s$, $t \in \mathbb{T}_a$, for locally integrable functions ϕ. Then

$$z(t) \le b(t) + Dz(t),$$

implies

$$z(t) \leq \sum_{k=0}^{n-1} D^k b(t) + D^n z(t).$$

Let us prove that

$$D^n z(t) \leq \int_0^t \frac{(h(t)\Gamma_q(\nu))^n}{\Gamma_q(n\nu)} (t-qs)^{\underline{(n\nu-1)}} z(s)\nabla s, \tag{2.1}$$

and $D^n z(t) \to 0$ as $n \to +\infty$ for each $t \in \mathbb{T}_a$.

We know this relation (2.1) is true for $n = 1$. Assume that it is true for some $n = k$, that is

$$D^k z(t) \leq \int_0^t \frac{(h(t)\Gamma_q(\nu))^k}{\Gamma_q(k\nu)} (t-qs)^{\underline{(k\nu-1)}} z(s)\nabla s.$$

If $n = k+1$, then the induction hypothesis implies

$$D^{k+1} z(t) \leq h(t) \int_0^t (t-qs)^{\underline{(\nu-1)}} \Big[\int_0^s \frac{(h(s)\Gamma_q(\nu))^k}{\Gamma_q(k\nu)} (s-q\tau)^{\underline{(k\nu-1)}} z(\tau)\nabla\tau \Big] \nabla s,$$

since $h(t)$ is nondecreasing, it follows that

$$D^{k+1} z(t) \leq \frac{\Big(h(t)\Big)^{k+1} \Big(\Gamma_q(\nu)\Big)^k}{\Gamma_q(k\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} \Big[\int_0^s (s-q\tau)^{\underline{(k\nu-1)}} z(\tau)\nabla\tau \Big] \nabla s.$$

By interchanging the order of integration (see [9]), we have

$$D^{k+1} z(t) \quad \leq \quad \frac{\Big(h(t)\Gamma_q(\nu)\Big)^{k+1}}{\Gamma_q((k+1)\nu)} \int_0^t (t-q\tau)^{\underline{((k+1)\nu-1)}} z(\tau)\nabla\tau,$$

where the integral

$$\begin{aligned} &\int_{\sigma(\tau)}^t (t-qs)^{\underline{(\nu-1)}} (s-q\tau)^{\underline{(k\nu-1)}} \nabla s \\ &= \int_{\sigma(\tau)}^t (t-qs)^{\underline{(\nu-1)}} (s-q\tau)^{\underline{(k\nu)}-1} \nabla s \\ &\leq (t-q\tau)^{\underline{(\nu-1)}} (t-q\tau)^{\underline{(k\nu)}} \int_0^1 (1-qz)^{\underline{(\nu-1)}} z^{k\nu-1} \nabla z, \end{aligned} \tag{2.2}$$

since $(t-s)^{\underline{(\nu)}}$ is decreasing about s, we have $(t-q\tau)^{\underline{(k\nu)}} \leq (t-q^\nu\tau)^{\underline{(k\nu)}}, \quad \nu > 1$, and by Proposition 2.3 and Definition 2.4, we get

$$\begin{aligned} &(t-q\tau)^{\underline{(\nu-1)}} (t-q\tau)^{\underline{(k\nu)}} \int_0^1 (1-qz)^{\underline{(\nu-1)}} z^{k\nu-1} \nabla z \\ &\leq (t-q\tau)^{\underline{((k+1)\nu-1)}} B_q(k\nu, \nu). \end{aligned}$$

The relation (2.1) is proved. By (2.1) and $h(t) \le M_0$, we have

$$D^n z(t) \le \int_0^t \frac{\left(a^\nu \Gamma_q(\nu) M_0\right)^n}{\Gamma_q(n\nu)} z(s)\nabla s.$$

Since $M_0 < \frac{1}{a^\nu (1-q)\Gamma_q(\nu)}$ and by Proposition 2.5, we get

$$\int_0^t \frac{\left(a^\nu \Gamma_q(\nu) M_0\right)^n}{\Gamma_q(n\nu)} z(s)\nabla s \to 0,$$

as $n \to \infty$, for $t \in \mathbb{T}_a$. Then the Lemma 2.14 is proved. □

Lemma 2.15. *Let $\nu \in (1,2)$ and $h : \mathbb{T}_a \to \mathbb{R}$ be jointly continuous. A function u given by*

$$u(t) = \begin{cases} u_0 + \overline{u}_0 t + G_1(t), & \text{for } t \in [0, t_1], \\ u_0 + \overline{u}_0 t + G_1(t) + \sum_{i=1}^{k} J_i(u(t_i))(t - \sigma(t_i)) + \sum_{i=1}^{k} I_i(u(t_i)) & \\ \quad + (q-1)\sum_{i=1}^{k} \sigma(t_i)(\overline{u}_0 + \sum_{j=1}^{i} J_j(u(t_j))), & \text{for } t \in (t_k, t_{k+1}], \end{cases} \tag{2.3}$$

is the unique solution of the following impulsive problem on time scales

$$\begin{cases} \nabla_q^\nu u(t) = h(t), \ \ t \in \mathbb{T}_a' := \mathbb{T}_a \backslash \{t_1, \dots, t_m\}, \\ \Delta\Big(u(t_k) - G_1(t_k)\Big) = I_k(u(t_k)), \\ \Delta\Big(\nabla_q u(t_k) - G_2(t_k)\Big) = J_k(u(t_k)), \\ u(0) = u_0, \ \ \nabla_q u(0) = \overline{u}_0, \end{cases} \tag{2.4}$$

where $\mathbb{T}_a = \{t : t = aq^n,\ n \in N_0\} \cup \{0\}$, $a \in \mathbb{R}^+$, $q \in (0,1)$, $N_0 = \{0, 1, 2, \cdots\}$, $k = 1, 2, 3, \cdots, m$, $I_k, J_k : \mathbb{R} \to \mathbb{R}$, t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$. $\sigma : \mathbb{T}_a \to \mathbb{T}_a$ is the forward jump operator $\sigma(t) := \inf\{s \in \mathbb{T}_a : s > t\}$. And for any function υ, we define

$$\Delta\upsilon(t_k) = \upsilon(\sigma(t_k)) - \upsilon(t_k),$$

$$G_1(t) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-1)}} h(s)\nabla s,$$

$$G_2(t) = \frac{1 - q^{\nu-1}}{(1-q)\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-2)}} h(s)\nabla s.$$

Proof. Assume the general solution u of the Eq. (2.4) is given by

$$u(t) = G_1(t) + A_k + B_k t, \quad t \in (t_k, t_{k+1}], \ \ k = 0, 1, 2, \cdots, m, \tag{2.5}$$

where $t_0 = 0,\ t_{m+1} = a$. Then, we have

$$\nabla_q u(t) = G_2(t) + B_k, \quad t \in (t_k, t_{k+1}], \ \ k = 0, 1, 2, \cdots, m. \tag{2.6}$$

Applying the cauchy conditions of (2.4), we get

$$A_0 = u_0, \qquad B_0 = \overline{u}_0. \tag{2.7}$$

Next, using the impulsive condition of (2.4), we find that

$$B_k = B_{k-1} + J_k(u(t_k)), \tag{2.8}$$

which by (2.7) implies

$$B_k = \overline{u}_0 + \sum_{i=1}^{k} J_i(u(t_i)), \quad k = 1, 2, \cdots, m. \tag{2.9}$$

Furthermore, using the impulsive condition of (2.4), we find that

$$A_k = A_{k-1} + \sigma(t_k)(q-1)B_{k-1} - \sigma(t_k)J_k(u(t_k)) + I_k(u(t_k)), \tag{2.10}$$

which implies

$$A_k = u_0 + \sum_{i=1}^{k} \Big(\sigma(t_i)(q-1)B_i - \sigma(t_i)J_i(u(t_i)) + I_i(u(t_i))\Big). \tag{2.11}$$

So by (2.9), (2.11), we have

$$\begin{aligned} A_k + B_k t = {} & u_0 + \overline{u}_0 t + \sum_{i=1}^{k}(t - \sigma(t_i))J_i(u(t_i)) + \sum_{i=1}^{k} I_i(u(t_i)) \\ & + \sum_{i=1}^{k} \sigma(t_i)(q-1)(\overline{u}_0 + \sum_{j=1}^{i} J_j(u(t_j))). \end{aligned}$$

Thus, we can get (2.3).

Conversely, assume that u satisfies (2.3). By a direct computation, it follows that the solution given by (2.3) satisfies (2.4). This completes the proof. □

3. Main results

In this section, we deal with the existence and uniqueness of solutions for the problem (1.3).

Before stating and proving the main results, we introduce the following hypotheses:

[H1] $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is jointly continuous.

[H2] For arbitrary $(t, u, v) \in \mathbb{T}_a \times \mathbb{R} \times \mathbb{R}$, there exist $L_1, L_2 > 0$, such that

$$|f(t, u, v)| \leq L_1|u| + L_2|v|.$$

[H3] There exist $q_1, q_2 \in (0, 1)$, real functions $h \in L^{\frac{1}{q_1}}(\mathbb{T}_a, \mathbb{R})$, $y \in L^{\frac{1}{q_2}}(\mathbb{T}_a, \mathbb{R})$ such that

$$\begin{aligned} & |f(t, u_1(\alpha(t)), u_1(\beta(t))) - f(t, u_2(\alpha(t)), u_2(\beta(t)))| \\ \leq & h(t)|u_1(\alpha(t)) - u_2(\alpha(t))| + y(t)|u_1(\beta(t)) - u_2(\beta(t))|, \end{aligned}$$

for all $t \in \mathbb{T}_a$ and $u_1(\cdot), u_2(\cdot) \in \mathbb{R}$.

[H4] There exist constants $L_3, L_4 > 0$, such that

$$\|I_k(u) - I_k(v)\| \leq L_3\|u - v\|, \quad \|J_k(u) - J_k(v)\| \leq L_4\|u - v\|,$$

for all $u, v \in \mathbb{R},$ and $k = 1, 2, \cdots, m.$
[H5] For arbitrary $u \in \mathbb{R},$ there exist constants $M_1, M_2 > 0$, such that

$$\|I_k(u)\| \leq M_1, \qquad \|J_k(u)\| \leq M_2, \quad k = 1, 2, \cdots, m.$$

Theorem 3.1. *Assume that [H1]-[H5] hold, and if* $L_1 + L_2 < \frac{1}{a^\nu(1-q)}$, *then the problem (1.3) has at least one solution on* $\mathbb{T}_a$.

Proof. By Lemma 2.15, we define an operator $\mathcal{T} : PC(\mathbb{T}_a, \mathbb{R}) \to PC(\mathbb{T}_a, \mathbb{R})$ by

$$\begin{aligned} Tu)(t) = u_0 + \overline{u}_0 t + \overline{G}_1(u,t) + \sum_{i=1}^{k} J_i(u(t_i))(t - \sigma(t_i)) + \sum_{i=1}^{k} I_i(u(t_i)) \\ + (q-1)\sum_{i=1}^{k} \sigma(t_i)(\overline{u}_0 + \sum_{j=1}^{i} J_j(u(t_j))), \end{aligned} \tag{3.1}$$

for $t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \cdots, m,$ where

$$\overline{G}_1(u,t) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-1)}} f(s, u(\alpha(s)), u(\beta(s)))\nabla s.$$

For the sake of convenience, we subdivide the proof into several steps.
Step 1. $\mathcal{T}$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(\mathbb{T}_a, \mathbb{R})$. Then for each $t \in (t_k,\ t_{k+1}]$, by conditions [H3], [H4], we have

$$\begin{aligned} &|(\mathcal{T}u_n)(t) - (\mathcal{T}u)(t)| \\ \leq &\frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-1)}} h(s)|u_n(\alpha(s)) - u(\alpha(s))|\nabla s \\ &+ \frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{\underline{(\nu-1)}} y(s)|u_n(\beta(s)) - u(\beta(s))|\nabla s \\ &+ L_4 \sum_{i=1}^{k} (t - \sigma(t_i))\|u_n(t_i) - u(t_i)\| + L_3 \sum_{i=1}^{k} \|u_n(t_i) - u(t_i)\| \\ &+ L_4(1-q) \sum_{i=1}^{k} \sigma(t_i)(\sum_{j=1}^{i} \|u_n(t_j) - u(t_j)\|). \end{aligned}$$

Further, we can obtain

$$\|(\mathcal{T}u_n)(t) - (\mathcal{T}u)(t)\|_{PC} \to 0, \quad n \to \infty.$$

Step 2. $\mathcal{T}$ maps bounded sets into bounded sets in $PC(\mathbb{T}_a, \mathbb{R})$.

For each $u \in B_\eta = \{u \in PC(\mathbb{T}_a, \mathbb{R}) \ : \ \|u\|_{PC} \leq \eta\},\ t \in (t_k,\, t_{k+1}]$, by [H2], [H5], we have

$$|(\mathcal{T}u)(t)| \leq |u_0| + a|\overline{u}_0| + (1-q)\sum_{i=1}^{k}\sigma(t_i)(|\overline{u}_0| + iM_2)$$
$$+ \frac{\eta(L_1+L_2)(1-q)a^\nu}{\Gamma_q(\nu)(1-q^\nu)} + M_2\sum_{i=1}^{k}(a-\sigma(t_i)) + mM_1.$$

Let $\ell := |u_0| + a|\overline{u}_0| + M_2\sum_{i=1}^{k}(a-\sigma(t_i)) + mM_1 + (1-q)\sum_{i=1}^{k}\sigma(t_i)(|\overline{u}_0| + iM_2) + \frac{\eta(L_1+L_2)(1-q)a^\nu}{\Gamma_q(\nu)(1-q^\nu)}$, we get

$$\|(\mathcal{T}u)(t)\|_{PC} \leq \ell.$$

Step 3. $\mathcal{T}$ maps bounded sets into equicontinuous sets of $PC(\mathbb{T}_a, \mathbb{R})$.
It is easy to know $\mathcal{T}$ is equicontinuous on interval $(t_k, t_{k+1}],\ k = 1, 2, \cdots, m$.
For any $0 \leq s_1 < s_2 \leq t_1$, $u \in B_\eta = \{u \in PC(\mathbb{T}_a, \mathbb{R}) \ : \ \|u\|_{PC} \leq \eta\}$, we have

$$|(\mathcal{T}u)(s_2) - (\mathcal{T}u)(s_1)|$$
$$\leq |\overline{u}_0||s_2 - s_1| + \frac{1}{\Gamma_q(\nu)}\left|\int_{s_1}^{s_2}(s_2 - qs)^{\underline{(\nu-1)}}f(s, u(\alpha(s)), u(\beta(s)))\nabla s\right|$$
$$+ \frac{1}{\Gamma_q(\nu)}\left|\int_0^{s_1}\left((s_2 - qs)^{\underline{(\nu-1)}} - (s_1 - qs)^{\underline{(\nu-1)}}\right)f(s, u(\alpha(s)), u(\beta(s)))\nabla s\right|,$$

considering $s_2 \to s_1$, we have

$$|(\mathcal{T}u)(s_2) - (\mathcal{T}u)(s_1)| \to 0.$$

Thus, we find that $\mathcal{T}$ is equicontinuous on $\mathbb{T}_a$.
Step 4. Now it remains to show that the set

$$E(\mathcal{T}) := \{u \in PC(\mathbb{T}_a, \mathbb{R}) \ : u = \lambda\mathcal{T}u, \quad \lambda \in (0,1)\}$$

is bounded.

Without loss of generality, for any $u \in E(\mathcal{T}), \quad t \in (t_k, t_{k+1}]$, by [H2], [H5], we have

$$|u(t)| \leq |u_0| + a|\overline{u}_0| + \frac{L_1}{\Gamma_q(\nu)}\int_0^t (t - qs)^{\underline{(\nu-1)}}|u(\alpha(s))|\nabla s$$
$$+ M_2\sum_{i=1}^{k}(a - \sigma(t_i)) + (1-q)\sum_{i=1}^{k}\sigma(t_i)(|\overline{u}_0| + iM_2)$$
$$+ mM_1 + \frac{L_2}{\Gamma_q(\nu)}\int_0^t (t - qs)^{\underline{(\nu-1)}}|u(\beta(s))|\nabla s,$$

where $k = 0, 1, 2, \cdots, m$.
By Lemma 2.14, there exists a $M_k > 0$ such that

$$|u(t)| \leq M_k, \qquad t \in (t_k, t_{k+1}].$$

Set $M = \max_{0\le k\le m} M_k$, thus for every $t \in \mathbb{T}_a$, we get

$$\|u(t)\|_{PC} \le M, \qquad t \in (t_k, t_{k+1}].$$

This shows that the set $E(\mathcal{T})$ is bounded.

As a consequence of Schaefer's fixed point theorem, we know that $\mathcal{T}$ has a fixed point which is a solution of the problem (1.3). The proof is complete. □

Theorem 3.2. *Assume that [H1],[H3] and [H4] hold, and if*

$$\begin{aligned} 1 > L_4 \sum_{i=1}^{k}(a-\sigma(t_i)) + mL_3 + (1-q)L_4\sum_{i=1}^{k}\sigma(t_i)i \\ + \frac{\|h\|_{L^{\frac{1}{q_1}}(\mathbb{T}_a)}}{\Gamma_q(\nu)}\left(\int_0^a \left[(a-qs)^{\underline{(\nu-1)}}\right]^{\frac{1}{1-q_1}}\nabla s\right)^{1-q_1} \\ + \frac{\|y\|_{L^{\frac{1}{q_2}}(\mathbb{T}_a)}}{\Gamma_q(\nu)}\left(\int_0^a \left[(a-qs)^{\underline{(\nu-1)}}\right]^{\frac{1}{1-q_2}}\nabla s\right)^{1-q_2}, \end{aligned} \tag{3.2}$$

then the problem (1.3) has an unique solution on $\mathbb{T}_a$.

Proof. Consider the operator $\mathcal{T} : PC(\mathbb{T}_a,\mathbb{R}) \to PC(\mathbb{T}_a,\mathbb{R})$ defined as (3.1), and transform the problem (1.3) into a fixed point problem of $\mathcal{T}$.

Step 1. $\mathcal{T}u \in PC(\mathbb{T}_a,\mathbb{R})$ for every $u \in PC(\mathbb{T}_a,\mathbb{R})$.

If $t = 0$, for any $\delta > 0$, we have

$$\begin{aligned} |(\mathcal{T}u)(\delta) - (\mathcal{T}u)(0)| &= |\overline{u}_0\delta + \overline{G}_1(u,\delta) + \delta\cdot\sum_{i=1}^{k}J_i(u(t_i))| \\ &\le |\overline{u}_0|\delta + \frac{1}{\Gamma_q(\nu)}\int_0^\delta (\delta-qs)^{\underline{(\nu-1)}}|f(s,u(\alpha(s)),u(\beta(s)))|\nabla s \\ &\quad + \delta\cdot\sum_{i=1}^{k}|J_i(u(t_i))|, \end{aligned}$$

then

$$|(\mathcal{T}u)(\delta) - (\mathcal{T}u)(0)| \to 0, \qquad as\ \delta \to 0.$$

Thus, we find that $\mathcal{T}u$ is continuous at 0. It is easy to see that $\mathcal{T}u \in C((t_k,t_{k+1}],\mathbb{R})$, $k = 0,1,\cdots,m$.

From the above discussion, we get $\mathcal{T}u \in PC(\mathbb{T}_a,\mathbb{R})$ for every $u \in PC(\mathbb{T}_a,\mathbb{R})$.

Step 2. $\mathcal{T}$ is a contraction operator on $PC(\mathbb{T}_a,\mathbb{R})$.

In fact, for arbitrary $u_1,u_2 \in PC(\mathbb{T}_a,\mathbb{R})$, by [H3], [H4] and Theorem 2.13, we obtain

$$\begin{aligned} &|(\mathcal{T}u_1)(t) - (\mathcal{T}u_2)(t)| \\ &\le \left\{\frac{\|h\|_{L^{\frac{1}{q_1}}(\mathbb{T}_a)}}{\Gamma_q(\nu)}\left(\int_0^a \left[(a-qs)^{\underline{(\nu-1)}}\right]^{\frac{1}{1-q_1}}\nabla s\right)^{1-q_1}\right. \end{aligned}$$

$$+\frac{\|y\|_{L^{\frac{1}{q_2}}(\mathbb{T}_a)}}{\Gamma_q(\nu)}\left(\int_0^a\left[(a-qs)^{\underline{(\nu-1)}}\right]^{\frac{1}{1-q_2}}\nabla s\right)^{1-q_2}$$
$$+L_4\sum_{i=1}^{k}(a-\sigma(t_i))+mL_3+(1-q)L_4\sum_{i=1}^{k}\sigma(t_i)i\Bigg\}\|u_1-u_2\|_{PC}.$$

Thus, due to (3.2), we know that $\mathcal{T}$ is a contraction mapping on $PC(\mathbb{T}_a,\mathbb{R})$.

By applying the well-known Banach's contraction mapping principle, we get that the operator $\mathcal{T}$ has a unique fixed point on $PC(\mathbb{T}_a,\mathbb{R})$. Therefore, the problem (1.3) has a unique solution. □

Before proving the next results, we introduce the following hypotheses.

[H2]′ For arbitrary $(t,u,v)\in\mathbb{T}_a\times\mathbb{R}\times\mathbb{R}$, there exist $C_f^1, C_f^2, M_f>0$, and $q_1,q_2\in[0,1)$ such that

$$|f(t,u,v)|\le C_f^1|u|^{q_1}+C_f^2|v|^{q_2}+M_f.$$

[H3]′ There exist $C_f^3, C_f^4>0$, such that

$$\begin{aligned}&|f(t,u_1(\alpha(t)),u_1(\beta(t)))-f(t,u_2(\alpha(t)),u_2(\beta(t)))|\\ \le\ & C_f^3|u_1(\alpha(t))-u_2(\alpha(t))|+C_f^4|u_1(\beta(t))-u_2(\beta(t))|,\end{aligned}$$

for each $t\in\mathbb{T}_a$, and all $u_1,u_2\in\mathbb{R}$.

[H4]′ There exist constants $C_I^k, C_J^k>0$, such that

$$\|I_k(u)-I_k(v)\|\le C_I^k\|u-v\|,\quad \|J_k(u)-J_k(v)\|\le C_J^k\|u-v\|,$$

for all $u,v\in\mathbb{R},$ and $k=1,2,\cdots,m.$

[H5]′ For arbitrary $u\in\mathbb{R},$ there exist constants $C_I, C_J>0$ and $q_3,q_4\in[0,1)$ such that

$$|I_k(u)|\le C_I|u|^{q_3},\qquad |J_k(u)|\le C_J|u|^{q_4},\quad k=1,2,\cdots,m.$$

Theorem 3.3. *Assume that [H1] and [H2]′-[H5]′ hold, and if*

$$\sum_{k=1}^{m}N_k\in(0,1), \tag{3.3}$$

where $N_k:=\sum\limits_{i=1}^{k}(a-\sigma(t_i))C_J^i+\sum\limits_{i=1}^{k}C_I^i+(1-q)\sum\limits_{i=1}^{k}\sigma(t_i)(\sum\limits_{j=1}^{i}C_J^j),$
then the promble (1.3) has at least one solution $u\in PC(\mathbb{T}_a,\mathbb{R})$ *and the set of the solutions of the problem (1.3) is bounded in* $PC(\mathbb{T}_a,\mathbb{R})$.

Proof. Now, we define the operators as follows:
$\mathcal{H}\ :\ PC(\mathbb{T}_a,\mathbb{R})\to PC(\mathbb{T}_a,\mathbb{R})$ given by

$$\begin{aligned}(\mathcal{H}u)(t)\ :=\ & u_0+\overline{u}_0t+(q-1)\sum_{i=1}^{k}\sigma(t_i)(\overline{u}_0+\sum_{j=1}^{i}J_j(u(t_j)))+\sum_{i=1}^{k}I_i(u(t_i))\\ &+\sum_{i=1}^{k}J_i(u(t_i))(t-\sigma(t_i)),\quad t\in(t_k,t_{k+1}],\ k=0,1,2,\cdots,m.\end{aligned}$$

$\mathcal{G}\ :\ PC(\mathbb{T}_a,\mathbb{R}) \to PC(\mathbb{T}_a,\mathbb{R})$ given by

$$(\mathcal{G}u)(t)\ :=\ \overline{G}_1(u,t), \qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\cdots,m,$$

where $\overline{G}_1(u,t)$ is defined as in (3.1).
Let $\mathcal{F}\ :\ PC(\mathbb{T}_a,\mathbb{R}) \to PC(\mathbb{T}_a,\mathbb{R})$ given by

$$\mathcal{F}u=\mathcal{H}u+\mathcal{G}u.$$

Thus, the existence of a solution for the problem (1.3) is equivalent to the existence of a fixed point for operator $\mathcal{F}$.
Step 1. The operator $\mathcal{H}$ is Lipschitz with constant $\kappa_1=\sum_{k=1}^{m}N_k$, by Proposition 2.10, consequently $\mathcal{H}$ is α-Lipschitz with the same constant $\kappa_1=\sum_{k=1}^{m}N_k$. Moreover, the operator $\mathcal{H}$ satisfies the following growth condition:

$$\begin{aligned}\|\mathcal{H}u\|_{PC} \le |u_0|+|\overline{u}_0|a+C_J\|u\|^{q_4}\sum_{i=1}^{k}(a-\sigma(t_i))+mC_I\|u\|^{q_3}\\ +(1-q)\sum_{i=1}^{k}\sigma(t_i)(|\overline{u}_0|+iC_J\|u\|^{q_4}).\end{aligned} \tag{3.4}$$

For every $t\in[0,t_1],\ u,v\in PC(\mathbb{T}_a,\mathbb{R})$, it is obvious that

$$|(\mathcal{H}u)(t)-(\mathcal{H}v)(t)|=0.$$

If $t\in(t_k,t_{k+1}],\ k=1,2,\cdots,m,\ u,v\in PC(\mathbb{T}_a,\mathbb{R})$, by [H4]$'$, we have

$$\begin{aligned}&|(\mathcal{H}u)(t)-(\mathcal{H}v)(t)|\\ &\le\Big[\sum_{i=1}^{k}(a-\sigma(t_i))C_J^i+\sum_{i=1}^{k}C_I^i+(1-q)\sum_{i=1}^{k}\sigma(t_i)(\sum_{j=1}^{i}C_J^j)\Big]\|u-v\|_{PC}.\end{aligned}$$

Let $N_k:=\sum_{i=1}^{k}(a-\sigma(t_i))C_J^i+\sum_{i=1}^{k}C_I^i+(1-q)\sum_{i=1}^{k}\sigma(t_i)(\sum_{j=1}^{i}C_J^j)$.
For every $u,v\in PC(\mathbb{T}_a,\mathbb{R}),\ t\in(t_k,t_{k+1}],\ k=1,2,3,\cdots,m$. Using [H4]$'$ step by step, and by Proposition 2.8 and Proposition 2.10, we know that $\mathcal{H}$ is α-Lipschitz with the constant $\kappa_1=\sum_{k=1}^{m}N_k$. And by [H5]$'$, we get (3.4).
Step 2. The operator $\mathcal{G}$ is compact, by Proposition 2.9, then $\mathcal{G}$ is α-Lipschitz with constant $\kappa_2=0$.
In order to prove the compactness of $\mathcal{G}$, we consider a bounded set $E\subseteq C((t_k,t_{k+1}),\mathbb{R})$,$k=0,1,2,\cdots,m$, and we will show that $\mathcal{G}(E)$ is relatively compact in $C((t_k,t_{k+1}])$ with the help of Theorem 2.11.
(i) For $t\in[0,t_1]$, let u_n be a sequence on $E\subseteq C([0,t_1],\mathbb{R})$, for every $u_n\in E$, by [H2]$'$, we have

$$\|\mathcal{G}u_n\|_{C([0,t_1],\mathbb{R})}\le(C_f^1\|u_n\|^{q_1}_{C([0,t_1],\mathbb{R})}+C_f^2\|u_n\|^{q_2}_{C([0,t_1],\mathbb{R})}+M_f)\frac{(1-q)a^{\nu}}{(1-q^{\nu})\Gamma_q(\nu)},$$

thus, the set $\mathcal{G}(E)$ is bounded in $C([0,t_1])$.
For each $(t_k, t_{k+1}], k = 1,2,3,\cdots,m$, repeating the above process again, one can obtain that the set $\mathcal{G}(E)$ is an uniformly bounded subset of $PC(\mathbb{T}_a,\mathbb{R})$.
(ii) For $t \in (t_k, t_{k+1}],\ k = 1,2,\cdots,m$, it is easy to know $\mathcal{G}u_n$ is equicontinuous. For any $0 \le s_1 < s_2 \le t_1$, $u_n \in E$, we have

$$\begin{aligned}&|(\mathcal{G}u_n)(s_2) - (\mathcal{G}u_n)(s_1)|\\ \le& \frac{1}{\Gamma_q(\nu)}\Big|\int_0^{s_1}\Big((s_2-qs)^{\underline{(\nu-1)}} - (s_1-qs)^{\underline{(\nu-1)}}\Big)f(s,u_n(\alpha(s)),u_n(\beta(s)))\nabla s\Big|\\ &+\frac{1}{\Gamma_q(\nu)}\Big|\int_{s_1}^{s_2}(s_2-qs)^{\underline{(\nu-1)}}f(s,u_n(\alpha(s)),u_n(\beta(s)))\nabla s\Big|,\end{aligned}$$

then

$$|(\mathcal{G}u_n)(s_2) - (\mathcal{G}u_n)(s_1)| \to 0, \qquad as\ s_2 \to s_1.$$

Thus, we find that $\mathcal{G}$ is equicontinuous on $\mathbb{T}_a$. From (i), (ii), we get the compactness of the operator $\mathcal{G}$ on $PC(\mathbb{T}_a,\mathbb{R})$.

By Proposition 2.9, we know that the operator $\mathcal{G}$ is α-Lipschitz with constant 0.

Step 3. The operator $\mathcal{G}$ is continuous. Moreover, by [H2]$'$, the operator $\mathcal{G}$ satisfies the following growth condition:

$$\|\mathcal{G}u\|_{PC} \le (C_f^1\|u\|_{PC}^{q_1} + C_f^2\|u\|_{PC}^{q_2} + M_f)\frac{(1-q)a^\nu}{(1-q^\nu)\Gamma_q(\nu)}. \tag{3.5}$$

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(\mathbb{T}_a,\mathbb{R})$. Then for each $t \in (t_k, t_{k+1}]$, by condition [H3]$'$, we have

$$\begin{aligned}&|(\mathcal{G}u_n)(t) - (\mathcal{G}u)(t)|\\ \le& \frac{C_f^3}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u_n(\alpha(s)) - u(\alpha(s))|\nabla s\\ &+\frac{C_f^4}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u_n(\beta(s)) - u(\beta(s))|\nabla s.\end{aligned}$$

Further, we can obtain

$$|(\mathcal{G}u_n)(t) - (\mathcal{G}u)(t)| \to 0, \quad n \to \infty.$$

We know that the operator $\mathcal{G}$ is continuous on $(t_k, t_{k+1}], k = 0,1,\cdots,m$.
Step 4. From Step 1 and Step 2, by Proposition 2.8 and condition (3.3), we obtain that the operator $\mathcal{F}$ is strict α-contraction with constant $\kappa = \sum\limits_{k=1}^{m} N_k$. Further, by Definition 2.7, we finally get that the operator $\mathcal{F}$ is a α-condensing map.
Step 5. Let $E(\mathcal{F}) := \{u \in PC(\mathbb{T}_a,\mathbb{R})\ :\ \exists\ \overline{\lambda} \in (0,1)\ \text{ such that }\ u = \overline{\lambda}\mathcal{F}u\}$. Consider every $u \in E(\mathcal{F})$, by (3.4),(3.5), we have

$$\begin{aligned} \|u\|_{PC} \leq \overline{\lambda}\Big(& C_J\|u\|_{PC}^{q_4}\sum_{i=1}^{k}(a-\sigma(t_i)) + (1-q)\sum_{i=1}^{k}\sigma(t_i)(|\overline{u}_0| + iC_J\|u\|_{PC}^{q_4}) \\ & + mC_I\|u\|_{PC}^{q_3} + (C_f^1\|u\|_{PC}^{q_1} + C_f^2\|u\|_{PC}^{q_2} + M_f)\frac{(1-q)a^\nu}{(1-q^\nu)\Gamma_q(\nu)} \\ & + |u_0| + |\overline{u}_0|a\Big) \\ := & \, m(\|u\|_{PC}). \end{aligned} \tag{3.6}$$

This inequality, together with $q_i \in [0,1),\ i = 1,2,3,4$, shows us that $E(\mathcal{F})$ is bounded in $PC(\mathbb{T}_a, \mathbb{R})$. If not, we suppose by contradiction $\xi := \|u\|_{PC} \to \infty$.

Dividing both sides of (3.6) by ξ, and taking $\xi \to \infty$, we get

$$1 \leq \lim_{\xi\to\infty}\frac{m(\xi)}{\xi} = 0,$$

this is a contraction.

From above, by Theorem 2.12, we deduce that the operator $\mathcal{F}$ has at least one fixed point and the set of the fixed points of $\mathcal{F}$ is bounded in $PC(\mathbb{T}_a, \mathbb{R})$. □

Theorem 3.4. *Assume that* $[H1]$, $[H2]'$-$[H5]'$ *and condition (3.3) hold, and if*

$$\frac{C_f^3 + C_f^4}{1 - N_k} < \frac{1}{a^\nu(1-q)},$$

then the problem (1.3) has a unique solution $u \in PC(\mathbb{T}_a, \mathbb{R})$.

Proof. By Theorem 3.3, the problem (1.3) has at least one solution. Now, let $u(.), v(.)$ be the solutions of problem (1.3) with the same initial values,

$$u(0) = v(0) = u_0, \quad \nabla_q u(0) = \nabla_q v(0) = \overline{u}_0,$$

by [H3]$'$, [H4]$'$, then

$$\begin{aligned} & |u(t) - v(t)| \\ \leq & N_k\|u-v\|_{PC} + \frac{C_f^3}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u(\alpha(s)) - v(\alpha(s))|\nabla s \\ & + \frac{C_f^4}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u(\beta(s)) - v(\beta(s))|\nabla s, \end{aligned} \tag{3.7}$$

then

$$\begin{aligned} \|u-v\|_{PC} \leq & N_k\|u-v\|_{PC} + \frac{C_f^3}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u(\alpha(s)) - v(\alpha(s))|\nabla s \\ & + \frac{C_f^4}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u(\beta(s)) - v(\beta(s))|\nabla s. \end{aligned} \tag{3.8}$$

From condition (3.3), (3.8), we obtain

$$\begin{aligned} &|u(t)-v(t)| \\ &\leq \frac{C_f^3}{(1-N_k)\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u(\alpha(s))-v(\alpha(s))|\nabla s \\ &\quad + \frac{C_f^4}{(1-N_k)\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}|u(\beta(s))-v(\beta(s))|\nabla s. \end{aligned} \tag{3.9}$$

Due to Lemma 2.14, we get

$$u(t)=v(t), \quad \text{for each } t\in\mathbb{T}_a, \text{ and all } u,v\in\mathbb{R}.$$

The proof is complete. □

Next, we give an example to illustrate the usefulness of our main results.

Example 3.5. Let us consider the following fractional impulsive problem with delay on time scales

$$\begin{cases} \nabla_{\frac{1}{3}}^{\frac{3}{2}} u(t) = \dfrac{u(\alpha(t))}{(1+e^t)(1+u(\alpha(t)))} + \dfrac{u(\beta(t))}{(1+e^{mt})(1+u(\beta(t)))}, \ t\in\mathbb{T}_1\backslash\{\frac{1}{9},\frac{1}{81}\}, \\ \Delta\Big(u(\frac{1}{9})-\overline{G}_1(\frac{1}{9})\Big) = \dfrac{1}{2(1+u(\frac{1}{9}))}, \ \Delta\Big(\nabla_q u(\frac{1}{9})-\overline{G}_2(\frac{1}{9})\Big) = \dfrac{2}{5(3+u(\frac{1}{9}))}, \\ \Delta\Big(u(\frac{1}{81})-\overline{G}_1(\frac{1}{81})\Big) = \dfrac{7}{8(2+u(\frac{1}{81}))}, \ \Delta\Big(\nabla_q u(\frac{1}{81})-\overline{G}_2(\frac{1}{81})\Big) = \dfrac{3}{4(9+u(\frac{1}{81}))}, \\ u(0)=4, \ \nabla_{\frac{1}{3}} u(0) = 11, \end{cases} \tag{3.10}$$

where $m>0$, is a constant, $\overline{G}_1(.),\overline{G}_2(.)$ are defined as in (1.4).
Let

$$f(t,u(\alpha(t)),u(\beta(t))) = \frac{u(\alpha(t))}{(1+e^t)(1+u(\alpha(t)))} + \frac{u(\beta(t))}{(1+e^{mt})(1+u(\beta(t)))}.$$

Obviously, for all $u\in C(\mathbb{T}_1,\mathbb{R}^+)$ and each $t\in\mathbb{T}_1$, we have

$$\begin{aligned} |f(t,u(\alpha(t)),u(\beta(t)))| &\leq \frac{|u(\alpha(t))|}{(1+e^t)|1+u(\alpha(t))|} + \frac{|u(\beta(t))|}{(1+e^{mt})|1+u(\beta(t))|} \\ &\leq \frac{1}{2}|u(\alpha(t))| + \frac{1}{2}|u(\beta(t))|, \end{aligned}$$

where $L_1=\frac{1}{2}, L_2=\frac{1}{2}$,
it is easy to know that $L_1+L_2 = 1 < \frac{3}{2} = \frac{1}{1^{\frac{3}{2}}(1-\frac{1}{3})}$.
For $u_1,u_2\in C(\mathbb{T}_1,\mathbb{R}^+)$ and $t\in\mathbb{T}_1$, we get

$$\begin{aligned} &|f(t,u_1(\alpha(t)),u_1(\beta(t))) - f(t,u_2(\alpha(t)),u_2(\beta(t)))| \\ &\leq \frac{1}{1+e^t}\frac{|u_1(\alpha(t))-u_2(\alpha(t))|}{(1+u_1(\alpha(t)))(1+u_2(\alpha(t)))} + \frac{1}{1+e^{mt}}\frac{|u_1(\beta(t))-u_2(\beta(t))|}{(1+u_1(\beta(t)))(1+u_2(\beta(t)))}, \end{aligned}$$

where $\frac{1}{1+e^t}\in L^{\frac{3}{2}}(\mathbb{T}_1,\mathbb{R})$, $\quad \frac{1}{1+e^{mt}}\in L^3(\mathbb{T}_1,\mathbb{R})$.
Set

$$I_1(u(\frac{1}{9})) = \frac{1}{2(1+u(\frac{1}{9}))}, \qquad I_2(u(\frac{1}{81})) = \frac{7}{8(2+u(\frac{1}{81}))},$$

we have

$$\|I_1(u(\frac{1}{9}))\| \leq \frac{1}{2}, \qquad \|I_2(u(\frac{1}{81}))\| \leq \frac{7}{16},$$

and for every $u, v \in C(\mathbb{T}_1, \mathbb{R}^+)$, we get

$$\|I_1(u(\frac{1}{9})) - I_1(v(\frac{1}{9}))\| = \|\frac{1}{2(1+u(\frac{1}{9}))} - \frac{1}{2(1+v(\frac{1}{9}))}\| \leq \frac{1}{2}\|u(\frac{1}{9}) - v(\frac{1}{9})\|,$$

$$\|I_2(u(\frac{1}{81}))-I_2(v(\frac{1}{81}))\| = \|\frac{7}{8(2+u(\frac{1}{81}))} - \frac{7}{8(2+v(\frac{1}{81}))}\| \leq \frac{7}{32}\|u(\frac{1}{81})-v(\frac{1}{81})\|.$$

Set

$$J_1(u(\frac{1}{9})) = \frac{2}{5(3+u(\frac{1}{9}))}, \qquad J_2(u(\frac{1}{81})) = \frac{3}{4(9+u(\frac{1}{81}))},$$

we have

$$\|J_1(u(\frac{1}{9}))\| \leq \frac{2}{15}, \qquad \|J_2(u(\frac{1}{81}))\| \leq \frac{1}{12},$$

and for every $u, v \in C(\mathbb{T}_1, \mathbb{R}^+)$, we get

$$\|J_1(u(\frac{1}{9})) - J_1(v(\frac{1}{9}))\| = \|\frac{2}{5(3+u(\frac{1}{9}))} - \frac{2}{5(3+v(\frac{1}{9}))}\| \leq \frac{2}{45}\|u(\frac{1}{9}) - v(\frac{1}{9})\|,$$

$$\|J_2(u(\frac{1}{81}))-J_2(v(\frac{1}{81}))\| = \|\frac{3}{4(9+u(\frac{1}{81}))} - \frac{3}{4(9+v(\frac{1}{81}))}\| \leq \frac{1}{108}\|u(\frac{1}{81})-v(\frac{1}{81})\|.$$

Thus, all the assumptions in Theorem 3.1 are satisfied. Eq. (3.10) has at least one solution.

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Zhi-Juan Gao is a postgraduate of Hebei Normal University. Her research interests are existence of solutions and qualitative behavior of solutions for differential equations.

Department of Mathematics, College of Mathematics and Information Sciences, Hebei Normal University, Shijiazhuang 050024, P. R. China.
e-mail: 1048241558@qq.com

Xu-Yang Fu is a postgraduate of Hebei Normal University. Her research interests are existence of solutions and qualitative behavior of solutions for differential equations.

Department of Mathematics, College of Mathematics and Information Sciences, Hebei Normal University, Shijiazhuang 050024, P. R. China.
e-mail: 563722089@qq.com

Qiao-Luan Li received M.Sc. from Hebei Normal University, and D.Sc. from Hebei Normal University. She is currently a associate professor at Hebei Normal University. Her research interests are existence of solutions and qualitative behavior of solutions for differential equations.

Department of Mathematics, College of Mathematics and Information Sciences, Hebei Normal University, Shijiazhuang 050024, P. R. China.
e-mail: qll71125@163.com